

Dr. Z's Math251 Handout #15.3 (2nd ed.) [Triple Integrals]

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**Problem Type 15.3a:** Evaluate the iterated integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z) dz dy dx \quad .$$

**Example Problem 15.3a:** Evaluate the iterated integral

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz dz dy dx \quad .$$

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**Steps**

**1.** You go from **inside to outside**. First isolate and compute the **inner** integral

$$\int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z) dz \quad ,$$

which in this problem is a  $z$ -integral, since in  $dz dy dx$ ,  $dz$  comes **first**. The answer should not have  $z$  in it, but in general depends on  $x$  and  $y$ .

**Example**

**1.**

$$\int_0^y 2xyz dz = xyz^2 \Big|_0^y = xy \cdot y^2 - 0 = xy^3 \quad .$$

**2. Compute the middle integral**

$$\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z) dz dy \quad ,$$

using what you already know from Step

1. This should be an expression in  $x$ .

**2.**

$$\begin{aligned} & \int_x^{2x} \int_0^y 2xyz dz dy \\ &= \int_x^{2x} \left[ \int_0^y 2xyz dz \right] dy \end{aligned}$$

from step 1, this equals:

$$= \int_x^{2x} xy^3 dy \quad .$$

This integral equals:

$$x \frac{y^4}{4} \Big|_x^{2x} = x \frac{(2x)^4}{4} - x \frac{(x)^4}{4} = \frac{15}{4} x^5 \quad .$$

**3.** Do the outside integral, by viewing it as

$$\int_a^b \left[ \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z) dz dy \right] dx \quad .$$

and using the result from step 2. The final answer should not depend on anything involving  $x, y, z$  (i.e. is a pure number or a constant).

**3.**

$$\begin{aligned} & \int_0^1 \int_x^{2x} \int_0^y 2xyz dz dy dx \\ &= \int_0^1 \left[ \int_x^{2x} \int_0^y 2xyz dz dy \right] dx \quad . \end{aligned}$$

By step 2, this is

$$= \int_0^1 \frac{15}{4} x^5 dx \quad .$$

This is a Calc I integral

$$= \frac{15}{4} \int_0^1 x^5 dx = \frac{15}{4} \frac{x^6}{6} \Big|_0^1 = \frac{5}{8} \quad .$$

**Ans.:**  $\frac{5}{8}$ .

**Problem Type 15.3b:** Evaluate the triple integral

$$\int \int \int_E F(x,y,z) dV \quad ,$$

where

$$E = \{ (x,y,z) \mid a \leq x \leq b, f_1(x) \leq y \leq f_2(x), g_1(x,y) \leq z \leq g_2(x,y) \} \quad .$$

**Example Problem 15.3b:** Evaluate the triple integral

$$\int \int \int_E yz \cos(x^5) dV \quad ,$$

where

$$E = \{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq x, x \leq z \leq 2x\} \quad .$$

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### Steps

**1.** Convert the volume integral into a triple iterated integral by transcribing the limits from left to right.

$$\int \int \int_E F(x, y, z) dV$$
$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x, y, z) dz dy dx \quad .$$

**2.** Compute the iterated integral like we did in 15.3a. First the inner integral, then the middle, then the outer.

### Example

**1.** Our volume integral equals the iterated integral

$$\int_0^1 \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx \quad .$$

**2.** The inner integral is:

$$\begin{aligned} \int_x^{2x} yz \cos(x^5) dz &= y \cos(x^5) \int_x^{2x} z dz \\ &= y \cos(x^5) \frac{z^2}{2} \Big|_x^{2x} = y \cos(x^5) \frac{(2x)^2 - x^2}{2} \\ &= \frac{3}{2} yx^2 \cos(x^5) \quad . \end{aligned}$$

**3.** Compute the middle integral

$$\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x, y, z) dz dy \quad ,$$

by writing it as

$$\int_{f_1(x)}^{f_2(x)} \left[ \int_{g_1(x,y)}^{g_2(x,y)} F(x, y, z) dz \right] dy \quad ,$$

and using the result from the previous step.

**3.**

$$\int_0^x \int_x^{2x} yz \cos(x^5) dz dy = \int_0^x \left[ \int_x^{2x} yz \cos(x^5) dz \right] dy$$

By step 2, this equals

$$\begin{aligned} \int_0^x \frac{3}{2} y x^2 \cos(x^5) dy &= \frac{3}{2} x^2 \cos(x^5) \int_0^x y dy \\ &= \frac{3}{2} x^2 \cos(x^5) \frac{y^2}{2} \Big|_0^x \\ &= \frac{3}{2} x^2 \cos(x^5) \frac{x^2 - 0^2}{2} = \frac{3}{4} x^4 \cos(x^5) \quad . \end{aligned}$$

**4.** Compute the outside integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x, y, z) dz dy dx \quad ,$$

by writing it as

$$\int_a^b \left[ \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x, y, z) dz dy \right] dx \quad ,$$

and using the result from the previous step.

