NAME: (print!) ____

Section: ____ **E-Mail address:** ____________________________

FULL SOLUTIONS to MATH 251 (4-6,11), Dr. Z. , Final Exam , Mon., Dec. 21, 2009, SEC 111, 12:00-3:00pm

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)
Do not write below this line

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1. (10 pts.) Find the curvature of the curve

$$\mathbf{r}(t) = \langle e^t, \sin t, \cos t \rangle$$

at the point $(1, 0, 1)$.

Ans.:: $\frac{\sqrt{6}}{4}$ (or $\frac{\sqrt{3}}{2\sqrt{2}}$)

$$\mathbf{r}'(t) = \langle e^t, \cos t, -\sin t \rangle$$
$$\mathbf{r}''(t) = \langle e^t, -\sin t, -\cos t \rangle$$

We need to know **what is the time** when the particle visits the point $(1, 0, 1)$. We have to solve

$$\langle e^t, \sin t, \cos t \rangle = \langle 1, 0, 1 \rangle \quad ,$$

This means, in particular, that $e^t = 1$ that forces $t = 0$. Plugging-in $t = 0$ into the other components gives

$$\langle e^0, \sin 0, \cos 0 \rangle = \langle 1, 0, 1 \rangle \quad ,$$

that is correct, so $t = 0$ is indeed the time.

Plugging-in $t = 0$ into $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$, we get

$$\mathbf{r}'(0) = \langle e^0, \cos 0, -\sin 0 \rangle = \langle 1, 1, 0 \rangle$$
$$\mathbf{r}''(0) = \langle 1, 0, -1 \rangle \quad .$$

Recall that the formula for the **curvature** is:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad ,$$

so

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} \quad .$$

We next must compute: $\mathbf{r}'(0) \times \mathbf{r}''(0)$.

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 1, 1, 0 \rangle \times \langle 1, 0, -1 \rangle = \langle -1, 1, -1 \rangle \quad ,$$

(You do it!). We have:

$$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = |\langle -1, 1, -1 \rangle| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3} \quad ,$$

and

$$|\mathbf{r}'(0)| = |\langle 1, 1, 0 \rangle| = \sqrt{1^2 + 1^2 + (0)^2} = \sqrt{2} \quad ,$$

so we have

$$\kappa(0) = \frac{\sqrt{3}}{\sqrt{2}^3} = \frac{\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{4} = \frac{\sqrt{6}}{4} \quad .$$

2. Sketch the region of integration and change the order of integration.

$$\int_0^1 \int_0^x F(x,y)\,dy\,dx \;+\; \int_1^2 \int_0^1 F(x,y)\,dy\,dx \;+\; \int_2^3 \int_0^{3-x} F(x,y)\,dy\,dx$$

Ans.:: $\int_0^1 \int_y^{3-y} F(x,y)\,dx\,dy$

If you draw the region (a trapezoid whose base is on the x-axis between $(0,0)$ and $(3,0)$, whose height is 1 and whose vertices are $(0,0)$, $(3,0)$, $(1,1)$, and $(2,1)$). It is a simple type II region (but a compound, type-I region). Its type-II description is

$$\{(x,y)\,|\,0 \le y \le 1\,,\; y \le x \le 3-y\} \quad ,$$

yielding the iterated integral

$$\int_0^1 \int_y^{3-y} F(x,y)\,dx\,dy \quad .$$

Comment: Quite a few people tackled each of the three double-integrals of the problem separately, getting

$$\int_0^1 \int_y^1 F(x,y)\,dx\,dy + \int_0^1 \int_1^2 F(x,y)\,dx\,dy + \int_0^1 \int_2^{3-y} F(x,y)\,dx\,dy \quad .$$

This is also correct, but not as simple as possible. These people got nine points.

