Dr. Z's Math251 Handout #14.7 (2nd ed.) [Optimization in Several Variables]

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Problem Type 14.7a: Find the local maximum and minimum values and saddle point(s) of the function $f(x,y)$.

Example Problem 14.7a: Find the local maximum and minimum values and saddle point(s) of the function

$$f(x,y) = x^3y + 12x^2 - 8y \quad .$$

Steps

1. Find the partial derivatives f_x and f_y. For future reference, also compute f_{xx}, f_{xy} and f_{yy}.

2. Set both f_x and f_y to zero. Then solve the system of two equations and two unknowns. The solutions are the **critical points**.

Example

1.

$$f_x = \frac{\partial}{\partial x}(x^3y+12x^2-8y) = 3x^2y+24x \quad ,$$

$$f_y = \frac{\partial}{\partial y}(x^3y+12x^2-8y) = x^3-8 \quad ,$$

$$f_{xx} = \frac{\partial}{\partial x}(3x^2y+24x) = 6xy+24 \quad ,$$

$$f_{xy} = \frac{\partial}{\partial y}(3x^2y+24x) = 3x^2 \quad ,$$

$$f_{yy} = \frac{\partial}{\partial y}(x^3-8) = 0 \quad .$$

2.

$$3x^2y + 24x = 0 \quad , \quad x^3 - 8 = 0 \quad .$$

Which is the same as

$$x(3xy+24) = 0 \quad , \quad x^3 - 8 = 0 \quad .$$

From the second equation we get $x = 2$, and plugging it into the first we get $2 \cdot (6y+24) = 0$, so $y = -4$. It turns out that in this problem there is only one critical point: $(2,-4)$.

3. Plug the point(s), one at a time, into f_{xx}, f_{xy}, f_{zz} and for each compute the **discriminant** $D = f_{xx}f_{yy} - [f_{xy}]^2$. At the examined point (a, b):

If $D > 0$ and $f_{xx} > 0$ then (a, b) is a **local minimum** and the local minimum value is $f(a, b)$.

If $D > 0$ and $f_{xx} < 0$ then (a, b) is a **local maximum** and the local maximum value is $f(a, b)$.

If $D < 0$, then (a, b) is neither max. nor min but a **saddle point**.

If $D = 0$ then we **don't know** (the test is inconclusive).

3.

$$f_{xx}(2, -4) = 6(2)(-4) + 24 = -24 \quad ,$$

$$f_{xy}(2, -4) = 3 \cdot 2^2 = 12 \quad ,$$

$$f_{yy}(2, -4) = 0 \quad .$$

Since $D = (-24) \cdot 0 - 12^2 = -144$ is **negative**, this is a **saddle point**.

Ans.: The function has no maximum values and no mimimum values. It has one saddle point at $(2, -4)$.

Problem Type 14.7b: Find the absolute maximum and minimum values of f on the set D

$$f(x,y) = Expression(x,y) \quad ,$$

$$S = \{(x,y) \,|\, a_1 \le x \le a_2 \,,\, b_1 \le y \le b_2 \,\} \quad .$$

Example Problem 14.7b: Find the absolute maximum and minimum values of f on the set S

$$f(x,y) = 4x + 6y - x^2 - y^2 \quad ,$$

$$S = \{(x,y) \,|\, 0 \le x \le 4 \,,\, 0 \le y \le 5 \,\} \quad .$$

Steps

1. First find the **critical points** by computing f_x, f_y, setting them both equal to 0, and solving for x and y. Only retain those points that belong to S. Then plug-in these point(s) into f, and keep them for comparison later on.

Example

1. $f_x = 4 - 2x$, $f_y = 6 - 2y$. Solving

$$4 - 2x = 0 \quad , \quad 6 - 2y = 0 \quad ,$$

gives one solution $x = 2, y = 3$. So $(2,3)$ is a critial point. Also $f(2,3) = 4 \cdot 2 + 6 \cdot 3 - 2^2 - 3^2 = 8 + 18 - 4 - 9 = 13$.

2. For each part of the boundary (in the case of a rectangle there are four sides), find the absolute max and min, like you did way back in Calc I.

2. On the LEFT side $x = 0$ and $0 \leq y \leq 5$. $f(0, y) = 6y - y^2$. Let's call this function, for now $F(y)$. $F'(y) = 6 - 2y$ which is 0 at $y = 3$. $F(3) = 6 \cdot 3 - 3^2 = 18 - 9 = 9$. At the endpoints $F(0) = 0, F(5) = 5$. So looking at the numbers $0, 3, 9$, the largest is 9 and the smallest is 0

abs. min. on Left Side: 0,

abs. max. on Left Side: 9.

