Dr. Z's Math251 Handout #14.4 (2nd ed.) [Differentiability, Linear Approximations, and Tangent Planes]

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Problem Type 14.4a: Find an equation of the tangent plane to the given surface at the specified point.

$$z = f(x,y) \quad , \quad (x_0, y_0, z_0) \quad .$$

Example Problem 14.4a: Find an equation of the tangent plane to the given surface at the specified point.

$$z = 9x^2 + y^2 + 6x - 3y + 5 \quad , \quad (1,2,18) \quad .$$

Steps

1. First make sure that $z_0 = f(x_0, y_0)$ or refuse to do the problem. Then take $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.

2. Plug in $x = x_0, y = y_0$ into f_x and f_y that you have just found.

3. An equation for the tangent plane for the given surface at the given point is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad .$$

Plug-in the x_0, y_0, z_0 from the data of the problem and $f_x(x_0, y_0), f_y(x_0, y_0)$ from step 2.

Example

1. $9 \cdot 1^2 + 2^2 + 6 \cdot 1 - 3 \cdot 2 + 5 = 18$, so the point $(1, 2, 18)$ indeed lies on the surface. Now

$$f_x = \frac{\partial}{\partial x}(9x^2+y^2+6x-3y+5) = 18x+6 \quad ,$$

$$f_y = \frac{\partial}{\partial y}(9x^2+y^2+6x-3y+5) = 2y-3 \quad .$$

2.

$$f_x(1,2) = 18 \cdot 1 + 6 = 24 \quad .$$

$$f_y(1,2) = 2 \cdot 2 - 3 = 1 \quad .$$

3. $z - 18 = 24(x-1) + (y-2)$. Or, in expanded form: $z = 24x + y - 8$.

Problem Type 14.4b: Explain why the function is differentiable at the given point. Then find the linearization of that function at the given point.

$$z = f(x,y) \quad , \quad (a,b) \quad .$$

Example Problem 14.4b: Explain why the function is differentiable at the given point. Then find the linearization of that function at the given point.

$$z = e^x \sin(xy) \quad , \quad (0, \pi/2) \quad .$$

Steps

1. Find the first partial derivatives f_x and f_y.

2. If both f_x and f_y are *continuous* at the designated point (a,b) (i.e. they are defined and do not blow up), then the function is *differentiable* at that point, and it is OK to have a linearization.

$$L(x,y) =$$

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad .$$

Example

1.

$$f_x = \frac{\partial}{\partial x} e^x \sin(xy) = (\frac{\partial}{\partial x} e^x) \sin(xy) + e^x (\frac{\partial}{\partial x} \sin(xy))$$

$$e^x \sin(xy) + e^x y \cos(xy) \quad ,$$

$$f_y = \frac{\partial}{\partial y} e^x \sin(xy) = e^x (\frac{\partial}{\partial y} \sin(xy)) = e^x x \cos(xy) \quad .$$

2. $f_x = e^x \sin(xy) + e^x y \cos(xy)$ and $f_y = e^x x \cos(xy)$ are both continuous at the point $(0, \pi/2)$ (they are continuous everywhere for that matter), so the function is differentiable there. Now

$$f(0, \pi/2) = e^0 \sin(0) = 0 \quad ,$$

$$f_x(0, \pi/2) = e^0 \sin(0) + e^0 \cdot (\pi/2) \cos(0) = \pi/2 \quad ,$$

$$f_y(0, \pi/2) = e^0 \cdot 0 \cdot \cos(0) = 0 \quad .$$

The linearization is

$$L(x,y) = 0 + (\pi/2) \cdot (x-0) + 0 \cdot (y - \pi/2) = (\pi/2) x \quad .$$

Ans.: The linearization of the function $e^x \sin(xy)$ at the point $(0, \pi/2)$ is $(\pi/2)x$.

Problem Type 14.4c: Use the linear apprimation of the function $f(x, y)$ at (a, b) to approximate $f(a_1, b_1)$, where (a_1, b_1) is "near" (a, b).

Example Problem 14.4b: Use the linear approximation of the function $f(x, y) = \sqrt{20 - x^2 - 2y^2}$ at $(3, 1)$ to approximate $f(3.05, .97)$.

Steps

1. The beginning is exactly as before. Just find the linearization of the function at the designated point (a, b).

So first find the first partial derivatives f_x and f_y.

Example

1.

$$f_x = \frac{\partial}{\partial x}(20-x^2-2y^2)^{1/2} = (1/2)(20-x^2-2y^2)^{-1/2}\cdot(-2x)$$

$$= \frac{-x}{\sqrt{20 - x^2 - 2y^2}} \quad ,$$

$$f_y = \frac{\partial}{\partial y}(20-x^2-2y^2)^{1/2} = (1/2)(20-x^2-2y^2)^{-1/2}\cdot(-4y)$$

$$= \frac{-2y}{\sqrt{20 - x^2 - 2y^2}} \quad .$$

2. If both f_x and f_y are *continuous* at the designated point (a,b) (i.e. they are defined and do not blow up), then the function is *differentiable* at that point, and it is OK to have a linearization.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad .$$

2. $f_x = \frac{-x}{\sqrt{20-x^2-2y^2}}$ and $f_y = \frac{-2y}{\sqrt{20-x^2-2y^2}}$ are both continuous at the point $(3,1)$ (the argument of the square-root is 9 which is positive and nothing blows up). Now

$$f(3,1) = \sqrt{20 - 3^2 - 2\cdot 1^2} = 3 \quad .$$

$$f_x(3,1) = \frac{-3}{\sqrt{20 - 3^2 - 2\cdot 1^2}} = -1 \quad ,$$

$$f_y(3,1) = \frac{-2\cdot 1}{\sqrt{20 - 3^2 - 2\cdot 1^2}} = -2/3 \quad .$$

and the linearization is

$$L(x,y) = 3 - (x-3) - (2/3)(y-1) \quad .$$

So the **linear approximation**, valid **near** $(3,1)$ is

$$f(x,y) \approx 3 - (x-3) - (2/3)\cdot(y-1) \quad .$$

3. Plug-in (a_1, b_1) into this approximation.

3.

$$f(3.05, .97) \approx 3 - (3.05-3) - (2/3)\cdot(.97-1) =$$

$$3 - .05 + .02 = 2.97 \quad .$$

Ans.: $f(3.05, .97) \approx 2.97$.

Problem from a previous Final

Find an equation of the tangent plane to the surface

$$z = e^{2x-3y} \quad ,$$

at the point $(3,2,1)$. Simplify as much as you can!

Ans.: $z = 2x - 3y + 1$ (or $2x - 3y - z = -1$).