3. (10 points) Find the absolute maximum value of the function $f(x,y) = x^2 + y^2 - 2x - 2y + 2$ in the triangular region

$$\{(x,y)\,|\,x \geq 0\,,\ y \geq 0\,,\ x+y \leq 4\,\} \quad .$$

Ans.:: 10 .

There are three types of finalists.

Critical points: First we look for critical points by solving $f_x = 0, f_y = 0$. Here $f_x = 2x-2, f_y = 2y-2$, so we have to solve $2x-2 = 0, 2y-2 = 0$ giving the solution $x = 1, y = 1$ yielding that the only critical point is $(1,1)$. Since $1+1 \leq 4$ and $1 \geq 0, 1 \geq 0$, it belongs to our region, so we include it as a **finalist**.

Vertices: $(0,0)$, $(4,0)$, $(0,4)$. These are automatically finalists.

Critical points from edges:

Left edge: $x = 0$, $0 \leq y \leq 4$. Plugging-in $x = 0$ into the function gives

$$f(0,y) = y^2 - 2y + 2 \quad ,$$

Since $(y^2 - 2y + 2)' = 2y - 2$ solving $2y - 2 = 0$ gives $y = 1$. So we must include $(0,1)$ as a finalist.

Bottom edge: $y = 0$, $0 \leq x \leq 4$. Plugging-in $y = 0$ into the function gives

$$f(x,0) = x^2 - 2x + 2 \quad ,$$

Since $(x^2 - 2x + 2)' = 2x - 2$ solving $2x - 2 = 0$ gives $x = 1$. So we must include $(1,0)$ as a finalist.

Right edge (hypoteneus): $x + y = 4$, i.e. $y = 4 - x$. So

$$f(x, 4-x) = x^2 + (4-x)^2 - 2(x + 4 - x) + 2 = 2x^2 - 8x + 18 \quad ,$$

since $(2x^2 - 8x + 18)' = 4x - 8$ we have to solve $4x - 8 = 0$ that gives $x = 2$. Plugging into $y = 4 - x$ gives $y = 2$, so $(2,2)$ is yet another finalist.

Now we are ready for the **final show-down**:

$$f(0,0) = 2 \quad , \quad f(1,0) = 1 \quad , \quad f(0,1) = 1 \quad ,$$

$$f(4,0)=10 \quad , \quad f(0,4)=10 \quad , \quad f(1,1)=0 \quad , \quad f(2,2)=-6 \quad .$$

The largest of these is 10 so the **absolute maximum value** is 10.

Comment: Many people got the right answer, but didn't get full credit, because they failed to consider all the finalists. In particular very few people found the finalist $(2,2)$ on the hypotenuse. They lucked out since it turned out that the absolute maximum value occurred elsewhere. If the problem would have been to find the absolute minimum value, they would have gotten the wrong answer, because the absolute minimum value, that happens to be -6, occurs at $(2,2)$.

4. (10 points) Find an equation of the tangent plane to the surface

$$z^5 + x^4 - y^3 = 1 \quad ,$$

at the point $(1, 1, 1)$

Ans.:: $4x - 3y + 5z = 6$

$(4(x-1) - 3(y-1) + 5(z-1) = 0$ is OK too, as is $z = (-4/5)x + (3/5)y + 6/5)$.

First, let's make sure that the problem makes sense. Plugging-in $(1, 1, 1)$ into the equation of the surface gives:

$$1^5 + 1^4 - 1^3 = 1 \quad ,$$

that is a true statement, so we must go on.

