

Dr. Z's Math251 Handout #13.3 (2nd ed.) [Arc Length and Speed]

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Problem Type 13.3a: Find the length of the curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $t_0 \leq t \leq t_1$.

Example Problem 13.3a: Find the length of the curve $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k}$, $1 \leq t \leq e$.

Steps

Example

1. Compute the derivative

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} \quad .$$

1.

$$\begin{aligned} \mathbf{r}'(t) &= (t^2)'\mathbf{i} + (2t)'\mathbf{j} + (\ln t)'\mathbf{k} \\ &= 2t\mathbf{i} + 2\mathbf{j} + \frac{1}{t}\mathbf{k} \quad . \end{aligned}$$

2. Find the **magnitude** of $\mathbf{r}'(t)$,

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \quad ,$$

and use algebra and/or trig to simplify as much as you can.

2.

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{(2t)^2 + 2^2 + \frac{1}{t^2}} \\ &= \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\frac{(2t^2 + 1)^2}{t^2}} = \frac{(2t^2 + 1)}{t} = 2t + \frac{1}{t} \quad . \end{aligned}$$

3. Integrate the expression that you got in step 2 from t_0 to t_1 .

$$\int_{t_0}^{t_1} |\mathbf{r}'(t)| dt$$

3.

$$\begin{aligned} \int_1^e |\mathbf{r}'(t)| dt &= \int_1^e \left[2t + \frac{1}{t}\right] dt = \left[t^2 + \ln t\right]_1^e \\ &= e^2 + \ln e - (1^2 + \ln 1) = e^2 + 1 - (1 + 0) = e^2 \quad . \end{aligned}$$

Ans.: The arc length of that curve is e^2 .

Problem Type 13.3b: Reparametrize the curve with respect to arc length measured from the point when $t = t_0$ in the direction of increasing t .

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad .$$

Example Problem 13.3b: Reparametrize the curve with respect to arc length measured from

the point when $t = 0$ in the direction of increasing t .

$$\mathbf{r}(t) = 5 \sin t \mathbf{i} + 3 \mathbf{j} + 5 \cos t \mathbf{k} \quad .$$

Steps

1. Compute $\mathbf{r}'(t)$, and then take its magnitude $|\mathbf{r}'(t)|$.

Example

1.

$$\mathbf{r}'(t) = (5 \sin t)' \mathbf{i} + 3' \mathbf{j} + (5 \cos t)' \mathbf{k} \quad .$$

$$(5 \cos t) \mathbf{i} - (5 \sin t) \mathbf{k} \quad .$$

So

$$|\mathbf{r}'(t)| = \sqrt{(5 \cos t)^2 + (-5 \sin t)^2} = \sqrt{25(\cos^2 t + \sin^2 t)} = 5 \quad .$$

2. Integrate it from t_0 to t_1 . Get an expression in terms of t_1 and call it s . Now change t_1 into t . Now solve for t in terms of s .

2.

$$s = \int_0^{t_1} 5 \, dt = 5t_1 \quad .$$

Changing the t_1 into t we get

$$s = 5t$$

and expressing t in terms of s , we get

$$t = s/5 \quad .$$

3. Go back to the original $\mathbf{r}(t)$ and replace t by the expression in s that you found in step 2.

3.

$$\mathbf{r}(t) = 5 \sin t \mathbf{i} + 3 \mathbf{j} + 5 \cos t \mathbf{k} \quad .$$

becomes

$$5 \sin(s/5) \mathbf{i} + 3 \mathbf{j} + 5 \cos(s/5) \mathbf{k} \quad .$$

This is the **Ans..**

A Problem from a Previous Final

Suppose that the **position** of a certain particle is given by

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle, 0 \leq t \leq \pi.$$

(a) (4 points) Find the **velocity** of the particle as a function of the time t .

(b) (9 points) Find the **length of the arc** traversed by the moving particle for $0 \leq t \leq \pi$.

Ans.: a) $\langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t \rangle$; b) $\sqrt{3}(e^\pi - 1)$.