Dr. Z’s Math251 Handout #14.4 (2nd ed.) [Differentiability, Linear Approximations, and Tangent Planes]

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**Problem Type 14.4a:** Find an equation of the tangent plane to the given surface at the specified point.

\[ z = f(x, y), \quad (x_0, y_0, z_0). \]

**Example Problem 14.4a:** Find an equation of the tangent plane to the given surface at the specified point.

\[ z = 9x^2 + y^2 + 6x - 3y + 5, \quad (1, 2, 18). \]

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**Steps**

1. First make sure that \( z_0 = f(x_0, y_0) \) or refuse to do the problem. Then take \( f_x = \frac{\partial f}{\partial x} \) and \( f_y = \frac{\partial f}{\partial y} \).

   \[ f_x = \frac{\partial}{\partial x}(9x^2 + y^2 + 6x - 3y + 5) = 18x + 6, \]
   \[ f_y = \frac{\partial}{\partial y}(9x^2 + y^2 + 6x - 3y + 5) = 2y - 3. \]

2. Plug in \( x = x_0, y = y_0 \) into \( f_x \) and \( f_y \) that you have just found.

   \[ f_x(1, 2) = 18 \cdot 1 + 6 = 24, \]
   \[ f_y(1, 2) = 2 \cdot 2 - 3 = 1. \]

3. An equation for the tangent plane for the given surface at the given point is

   \[ z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \]

   Plug-in the \( x_0, y_0, z_0 \) from the data of the problem and \( f_x(x_0, y_0), f_y(x_0, y_0) \) from step 2.

3. \( z - 18 = 24(x - 1) + (y - 2). \) Or, in expanded form: \( z = 24x + y - 8. \)
**Problem Type 14.4b:** Explain why the function is differentiable at the given point. Then find the linearization of that function at the given point.

\[ z = f(x, y) \quad (a, b) \quad . \]

**Example Problem 14.4b:** Explain why the function is differentiable at the given point. Then find the linearization of that function at the given point.

\[ z = e^x \sin(xy) \quad (0, \pi/2) \quad . \]

**Steps**

1. Find the first partial derivatives \( f_x \) and \( f_y \).

\[
L(x, y) = \left[ f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \right]
\]

2. If both \( f_x \) and \( f_y \) are continuous at the designated point \((a, b)\) (i.e. they are defined and do not blow up), then the function is differentiable at that point, and it is OK to have a linearization.

**Example**

\[
f_x = \frac{\partial}{\partial x} e^x \sin(xy) = \left( \frac{\partial}{\partial x} e^x \right) \sin(xy) + e^x \left( \frac{\partial}{\partial x} \sin(xy) \right) = e^x \sin(xy) + e^x y \cos(xy),
\]

\[
f_y = \frac{\partial}{\partial y} e^x \sin(xy) = e^x \left( \frac{\partial}{\partial y} \sin(xy) \right) = e^x x \cos(xy).
\]

2. \( f_x = e^x \sin(xy) + e^x y \cos(xy) \) and \( f_y = e^x x \cos(xy) \) are both continuous at the point \((0, \pi/2)\) (they are continuous everywhere for that matter), so the function is differentiable there. Now

\[
f(0, \pi/2) = e^0 \sin(0) = 0, \quad f_x(0, \pi/2) = e^0 \sin(0) + e^0 \cdot (\pi/2) \cos(0) = \pi/2, \quad f_y(0, \pi/2) = e^0 \cdot 0 \cdot \cos(0) = 0.
\]

The linearization is

\[
L(x, y) = 0 + (\pi/2) \cdot (x - 0) + 0 \cdot (y - \pi/2) = (\pi/2)x
\]

**Ans.:** The linearization of the function \( e^x \sin(xy) \) at the point \((0, \pi/2)\) is \((\pi/2)x\).
**Problem Type 14.4c:** Use the linear approximation of the function $f(x, y)$ at $(a, b)$ to approximate $f(a_1, b_1)$, where $(a_1, b_1)$ is “near” $(a, b)$.

**Example Problem 14.4b:** Use the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 2y^2}$ at $(3, 1)$ to approximate $f(3.05, .97)$.

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**Steps**

1. The beginning is exactly as before. Just find the linearization of the function at the designated point $(a, b)$.

So first find the first partial derivatives $f_x$ and $f_y$.

**Example**

1. 

$$f_x = \frac{\partial}{\partial x} (20 - x^2 - 2y^2)^{1/2} = (1/2)(20 - x^2 - 2y^2)^{-1/2} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{20 - x^2 - 2y^2}} ,$$

$$f_y = \frac{\partial}{\partial y} (20 - x^2 - 2y^2)^{1/2} = (1/2)(20 - x^2 - 2y^2)^{-1/2} \cdot (-4y)$$

$$= \frac{-2y}{\sqrt{20 - x^2 - 2y^2}} .$$
2. If both $f_x$ and $f_y$ are \textit{continuous} at the designated point $(a, b)$ (i.e. they are defined and do not blow up), then the function is \textit{differentiable} at that point, and it is OK to have a linearization.

$$L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

2. $f_x = \frac{-x}{\sqrt{20-x^2-2y^2}}$ and $f_y = \frac{-2y}{\sqrt{20-x^2-2y^2}}$ are both continuous at the point $(3, 1)$ (the argument of the square-root is 9 which is positive and nothing blows up). Now

$$f(3, 1) = \sqrt{20 - 3^2 - 2 \cdot 1^2} = 3$$

$$f_x(3, 1) = \frac{-3}{\sqrt{20 - 3^2 - 2 \cdot 1^2}} = -1$$

$$f_y(3, 1) = \frac{-2 \cdot 1}{\sqrt{20 - 3^2 - 2 \cdot 1^2}} = -2/3$$

and the linearization is

$$L(x, y) = 3 - (x - 3) - (2/3)(y - 1)$$

So the \textbf{linear approximation}, valid near $(3, 1)$ is

$$f(x, y) \approx 3 - (x - 3) - (2/3) \cdot (y - 1)$$

3. Plug-in $(a_1, b_1)$ into this approximation.

$$f(3.05, .97) \approx 3 - (3.05 - 3) - (2/3) \cdot (.97 - 1) =$$

$$3 - .05 + .02 = 2.97$$

\textbf{Ans.:} $f(3.05, .97) \approx 2.97$.

\textbf{Problem from a previous Final}

Find an equation of the tangent plane to the surface

$$z = e^{2x-3y}$$

at the point $(3, 2, 1)$. Simplify as much as you can!

\textbf{Ans.:} $z = 2x - 3y + 1$ (or $2x - 3y - z = -1$).