Recall that for an implicitly-defined surface $F(x, y, z) = 0$ we take the gradient $\bigtriangledown F$ and plug-in the specific point (x_0, y_0, z_0) in order to get the normal direction to the tangent plane.

$$\bigtriangledown F = \langle 4x^3, -3y^2, 5z^4 \rangle \quad .$$

Plugging-in $x = 1, y = 1, z = 1$, we get:

$$\mathbf{n} = \langle 4 \cdot 1^3, -3 \cdot 1^2, 5 \cdot 1^4 \rangle = \langle 4, -3, 5 \rangle.$$

The equation of the tangent plane is:

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \quad ,$$

that in this problem becomes:

$$\langle 4, -3, 5 \rangle \cdot \langle x - 1, y - 1, z - 1 \rangle = 0 \quad ,$$

that gives

$$4(x-1) - 3(y-1) + 5(z-1) = 0 \quad ,$$

which is an OK answer. Opening-up parentheses, we get:

$$4x - 3y + 5z = 6 \quad .$$

5. (10 points) Compute $f_{xy}(1,1)$ if $f(x,y) = e^{x^2+xy+y^3}$.

Ans.: $13e^3$.

By the chain rule:

$$f_x(x,y) = e^{x^2+xy+y^3}(2x+y)$$

By the product rule followed by the chain rule:

$$f_{xy}(x,y) = e^{x^2+xy+y^3}(x+3y^2)(2x+y) + e^{x^2+xy+y^3} \cdot 1 = e^{x^2+xy+y^3}((x+3y^2)(2x+y)+1) \quad .$$

Now we plug-in $x = 1, y = 1$, and get:

$$f_{xy}(1,1) = e^{1^2+1\cdot 1+1^3}((1+3\cdot 1^2)(2\cdot 1+1)+1) = e^3 \cdot 13 = 13e^3 \quad .$$

6. (10 points) Use the linearization of $f(x,y) = \sqrt{x^2+y^2+1}$ to approximate $f(2.01, 1.98)$.

Ans.: $\frac{449}{150}$ or $2\frac{149}{150}$.

Recall the **linearlization** formula:

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \quad .$$

Here $x_0 = 2$ and $y_0 = 2$. We have

$$f_x(x,y) = \frac{1}{2}(x^2+y^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+y^2+1}} \quad ,$$

$$f_y(x,y) = \frac{1}{2}(x^2+y^2+1)^{-\frac{1}{2}}(2y) = \frac{y}{\sqrt{x^2+y^2+1}} \quad ,$$

So

$$f(x_0,y_0) = f(2,2) = \sqrt{2^2+2^2+1} = \sqrt{9} = 3 \quad .$$

$$f_x(x_0,y_0) = f(2,2) = \frac{2}{\sqrt{2^2+2^2+1}} = \frac{2}{3} \quad ,$$

$$f_y(x_0,y_0) = f(2,2) = \frac{2}{\sqrt{2^2+2^2+1}} = \frac{2}{3} \quad .$$

And so the linearization (at $(2,2)$) is:

$$L(x,y) = 3 + \frac{2}{3}(x-2) + \frac{2}{3}(y-2) \quad .$$

Finally, to get an approximation for $f(2.01, 1.98)$ we plug-in $x = 2.01, y = 1.98$:

$$L(2.01, 1.98) = 3 + \frac{2}{3}(2.01-2) + \frac{2}{3}(1.98-2) = 3 + \frac{2}{3}\cdot\frac{1}{100} + \frac{2}{3}\cdot\frac{-2}{100} =$$

$$3 - \frac{2}{3}\cdot\frac{1}{100} = 3 - \frac{1}{150} = \frac{449}{150} \quad .$$

7. (10 points) Does the following limit exist? If it does, find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3+z^3}{2x^3+3y^3+4z^3}$$

Ans.: Does Not Exist .

The easiest way to prove that the limit does not exist, is to pick two different straight-lines approaching the origin. One could be along the x-axis, setting $y=0, z=0$, getting that the limit of the function as (x,y,z) approaches $(0,0,0)$ moving along the x-axis is:

$$\lim_{(x,y,z)\to(0,0,0), y=0, z=0} \frac{x^3}{2x^3} = \frac{1}{2} \quad .$$

Approaching along the y-axis, we get

$$\lim_{(x,y,z)\to(0,0,0), x=0, z=0} \frac{y^3}{3y^3} = \frac{1}{3} \quad .$$

Since these numbers are **different**, there is no way that the limit could exist.

