Dr. Z’s Math251 Handout #14.7 (2nd ed.) [Optimization in Several Variables]

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**Problem Type 14.7a:** Find the local maximum and minimum values and saddle point(s) of the function \( f(x, y) \).

**Example Problem 14.7a:** Find the local maximum and minimum values and saddle point(s) of the function
\[
f(x, y) = x^3y + 12x^2 - 8y .
\]

**Steps**

1. Find the partial derivatives \( f_x \) and \( f_y \).
   For future reference, also compute \( f_{xx} \), \( f_{xy} \) and \( f_{yy} \).

   \[
   \begin{align*}
   f_x &= \frac{\partial}{\partial x} (x^3y + 12x^2 - 8y) = 3x^2y + 24x , \\
   f_y &= \frac{\partial}{\partial y} (x^3y + 12x^2 - 8y) = x^3 - 8 , \\
   f_{xx} &= \frac{\partial}{\partial x} (3x^2y + 24x) = 6xy + 24 , \\
   f_{xy} &= \frac{\partial}{\partial y} (3x^2y + 24x) = 3x^2 , \\
   f_{yy} &= \frac{\partial}{\partial y} (x^3 - 8) = 0 .
   \end{align*}
   \]

2. Set both \( f_x \) and \( f_y \) to zero. Then solve the system of two equations and two unknowns. The solutions are the **critical points**.

   \[
   \begin{align*}
   3x^2y + 24x &= 0 , \\
   x^3 - 8 &= 0 .
   \end{align*}
   \]

   Which is the same as
   \[
   x(3xy + 24) = 0 , \\
   x^3 - 8 = 0 .
   \]

   From the second equation we get \( x = 2 \), and plugging it into the first we get
   \[
   2 \cdot (6y + 24) = 0 , \text{ so } y = -4 .
   \]

   It turns out that in this problem there is only one critical point: \( (2, -4) \).
3. Plug the point(s), one at a time, into $f_{xx}, f_{xy}, f_{zz}$ and for each compute the discriminant $D = f_{xx}f_{yy} - [f_{xy}]^2$. At the examined point $(a, b)$:

If $D > 0$ and $f_{xx} > 0$ then $(a, b)$ is a **local minimum** and the local minimum value is $f(a, b)$.

If $D > 0$ and $f_{xx} < 0$ then $(a, b)$ is a **local maximum** and the local maximum value is $f(a, b)$.

If $D < 0$, then $(a, b)$ is neither max. nor min but a **saddle point**.

If $D = 0$ then we **don’t know** (the test is inconclusive).

3. 

$$f_{xx}(2, -4) = 6(2)(-4) + 24 = -24$$

$$f_{xy}(2, -4) = 3 \cdot 2^2 = 12$$

$$f_{yy}(2, -4) = 0$$

Since $D = (-24) \cdot 0 - 12^2 = -144$ is negative, this is a **saddle point**.

**Ans.:** The function has no maximum values and no minimum values. It has one saddle point at $(2, -4)$. 
**Problem Type 14.7b**: Find the absolute maximum and minimum values of \( f \) on the set \( D \)

\[
f(x, y) = \text{Expression}(x, y),
\]

\[
S = \{(x, y) | a_1 \leq x \leq a_2, b_1 \leq y \leq b_2\},
\]

**Example Problem 14.7b**: Find the absolute maximum and minimum values of \( f \) on the set \( S \)

\[
f(x, y) = 4x + 6y - x^2 - y^2,
\]

\[
S = \{(x, y) | 0 \leq x \leq 4, 0 \leq y \leq 5\}.
\]

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**Steps**

1. First find the **critical points** by computing \( f_x \), \( f_y \), setting them both equal to 0, and solving for \( x \) and \( y \). Only retain those points that belong to \( S \). Then plug-in these point(s) into \( f \), and keep them for comparison later on.

**Example**

1. \( f_x = 4 - 2x, \ f_y = 6 - 2y \). Solving

\[
4 - 2x = 0, \quad 6 - 2y = 0,
\]

gives one solution \( x = 2, y = 3 \). So \((2, 3)\) is a critical point. Also \( f(2, 3) = 4 \cdot 2 + 6 \cdot 3 - 2^2 - 3^2 = 8 + 18 - 4 - 9 = 13 \).
2. For each part of the boundary (in the case of a rectangle there are four sides), find the absolute max and min, like you did way back in Calc I.

2. On the LEFT side $x = 0$ and $0 \leq y \leq 5$. \(f(0, y) = 6y - y^2\). Let’s call this function, for now \(F(y)\). \(F'(y) = 6 - 2y\) which is 0 at \(y = 3\). \(F(3) = 6 \cdot 3 - 3^2 = 18 - 9 = 9\). At the endpoints \(F(0) = 0, F(5) = 5\). So looking at the numbers 0, 3, 9, the largest is 9 and the smallest is 0

abs. min. on Left Side: 0,

abs. max. on Left Side: 9.