On the RIGHT side $x = 4$ and $0 \leq y \leq 5$. $f(4, y) = 16 + 6y - 16 - y^2 = 6y - y^2$. Let's call this function, for now $F(y)$. $F'(y) = 6 - 2y$ which is 0 at $y = 3$. $F(3) = 6 \cdot 3 - 3^2 = 18 - 9 = 9$. At the endpoints $F(0) = 0, F(5) = 5$. So looking at the numbers $0, 5, 9$, the largest is 9 and the smallest is 0

abs. min. on Right Side: 0,

abs. max. on Right Side: 9.

On the DOWN side $y = 0$ and $0 \leq x \leq 4$. $f(x, 0) = 4x - x^2$. Let's call this function, for now $F(x)$. $F'(x) = 4 - 2x$ which is 0 at $x = 2$, and $F(2) = 4$. At the endpoints $F(0) = 0, F(4) = 0$. So looking at the numbers $4, 0, 0$, the largest is 4 and the smallest is 0

abs. min. on DOWN Side: 0,

abs. max. on DOWN Side: 4.

On the UP side $y = 5$ and $0 \leq x \leq 4$. $f(x, 5) = 4x - x^2 + 5$. Let's call this function, for now $F(x) = 4 * x - x^2 + 5$. $F'(x) = 4 - 2x$ which is 0 at $x = 2$, and $F(2) = 9$. At the endpoints $F(0) = 5, F(4) = 5$. So looking at the numbers $4, 5, 9$, the largest is 9 and the smallest is 4

abs. min. on UP Side: 4,

abs. max. on UP Side: 9.

3. Now gather all these champions (in both min. and max. catergories) plus those came from the critical points inside the region and find the largest value, this is your **absolute maximum value** and the smallest, this is your **absolute minimum value**.

3. For abs. min the contdenders are $0, 0, 0, 4, 13$ so the **absolute minimum value** is 0.

For abs. max the contdenders are $9, 9, 4, 5, 13$ so the **absolute maximum value** is 13.

Problem Type 14.7c: Find the point on the surface $F(x, y, z) = k$ that is closest to the origin.

Example Problem 14.7c: Find the point on the surface $x^2y^2z = 1$ that is closest to the origin.

Steps

1. It is more convenient to consider the distance-squared, which is $x^2 + y^2 + z^2$. Take one of the variables (say z) (whatever is convenient) and express it in terms of the other two (say x, y), and plug it into $x^2 + y^2 + z^2$ getting a function, let's call it $f(x, y)$

Example

1. $z = 1/(x^2y^2)$ so $z^2 = x^{-4}y^{-4}$ and the distance-squared, in terms of x, y is $f(x, y) = x^2 + y^2 + x^{-4}y^{-4}$.

2. Find the critical points by taking f_x, f_y and setting them equal to zero, and solving for x and y.

2.

$$f_x = \frac{\partial}{\partial x}(x^2+y^2+x^{-4}y^{-4}) = 2x-4x^{-5}y^{-4} \quad ,$$

$$f_y = \frac{\partial}{\partial y}(x^2+y^2+x^{-4}y^{-4}) = 2y-4x^{-4}y^{-5} \quad .$$

We have to solve

$$2x-4x^{-5}y^{-4} = 0 \quad , \quad 2y-4x^{-4}y^{-5} = 0 \quad ,$$

which is the same

$$x^6y^4 = 2 \quad , \quad x^4y^6 = 2 \quad .$$

Dividing the first by the second we get $x^2/y^2 = 1$ so $x^2 = y^2$ and $x^{10} = 2$ and we get $x^2 = y^2 = 2^{1/5}$ so $x = \pm 2^{1/10}$, $y = \pm 2^{1/10}$).

3. To get the z coordinates for each of these points plug into $f(x,y)$.

3. $z = 1/(x^2y^2)$ so for each of the four possibilities $z = 1/2^{2/5} = 2^{-2/5}$.

Ans.: The points on the surface closest to the origin are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$.

Problem from a Previous Final

Find the local maximum and minimum points, the local maximum and minimum values, and saddle points of the function

$$f(x,y) = 4x^2 + y^2 + 2x^2y - 1 \quad .$$

Ans.: $(0,0)$ is the local minimum point

-1 is the local minimum value

$(\sqrt{2}, -2), (-\sqrt{2}, -2)$ are saddle points.

Another Problem from a Previous Final

Find the local maximum and minimum point(s), the local maximum and minimum values, and

saddle point(s) of the function

$$f(x, y) = 6y^2 - 2y^3 + 3x^2 + 6xy \quad .$$

Ans. Local max: none. Local Min: location: $(0, 0)$, value: 0. Saddle point: $(-1, 1)$.

Added Oct. 8, 2020: I thank Zixin Qu and Shubin Xie for correcting some typos