Another way is to set $y=kx, z=mx$ getting that the limit along an arbitrary line is

$$\lim_{(x,y,z)\to(0,0,0), y=kx, z=mx} \frac{x^3+(kx)^3+(mx)^3}{2x^3+3(kx)^3+4(mx)^3} = \frac{1+k^3+m^3}{2+3k^3+4m^3} \quad .$$

This obviously depends on k and m, so there is no consensus, and the limit does not exist.

8. (10 points) Find the length of the curve

$$\mathbf{r}(t) = \langle\, 4t\,,\, 3\sin t\,,\, 3\cos t\,\rangle \quad , \quad 0 \le t \le \pi \quad .$$

Ans.: 5π

$$\mathbf{r}'(t) = \langle\, 4\,,\, 3\cos t\,,\, -3\sin t\,\rangle \quad ,$$

and

$$|\mathbf{r}'(t)| = |\langle\, 4\,,\, 3\cos t\,,\, -3\sin t\,\rangle| =$$

$$\sqrt{4^2 + (3\cos t)^2 + (3\sin t)^2} = \sqrt{16 + 9(\cos^2 t + \sin^2 t)} = \sqrt{16+9} = \sqrt{25} = 5 \quad .$$

So $ds = 5\,dt$, and

$$arclength = \int_C ds = \int_0^\pi 5dt = 5\pi \quad .$$

9. (10 points) A certain particle has acceleration

$$\mathbf{a}(t) = \langle\, 16e^{4t} \,,\, 9\sin(3t) \,,\, 6t \,\rangle \quad ,$$

and at $t = 0$ its velocity is $\langle 4, -3, 0\rangle$ and its position vector is $\langle 1, 0, 0\rangle$, find its position vector at any time t.

Ans.: $\langle e^{\pi}, \frac{-\sqrt{2}}{2}, \frac{\pi^3}{64}\rangle$.

We first need the velocity $\mathbf{v}(t)$, obtained by integrating $\mathbf{a}(t)$:

$$\mathbf{v}(t) = \int \langle\, 16e^{4t} \,,\, 9\sin(3t) \,,\, 6t \,\rangle \, dt = \langle\, 4e^{4t} \,,\, -3\cos(3t) \,,\, 3t^2 \,\rangle \, + \mathbf{C} \quad .$$

By-plugging in $t = 0$ we get $\mathbf{C} = \mathbf{0}$ and so

$$\mathbf{v}(t) = \langle\, 4e^{4t} \,,\, -3\cos(3t) \,,\, 3t^2 \,\rangle + \mathbf{0} = \langle\, 4e^{4t} \,,\, -3\cos(3t) \,,\, 3t^2 \,\rangle \quad .$$

To get the position, we integrate the velocity:

$$\mathbf{r}(t) = \int \langle\, 4e^{4t} \,,\, -3\cos(3t) \,,\, 3t^2 \,\rangle \, dt = \langle\, e^{4t} \,,\, -sin(3t) \,,\, t^3 \,\rangle + \mathbf{C} \quad ,$$

By-plugging in $t = 0$ we get $\mathbf{C} = \mathbf{0}$ and so

$$\mathbf{r}(t) = \langle\, e^{4t} \,,\, -sin(3t) \,,\, t^3 \,\rangle \quad .$$

Finally, plugging-in $t = \pi/4$, we get:

$$\mathbf{r}(\pi/4) = \langle\, e^{\pi} \,,\, -sin(3\pi/4) \,,\, (\pi/4)^3 \,\rangle = \langle e^{\pi}, \frac{-\sqrt{2}}{2}, \frac{\pi^3}{64}\rangle \quad .$$

Comment: People who left the answer as $\langle\, e^{\pi} \,,\, -sin(3\pi/4) \,,\, (\pi/4)^3 \,\rangle$ got eight points.

