Problem Type 14.5a: Find the directional derivative of the function \( f(x,y,z) \) at the point \((x_0,y_0,z_0)\) in the direction \((v_1,v_2,v_3)\).

Example Problem 14.5a: Find the directional derivative of the function \( f(x,y,z) = \ln(x^2 + y^2 + z^2) \) at the point \((2,1,3)\) in the direction \((1,2,2)\).

Steps

1. Find the gradient \( \nabla f = (f_x, f_y, f_z) \) by taking all the first partial derivatives.
   Also find the unit vector in the direction of \((v_1,v_2,v_3)\) by dividing by its length.

   Example

   \[
   f_x = \frac{2x}{x^2 + y^2 + z^2}, \quad f_y = \frac{2y}{x^2 + y^2 + z^2}, \quad f_z = \frac{2z}{x^2 + y^2 + z^2}.
   \]

   \[
   \nabla f = \left( \frac{2x}{x^2 + y^2 + z^2}, \frac{2y}{x^2 + y^2 + z^2}, \frac{2z}{x^2 + y^2 + z^2} \right)
   \]

   \[
   |(1,2,2)| = \sqrt{1^2 + 2^2 + 2^2} = 3, \text{ so }
   \]

   \[
   u = \frac{1}{3}(1,2,2) = \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right)
   \]

2. Plug-in \( x = x_0, y = y_0, z = z_0 \) into \( \nabla f \).

   Example

   \[
   \nabla f(2,1,3) = \left( \frac{2 \cdot 2}{2^2 + 1^2 + 3^2}, \frac{2 \cdot 1}{2^2 + 1^2 + 3^2}, \frac{2 \cdot 3}{2^2 + 1^2 + 3^2} \right)
   \]

   \[
   = \left( \frac{2}{7}, \frac{1}{7}, \frac{3}{7} \right).
   \]
3. Take the dot product $\nabla f . \mathbf{u}$.

$$\begin{align*}
\left(\frac{2}{7}, \frac{1}{7}, \frac{3}{7}\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \\
= \frac{2}{7} \cdot \frac{1}{3} + \frac{1}{7} \cdot \frac{2}{3} + \frac{3}{7} \cdot \frac{2}{3} = \frac{10}{21}
\end{align*}$$

Ans.: The requested directional derivative is $\frac{10}{21}$.

**Problem Type 14.5b:** Find the maximum rate of change of $f$ at the given point and the direction in which it occurs.

$$f(x, y) = Expression(x, y) \quad (x_0, y_0).$$

**Example Problem 14.5b:** Find the maximum rate of change of $f$ at the given point and the direction in which it occurs.

$$f(x, y) = \sin(xy) \quad (1, 0).$$

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**Steps**

1. Find the gradient

$$\nabla f = (f_x, f_y)$$

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**Example**

1. $f_x = y \cos(xy), f_y = x \cos(xy)$. So

$$\nabla f = (f_x, f_y) = (y \cos(xy), x \cos(xy)).$$

2. Plug-in $x = x_0, y = y_0$ into $\nabla f$.

$$\nabla f(1, 0) = (0 \cdot \cos(0), 1 \cdot \cos(0)) = (0, 1).$$

3. The maximum rate of change of $f$ is simply the length of $\nabla f$ at the designated point. The direction in which it occurs is that direction. So find the unit vector in that direction.

$$|\langle 0, 1 \rangle| = \sqrt{0^2 + 1^2} = 1.$$

$\langle 0, 1 \rangle$ is already a unit vector, so the direction is $\langle 0, 1 \rangle$.

Ans.: The maximum rate of change is 1 in the direction $\langle 0, 1 \rangle$ (or $\mathbf{j}$).
A Problem from a Previous Final

Let

\[ f(x, y, z) = -x^2 + y^2 + z^2 - 1. \]

(a) (2 points) Compute \( \nabla f \).

(b) (5 points) Find a normal to the level surface \( f(x, y, z) = 0 \) at the point \((1, 1, 1)\), and give an equation for the tangent plane to that surface at that point.

(c) (6 points) Compute the directional derivative of \( f(x, y, z) \) at the point \((1, 1, 1)\) in the direction \( \langle 1, 2, 2 \rangle \).

Ans.: a) \( \langle -2x, 2y, 2z \rangle \); b) \( z = x - y + 1 \); c) 2.