

Dr. Z's Math251 Handout #13.4 (2nd ed.) [Curvature]

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Problem Type 13.4a: Find the curvature for

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad .$$

Example Problem 13.4a: Find the curvature for

$$\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k} \quad .$$

Steps

1. Compute $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

2. Compute the **cross product**

$$\mathbf{r}'(t) \times \mathbf{r}''(t).$$

Example

1.

$$\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \quad .$$

$$\mathbf{r}''(t) = 2\mathbf{k} \quad .$$

2. $\mathbf{r}'(t) \times \mathbf{r}''(t)$ equals

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix} =$$

$$\mathbf{i} \begin{vmatrix} 2 & 2t \\ 0 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ = 4\mathbf{i} - 2\mathbf{j}$$

3. Find the magnitude of the vector that you found in step 2 (namely $\mathbf{r}'(t) \times \mathbf{r}''(t)$). Also find the magnitude of $\mathbf{r}'(t)$, and finally use the formula for the curvature

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

3.

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{4^2 + 2^2 + 0^2} = \sqrt{20} \quad .$$

$$\begin{aligned} |\mathbf{r}'(t)| &= |\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}| \\ &= \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \quad . \end{aligned}$$

Finally,

$$\kappa(t) = \frac{\sqrt{20}}{(\sqrt{5 + 4t^2})^3}$$

This is the **Ans.**.

Problem from a previous Final Exam

Find the curvature of the curve

$$\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$$

at the point $(1, 1, \frac{2}{3})$.

Ans.: $\frac{2}{9}$.