10. (10 points) Let $\Phi(u,v) = (3u + v, u - 2v)$. Use the Jacobian to determine the area of $\Phi(\mathcal{R})$ for $\mathcal{R} = [2,5] \times [1,7]$.

Ans.: 126

The Jacobian is $(x_u)(y_v) - (x_v)(y_u) = (3)(-2) - (1)(1) = -7$, so the **magnification factor** is its absolute value, $|-7| = 7$. The area of the rectangle $\mathcal{R}$ is $(5-2) \cdot (7-1) = 3 \cdot 6 = 18$, so the area of $\Phi(\mathcal{R})$ equals $18 \cdot 7 = 126$.

Comments:

1. Some people didn't know what $[2,5] \times [1,7]$ means. It is the rectangle

$$\{(x,y)\,|\,2 \leq x \leq 5\,,\,1 \leq y \leq 7\,\} \quad .$$

One or two people forgot to take the absolute value, giving the answer of -126, shame on you! How many times did I tell you **area can never be negative!**.

11. (10 pts.) Compute *curl* **F** if

$$\mathbf{F} \;=\; \langle\, 2e^{2x+3y+5z} + y^2 + 2y + 1\,,\; 3e^{2x+3y+5z} + 2xy + 3x + 1\,,\; 5e^{2x+3y+5z}\,\rangle$$

Ans.: $\langle 0, 0, 1\rangle$.

$$curl\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2e^{2x+3y+5z} + y^2 + 2y + 1 & 3e^{2x+3y+5z} + 2xy + 3x + 1 & 5e^{2x+3y+5z} \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial}{\partial y} 5e^{2x+3y+5z} - \frac{\partial}{\partial z}(3e^{2x+3y+5z} + 2xy + 3x + 1)\right)$$

$$-\mathbf{j}\left(\frac{\partial}{\partial x} 5e^{2x+3y+5z} - \frac{\partial}{\partial z}(2e^{2x+3y+5z} + y^2 + 2y + 1)\right)$$

$$+\mathbf{k}\left(\frac{\partial}{\partial x}(3e^{2x+3y+5z} + 2xy + 3x + 1) - \frac{\partial}{\partial y}(2e^{2x+3y+5z} + y^2 + 2y + 1)\right)$$

$$= \mathbf{i}\,((15e^{2x+3y+5z} - 15e^{2x+3y+5z}) - \mathbf{j}\,(10e^{2x+3y+5z} - 10e^{2x+3y+5z})$$

$$+\mathbf{k}\,(6e^{2x+3y+5z} + 2y + 3) - (6e^{2x+3y+5z} + 2y + 2)$$

$$= \mathbf{i}\cdot 0 - \mathbf{j}\cdot + \mathbf{k}\cdot 1 \;=\; \langle 0, 0, 1\rangle \quad .$$

12. (10 points) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x,y,z) = y\mathbf{i} + z\mathbf{j} + y\mathbf{k} \quad ,$$

$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k} \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: $\frac{8}{3}$

Here $\mathbf{r}(t) = \langle t^2, t, 2t \rangle$, so $\mathbf{r}'(t) = \langle 2t, 1, 2 \rangle$. Since $x = t^2, y = t, z = 2t$ on C, translating $\mathbf{F}$ to the t-language gives $\mathbf{F} = \langle t, 2t, t \rangle$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t, 2t, t \rangle \cdot \mathbf{r}'(t) dt = \int_0^1 \langle t, 2t, t \rangle \cdot \langle 2t, 1, 2 \rangle dt =$$

$$\int_0^1 ((t)(2t) + (2t)(1) + t(2)) dt = \int_0^1 (2t^2 + 2t + 2t) dt = \int_0^1 (2t^2 + 4t) dt =$$

$$\frac{2t^3}{3} + 4\frac{t^2}{2}\Big|_0^1 = \frac{2t^3}{3} + 2t^2\Big|_0^1 = \frac{2}{3} + 2 \cdot 1^2 - 0 = \frac{2}{3} + 2 = \frac{8}{3}$$

13. (10 points) Evaluate

$$\int\int\int_E \frac{1}{\sqrt{x^2+y^2}}\, dV \quad ,$$

where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.