**4.**

$$\int_0^1 \int_0^x \int_x^{2x} yz \cos(x^5) dz dy dx$$

$$= \int_0^1 \left[ \int_0^x \int_x^{2x} yz \cos(x^5) dz dy \right] dx \quad .$$

By step 3, this equals

$$= \int_0^1 \frac{3}{4} x^4 \cos(x^5) dx = \frac{3}{4} \int_0^1 x^4 \cos(x^5) dx \quad .$$

Making the  $u$ -substitution  $u = x^5$ , gives that this equals

$$= \frac{3}{4} \cdot \frac{1}{5} \sin(x^5) \Big|_0^1 = \frac{3}{20} (\sin(1) - \sin(0)) = \frac{3 \sin(1)}{20} \quad .$$

**Ans.:**  $\frac{3 \sin(1)}{20}$ .

**Problem Type 15.3c:** Evaluate the triple integral

$$\int \int \int_E F(x, y, z) dV \quad ,$$

where  $E$  is bounded by the surfaces  $z = g_1(x, y)$   $z = g_2(x, y)$  and possibly some planes of the form  $x = a$ .

**Example Problem 15.3c:** Evaluate the triple integral

$$\int \int \int_E x^2 y^2 dV \quad ,$$

where  $E$  is bounded by the parabolic cylinder  $z = 1 - y^2$  and the planes  $z = 0$ ,  $x = 1$  and  $x = -1$ .

## Steps

1. Express the solid region  $E$  in the form

$$E = \{(x, y, z) | (x, y) \in D,$$

$$u_1(x, y) \leq z \leq u_2(x, y)\} \quad ,$$

where  $D$  is the projection of  $E$  on the  $xy$ -plane. To find  $D$ , set the two surfaces equal to each other i.e. solve  $g_1(x, y) = g_2(x, y)$  and add to it the other restrictions only involving  $x$  and/or  $y$ . After you figure out  $D$ , express  $E$  in the form

$$E = \{(x, y, z) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x),$$

$$u_1(x, y) \leq z \leq u_2(x, y)\} \quad .$$

## Example

1. To get the projection  $D$  solve  $1 - y^2 = 0$ , which gives  $y = -1$  and  $y = 1$ . Together with  $x = 1$  and  $x = -1$ , the region  $D$  is

$$D = \{(x, y) | -1 \leq x \leq 1, -1 \leq y \leq 1\} \quad .$$

And so

$$E = \{(x, y, z) | (x, y) \in D, 0 \leq z \leq 1 - y^2\} \quad ,$$

and finally (combining it with  $D$ ), we get

$$E = \{(x, y, z) | -1 \leq x \leq 1, -1 \leq y \leq 1, 0 \leq z \leq 1 - y^2\} \quad .$$

**2.** Convert the triple integral into an iterated integral.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} F(x, y, z) dz dy dx \quad .$$

Now evaluate that iterated integral by going from the inside to the outside.

**2.**

$$\int_{-1}^1 \int_{-1}^1 \int_0^{1-y^2} x^2 y^2 dz dy dx \quad .$$

The inner integral is:

$$\int_0^{1-y^2} x^2 y^2 dz = x^2 y^2 \int_0^{1-y^2} dz = x^2 y^2 (1-y^2) = x^2 (y^2 - y^4)$$

The middle integral is:

$$\begin{aligned} \int_{-1}^1 \int_0^{1-y^2} x^2 y^2 dz dy &= \int_{-1}^1 \left[ \int_0^{1-y^2} x^2 y^2 dz \right] dy \\ &= \int_{-1}^1 x^2 (y^2 - y^4) dy = x^2 \left( \frac{y^3}{3} - \frac{y^5}{5} \right) \Big|_{-1}^1 = \frac{4}{15} x^2 \quad . \end{aligned}$$

The outer integral (i.e. the whole thing) is

$$\begin{aligned} \int_{-1}^1 \int_{-1}^1 \int_0^{1-y^2} x^2 y^2 dz dy dx \\ = \int_{-1}^1 \left[ \int_{-1}^1 \int_0^{1-y^2} x^2 y^2 dz dy \right] dx \quad . \end{aligned}$$

By the previous result, this is:

$$= \int_{-1}^1 \frac{4}{15} x^2 dx = \frac{4}{15} \frac{x^3}{3} \Big|_{-1}^1 = \frac{4}{15} \frac{2}{3} = \frac{8}{45} \quad .$$

**Ans.:**  $\frac{8}{45}$ .

### A Problem from a Previous Final

Find the volume of the solid bounded by the cylinder  $y = x^2$  and the planes  $z = 0$  and  $y + z = 1$ . Simplify as much as you can.

**Ans.**  $\frac{8}{15}$ .