Dr. Z's Math251 Handout #15.2 (2nd ed.) [Double Integrals over General Regions]

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Problem Type 15.2a: Evaluate the double integral

$$\int\int_D F(x,y)\,dA \quad ; \quad D = \{\,(x,y)\,|\, a \leq x \leq b, f(x) \leq y \leq g(x)\,\} \quad .$$

Or

$$\int\int_D F(x,y)\,dA \quad ; \quad D = \{\,(x,y)\,|\, a \leq y \leq b, f(y) \leq x \leq g(y)\,\} \quad .$$

Example Problem 15.2a: Evaluate the double integral

$$\int\int_D e^{y^2}\,dA \quad ; \quad D = \{\,(x,y)\,|\, 0 \leq y \leq 1, 0 \leq x \leq y\,\} \quad .$$

Steps

1. Set up an **iterated integral** with the variable (x or y) that is the 'boss' the *outside* integral and the the other variable the *inner* integral.

For the first kind (Type I region) we have

$$\int\int_D F(x,y)\,dA = \int_a^b \int_{f(x)}^{g(x)} F(x,y)\,dy\,dx$$

where the x-integral is on the *outside* and the y-integral on the *inside*.

For the second kind (Type II region) we have

$$\int\int_D F(x,y)\,dA = \int_a^b \int_{f(y)}^{g(y)} F(x,y)\,dx\,dy$$

where the y-integral is on the *outside* and the x-integral on the *inside*.

Example

1. Here we have a type II region, and the iterated integral is

$$\int\int_D e^{y^2}\,dA = \int_0^1 \int_0^y e^{y^2}\,dx\,dy \quad .$$

2. Evaluate the iterated integral, by first doing the inner integral, getting an expression in the outer variable, and then doing the outer integral.

2. The inner integral is

$$\int_0^y e^{y^2}\,dx = e^{y^2}\int_0^y dx$$

$$= e^{y^2}\cdot[y-0] = ye^{y^2} \quad .$$

The outer integral is

$$\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left[\int_0^y e^{y^2}\,dx\right] dy$$

$$= \int_0^1 ye^{y^2}dy = (1/2)e^{y^2}\Big|_0^1 = (1/2)[e^{1^2}-e^{0^2}] = (e-1)/2 \quad .$$

Ans.: $(e-1)/2$.

Problem Type 15.2b: Find the volume of the solid that lies under the surface $z = F(x,y)$ and above the region bounded by $y = f(x)$ and $y = g(x)$.

Example Problem 15.2b: Find the volume of the solid that lies under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^2$.

Steps

1. Here we have an extra step of finding the region, and expressing it either as type I or type II. It helps to sketch the region. If the bounding curves are of the form $y = f(x), y = g(x)$ then it is going to be a type I region. If the bounding curves are of the form $x = f(y), x = g(y)$ then it is going to be a type II region.

To get the a and b in $a \le x \le b$, we solve $f(x) = g(x)$, and usually get two roots. These are our a and b. Then by looking at the sketch (or by plugging-in a random value) decide who is the top and who is at the bottom.

Example

1. Here the bounding curves of our region are $y = x$ and $y = x^2$. Setting them equal, we have to solve $x = x^2$ which is the same as $x - x^2 = 0$ which is the same as $x(1 - x) = 0$ and we get $x = 0$ and $x = 1$. Now from a diagram (or plug in $x = 1/2$ and see that $(1/2)^2 < 1/2$) we see that $y = x$ is at the **top** and $y = x^2$ is at the **bottom**). So our region is

$$D = \{\,(x,y)\,|\,0 \le x \le 1\,,\,x^2 \le y \le x\,\} \quad .$$

2. Compute the area integral

$$\int\int_D f(x,y)\,dA \quad ,$$

by first converting it to an iterated intergral, and then evaluating it step-by-step. First the inner integral, and then the outer integral.

Note: The answers should **always** be positive. Volumes are never negative! If you get a negative number it means that you messed up somewhere.

2. Solving for z the plane $x+2y-z=0$ is really $z=x+2y$ so our **integrand** is $f(x,y)=x+2y$. The required volume is

$$\int\int_D (x+2y)\,dA \quad ,$$

which equals the iterated integral

$$\int_0^1 \int_{x^2}^{x} (x+2y)\,dy\,dx \quad .$$

First we do the inner integral

$$\int_{x^2}^{x} (x+2y)\,dy = xy+y^2\Big|_{x^2}^{x}$$

$$= x\cdot x + x^2 - (x\cdot x^2 + (x^2)^2) = 2x^2 - x^3 - x^4 \quad .$$

Having done the inner integral, we are ready for the outer intgeral

$$\int_0^1 \int_{x^2}^{x} (x+2y)\,dy\,dx$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x+2y)\,dy\right]\,dx$$

$$= \int_0^1 (2x^2 - x^3 - x^4)\,dx$$

$$= 2x^3/3 - x^4/4 - x^5/5\Big|_0^1 = 2/3 - 1/4 - 1/5 = 13/60 \quad .$$

Ans.: 13/60.

Problem Type 15.2c: Sketch the region of integration and change the order of integration.

$$\int_a^b \int_{f_1(x)}^{f_2(x)} F(x,y)\,dy\,dx$$

where $F(x,y)$ is not given specifically (i.e. is left as an abstract function).

Example Problem 15.2c: Sketch the region of integration and change the order of integration.

$$\int_0^1 \int_{4x}^{4} F(x,y)\,dy\,dx$$

Steps

Example

1. Sketch the type I region

$$D = \{ (x,y) \,|\, a \le x \le b\,,\; f_1(x) \le y \le f_2(x) \}$$

and express it as a type II region

$$D = \{ (x,y) \,|\, c \le y \le d\,,\; g_1(y) \le x \le g_2(y) \}$$

1. The region of integration, D is

$$D = \{ (x,y) \,|\, 0 \le x \le 1\,,\; 4x \le y \le 4 \} \quad .$$

This is a triangle whose vertices are $(0,0)$ $(0,1)$ and $(1,4)$. Viewing it from the point of view of the y-axis, the **main road** is $0 \le y \le 4$ and the "side streets" are horizontal line-segments stretching from $x = 0$ to $x = y/4$. So our region D written as a type II integral is:

$$D = \{ (x,y) \,|\, 0 \le y \le 4\,,\; 0 \le x \le y/4 \} \quad .$$

2. Set-up the iterated integral

$$\int_c^d \int_{g_1(x)}^{g_2(x)} F(x,y)\, dx\, dy \quad .$$

Of course you can't evaluate it, since you were not given $F(x,y)$.

2.

$$\int_0^4 \int_0^{y/4} F(x,y)\, dx\, dy \quad .$$

This is the **Ans.** .

A Problem from a previous Final

Change the order of integration in

$$\int_1^2 \int_0^{\ln x} f(x,y)\, dy\, dx \quad .$$

Ans.:

$$\int_0^{\ln 2} \int_{e^y}^{2} f(x,y)\, dx\, dy \quad .$$

Another Problem from a previous Final

Evaluate

$$\int_0^4 \int_{y/4}^1 \frac{12}{(x^2+1)^4}\, dx\, dy \quad ,$$

by inverting the order of integration and evaluating the new iterated integral.

Ans.: 7.