Ans.: 2π .

We use **cylindrical coordinates**. Recall that (just like in polar) $x^2+y^2=r^2$. Also $dV = r\,dz\,dr\,d\theta$. In cylindrical language the surface $z^2=4x^2+4y^2$ means $z^2=4r^2$ which means $z=2r$ or $z=-2r$. But since our region is **above** the xy-plane, the limits in the z-integration are $0 \le z \le 2r$ (i.e. the bottom half of the cone, namely $z=-2r$ is irrelevant). The projection on the xy-plane is the disc $x^2+y^2 \le 1$, that means $0 \le r \le 1$, and since we are having the **whole** disc, we have the full range of θ: $0 \le \theta \le 2\pi$.

Our integral becomes:

$$\int_0^{2\pi}\int_0^1\int_0^{2r} \frac{1}{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_0^{2r} dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(\int_0^{2r} dz\right) dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 2r\,dr\,d\theta = \int_0^{2\pi} (\int_0^1 2r\,dr)\,d\theta = \int_0^{2\pi} (r^2\Big|_0^1)\,d\theta = \int_0^{2\pi} (1)\,d\theta = 2\pi \quad .$$

14. (10 points) Evaluate the iterated integral

$$\int_0^2 \int_x^{3x} \int_0^{x+y} x \, dz \, dy \, dx \quad .$$

Ans.: 24

The **inside integral** is:

$$\int_0^{x+y} x \, dz = xz\Big|_{z=0}^{z=x+y} = x(x+y-0) = x^2 + xy \quad .$$

The **middle integral** is:

$$\int_x^{3x} (x^2+xy) \, dy = x^2y + \frac{xy^2}{2}\Big|_{y=x}^{y=3x} = x^2(2x) + \frac{x}{2}((3x)^2 - x^2) = 2x^3 + \frac{x}{2}(8x^2) = 2x^3 + 4x^3 = 6x^3 \quad .$$

The **outside integral** is:

$$\int_0^2 6x^3 \, dx = 6\frac{x^4}{4}\Big|_{x=0}^{x=2} = \frac{3}{2}(2^4 - 0^4) = \frac{3}{2}(16) = 24 \quad .$$

15. (10 points) Use the given transformation to evaluate the integral

$$\int\int_R (2x+y)^2\,dA \quad ,$$

where R is the triangular region with vertices $(0,0)$,$(2,-3)$, $(3,-5)$; $x = 3u - v$, $y = -5u + 2v$.

Ans.: $\frac{1}{4}$

First we compute the **Jacobian** $(3)(2) - (-1)(-5) = 1$.
We need to find the three vertices of the triangle in the uv-plane.

For $(0,0)$ we solve $3u - v = 0, -5u + 2v = 0$, getting $u = 0, v = 0$, so we have the point $(0,0)$ in the uv-plane.

For $(2,-3)$ we solve $3u - v = 2, -5u + 2v = -3$, getting $u = 1, v = 1$,(you do it!) so we have the point $(1,1)$ in the uv-plane.

For $(3,-5)$ we solve $3u - v = 3, -5u + 2v = -5$, getting $u = 1, v = 0$,(you do it!) so we have the point $(1,0)$ in the uv-plane.

