Dr. Z's Math251 Handout #14.2 (2nd ed.) [Limits and Continuity in Several Variables]

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Problem Type 14.2a: Find the limit if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad .$$

Example Problem 14.2a: Find the limit if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{3x^2+4y^2} \quad .$$

Steps

1. If the function is *continuous* at the point, which is the case if both top and bottom are 'nice' functions like polynomials or exponentials or sines or cosines **and** the denominator does not vanish at the designated point (a,b), then just plug-it in and if the answer makes sense, then $f(a,b)$ is the limit.

If the denominator does vanish but the numerator does not, the limit automatically *does not exist*, end of story.

Unfortunately, in most cases that you are likely to get, both top and bottom are zero at the designated point (a,b), so you must go on to the next step.

Example

1. Both top and bottom vanish when we plug-in $(x,y)=(0,0)$ so we must go on.

2. The limit may still exist, but proving it formally is beyond the scope of this class, so chances are that the limit does not exist, so try to prove that the limit does **not** exist by finding limits along straight lines. $(y-b) = c(x-a)$.

2. Plugging $y = cx$ in $f(x,y)$ we get that the limit as $(x,y) \to (0,0)$ along the line $y = cx$ is

$$\lim_{x\to 0} \frac{2x^2}{3x^2 + 4(cx)^2} = \lim_{x\to 0} \frac{2x^2}{(3+4c^2)x^2} =$$

$$\lim_{x\to 0} \frac{2}{(3+4c^2)} = \frac{2}{(3+4c^2)} \quad .$$

Since this **depends** on the slope c, we get **different** limits for different lines, hence it is not always the same hence the **limit** (proper) **does not exist**.

Ans.: The limit does not exist since you get different limits when you approach the point $(0,0)$ on different lines.

Problem Type 14.2b: Find the limit if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad .$$

Example Problem 14.2b: Find the limit if it exists, or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2}{\sqrt{x^2+y^2}} \quad .$$

Steps

Example

1. If the function is *continuous* at the point, which is the case if both top and bottom are 'nice' functions like polynomials or exponentials or sines or cosines **and** the denominator does not vanish at the designated point (a, b), then just plug-it in and if the answer makes sense, then $f(a, b)$ is the limit.

If the denominator does vanish but the numerator does not, the limit automatically *does not exist*, end of story.

Unfortunately, in most cases that you are likely to get both top and bottom are zero at the designated point (a, b), so you must go on to the next step.

1. Both top and bottom vanish when we plug-in $(x, y) = (0, 0)$ so we must go on.

2. The limit may still exist, but proving it formally is beyond the scope of this class, so chances are that the limit does not exist, so try to prove that the limit does **not** exist by finding limits along straight lines $(y - b) = c(x - a)$, hoping that you would get an expression that **depends on c**, thereby showing that there are "different limits for different lines" and hence the **limit** itself definitely **does not exist**.

Alas, if the limit does **not** depend on c, then it is a **plain number**, and that number is the **candidate limit**.

2. Plugging $y = cx$ in $f(x, y)$ we get that the limit as $(x, y) \to (0, 0)$ along the line $y = cx$ is

$$\lim_{x\to 0} \frac{2x^2}{\sqrt{x^2 + (cx)^2}}$$

$$= \lim_{x\to 0} \frac{2x^2}{(\sqrt{(1 + c^2)x^2})} = \lim_{x\to 0} \frac{2x^2}{(\sqrt{(1 + c^2)}|x|)}$$

$$= \lim_{x\to 0} \frac{2x^2}{(\sqrt{(1 + c^2)}|x|)} = \lim_{x\to 0} \frac{2|x|}{(\sqrt{(1 + c^2)})} = 0 \quad .$$

Since this does **not** depend on the slope c, there is a **good chance** that the limit exists and is 0, but this is **not** a proof, since if you approach the point in question $(0, 0)$ via other paths you might get different values, and then the limit would not exist.

3. Proving formally that the above **candidate limit** is indeed correct, using $\epsilon - \delta$, is beyond the scope of this class. So we have to resort to a *trick*, that often works: converting to *polar coordinates*,

$$x = r\cos\theta \quad y = r\sin\theta \quad .$$

3. The function becomes (since $r^2 = x^2 + y^2$)

$$f(r\cos\theta, r\sin\theta) = \frac{2(r\cos\theta)^2}{r} = 2r\cos^2\theta \quad ,$$

and the limit becomes

$$\lim_{r\to 0} 2r\cos^2\theta \quad .$$

Now everything is nice and continuous (nothing blows-up) so just plug-it in $r = 0$ and get $2\cdot 0\cdot\cos^2\theta = 0$, and the limit is indeed 0.

Ans.: The limit exists and equals 0.

Note: The *origin* is $r = 0$ *not* $(r, \theta) = (0, 0)$ so you **can't** plug-in $r = 0$ **and** $\theta = 0$. It might happen that when you plug-in $r = 0$ and $\theta = 0$ you would get 0 but the limit still does not exist.

Problem Type 14.2c: Determine the set of points at which the given function $f(x, y)$ is continuous

Example Problem 14.2c: Determine the set of points at which the function is continuous:

$$(a) f(x, y) = \frac{1}{x + y + 2} \quad ,$$

$$(b) f(x, y) = \sqrt{x + y - 3} \quad ,$$

$$(c) f(x, y) = \frac{\ln(x + y)}{y - 3} \quad .$$

Steps

Example

1. The function is continuous whenever "it makes sense" and does not blow-up. Remember that whenever you have something at the bottom, you must insist that it is **not zero** for it not to make trouble. Also $\sqrt{w}$ and $\ln w$ makes sense only when $w > 0$.

1. (a) The function is continuous whenever the bottom is not zero, i.e. $x + y + 2 \neq 0$. **Ans. to (a):** The function is continuous everywhere **except the line** $x + y + 2 = 0$.

(b) Because of the square-root, the function is continuous whenever $x+y-3 > 0$.

Ans. to (b): The function is continuous in the half place $x + y > 3$.

(c) To make the numerator happy we must insist that $x + y > 0$ and to make the denominator happy we must demand that $y \neq 3$.

Ans. to (c): The half-plane $x + y > 0$ without the line $y = 3$.

Problem from a previous Final

Compute the limit

$$\lim_{(x,y,z)\to(1,1,1)} e^{-xy}\sin(\pi z/2) \quad ,$$

or prove that it does not exist.

Ans.: It exists and equals $\frac{1}{e}$.