On the RIGHT side $x = 4$ and $0 \leq y \leq 5$. \(f(4, y) = 16 + 6y - 16 - y^2 = 6y - y^2\). Let’s call this function, for now \(F(y)\). \(F'(y) = 6 - 2y\) which is 0 at \(y = 3\). \(F(3) = 6 \cdot 3 - 3^2 = 18 - 9 = 9\). At the endpoints \(F(0) = 0, F(5) = 5\). So looking at the numbers 0, 5, 9, the largest is 9 and the smallest is 0

abs. min. on Right Side: 0,

abs. max. on Right Side: 9.

On the DOWN side $y = 0$ and $0 \leq x \leq 4$. \(f(x, 0) = 4x - x^2\). Let’s call this function, for now \(F(x)\). \(F'(x) = 4 - 2x\) which is 0 at \(x = 2\), and \(F(2) = 4\). At the endpoints \(F(0) = 0, F(4) = 0\). So looking at the numbers 4, 0, 0, the largest is 4 and the smallest is 0

abs. min. on DOWN Side: 0,

abs. max. on DOWN Side: 4.

On the UP side $y = 5$ and $0 \leq x \leq 4$. \(f(x, 5) = 4x - x^2 + 5\). Let’s call this function, for now \(F(x) = 4x - x^2 + 5\). \(F'(x) = 4 - 2x\) which is 0 at \(x = 2\), and \(F(2) = 9\). At the endpoints \(F(0) = 5, F(4) = 5\). So looking at the numbers 4, 5, 9, the largest is 9 and the smallest is 4

abs. min. on UP Side: 4,

abs. max. on UP Side: 9.
3. Now gather all these champions (in both min. and max. categories) plus those came from the critical points inside the region and find the largest value, this is your **absolute maximum value** and the smallest, this is your **absolute minimum value**.

3. For abs. min the contenders are 0, 0, 0, 4, 13 so the **absolute minimum value** is 0.
For abs. max the contenders are 9, 9, 4, 5, 13 so the **absolute maximum value** is 13.

**Problem Type 14.7c:** Find the point on the surface $F(x, y, z) = k$ that is closest to the origin.

**Example Problem 14.7c:** Find the point on the surface $x^2y^2z = 1$ that is closest to the origin.

**Steps**

1. It is more convenient to consider the distance-squared, which is $x^2 + y^2 + z^2$. Take one of the variables (say $z$) (whatever is convenient) and express it in terms of the other two (say $x, y$), and plug it into $x^2 + y^2 + z^2$ getting a function, let’s call it $f(x, y)$.

**Example**

1. $z = 1/(x^2y^2)$ so $z^2 = x^{-4}y^{-4}$ and the distance-squared, in terms of $x, y$ is $f(x, y) = x^2 + y^2 + x^{-4}y^{-4}$. 
2. Find the critical points by taking $f_x, f_y$ and setting them equal to zero, and solving for $x$ and $y$.

$$f_x = \frac{\partial}{\partial x}(x^2+y^2+x^{-4}y^{-4}) = 2x-4x^{-5}y^{-4},$$

$$f_y = \frac{\partial}{\partial y}(x^2+y^2+x^{-4}y^{-4}) = 2y-4x^{-4}y^{-5}.$$

We have to solve

$$2x-4x^{-5}y^{-4} = 0, \quad 2y-4x^{-4}y^{-5} = 0,$$

which is the same

$$x^6y^4 = 2, \quad x^4y^6 = 2.$$

Dividing the first by the second we get $x^2/y^2 = 1$ so $x^2 = y^2$ and $x^{10} = 2$ and we get $x^2 = y^2 = 2^{1/5}$ so $x = \pm 2^{1/10}, y = \pm 2^{1/10}$.

3. To get the $z$ coordinates for each of these points plug into $f(x, y)$.

$$z = 1/(x^2y^2) \text{ so for each of the four possibilities } z = 1/2^{2/5} = 2^{-2/5}.$$

Ans.: The points on the surface closest to the origin are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$.

Problem from a Previous Final

Find the local maximum and minimum points, the local maximum and minimum values, and saddle points of the function

$$f(x, y) = 4x^2 + y^2 + 2x^2y - 1.$$ 

Ans.: $(0, 0)$ is the local minimum point

$-1$ is the local minimum value

$(\sqrt{2}, -2), (-\sqrt{2}, -2)$ are saddle points.

Another Problem from a Previous Final

Find the local maximum and minimum point(s), the local maximum and minimum values, and
saddle point(s) of the function

\[ f(x, y) = 6y^2 - 2y^3 + 3x^2 + 6xy \, . \]

**Ans.** Local max: none. Local Min: location: (0,0), value: 0. Saddle point: (−1,1).

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