This is a much simpler region, the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$. Its type-I description is:

$$\{(u,v)\,|\,0 \le u \le 1, 0 \le v \le u\} \quad .$$

The **integrand** in the uv-language is:

$$(2x+y)^2 = (2(3u-v) + (-5u+2v))^2 = u^2 \quad .$$

So the xy-integral becomes the following uv-integral

$$\int_0^1 \int_0^u |1| \cdot u^2\,dv\,du = \int_0^1 \int_0^u u^2\,dv\,du \quad .$$

The **inside integral** is

$$\int_0^u u^2\,dv = u^2(v\Big|_{v=0}^{v=u}) = u^2(u-0) = u^3 \quad .$$

The **outside integral** is

$$\int_0^1 u^3 = \frac{u^4}{4}\Big|_0^1 = \frac{1^4}{4} - \frac{1^0}{4} = \frac{1}{4} \quad .$$

16. (10 points) Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 10(x^2+y^2)^4 \, dy \, dx$$

Ans.: $\frac{\pi}{2}$.

We must use **polar coordinates** (since the question asked for it). The region of integration is (in type-I format)

$$\{(x,y) \,|\, 0 \le x \le 1 \,,\, -\sqrt{1-x^2} \le y \le 0\} \quad .$$

If you draw it (do it!), this emerges as a quarter-disc, center the origin, radius 1, that lives in the **fourth-quadrant** (the lower-right quadrant). This means that the limits of θ are $-\pi/2 \le \theta \le 0$ (or, equally correctly $3\pi/2 \le \theta \le 2\pi$). The limits in the r-integration are $0 \le r \le 1$ (since the radius of the disc is 1). Recall that $dy\,dx = dA = r\,d\,r d\,\theta$ and that $x^2+y^2 = r^2$, so we have that our integral is

$$\int_{-\pi/2}^0 \int_0^1 10(r^2)^4 \, r \, dr \, d\theta = \int_{-\pi/2}^0 \int_0^1 10r^9 \, r \, dr \, d\theta = (\int_{-\pi/2}^0 d\theta)(\int_0^1 10r^{10} \, dr\,) =$$

$$(\theta \Big|_{-\pi/2}^0)(r^{10}\Big|_0^1) = (0 - -\pi/2)(1^{10} - 0^{10}) = \frac{\pi}{2} \quad .$$

17. (10 points) Compute the (vector-field) line-integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where C is an ellipse with semi-major-axis 4 and semi-minor-axis 3, traveled **clockwise**, and the vector-field $\mathbf{F}$ is

$$\mathbf{F} = \langle\, 2y \,,\, 3x \,\rangle \quad .$$

(Reminder: the area of an ellipse with semi-major axis a and semi-minor-axis b equals πab).

Ans.: -12π .

Since an ellipse is a **closed curve**, we can use **Green's Theorem**: if $\mathbf{F} = \langle P, Q \rangle$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} \, = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad ,$$

where R is the **inside** region of the curve, and the line-integration is performed **counterclockwise**. Since in this problem we are told that the line-integration is performed clockwise, we need to stick a **minus sign** in front, getting (here $P = 2y$, $Q = 3x$):

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int\int_R \left(\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y) \right) = -\int\int_R (3-2) = -area(R) \quad .$$

The area of the ellipse is $3 \cdot 4 \cdot \pi = 12\pi$, so we get that the answer is -12π.

Comment: People who forgot to stick the minus sign (and got 12π) got six points.

18. (10 points) Compute the vector-field surface integral:

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle 5x \, , \, 2y \, , \, 3z \, \rangle$$

and where S is the boundary of a pyramid whose base is a 2×2 square and whose height is 3, with the normal pointing **outward**.

(Reminder: the volume of a pyramid is the area of the base times the height divided by 3)

Ans.: 40 .

We use the **divergence theorem**:

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E (div\mathbf{F}) \, dV \quad ,$$

where E is the interiour of the surface (in this problem the pyramid).

The divergence is $P_x + Q_y + R_z = 5 + 2 + 3 = 10$, so we have

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int\int_E (10) \, dV = 10 \int\int\int_E 1 \, dV = 10 Volume(E) \quad .$$

By the hint, the volume is $(2 \cdot 2 \cdot 3)/3 = 4$, so the answer is $10 \cdot 4 = 40$.

19. (10 points) Find the maximum value of the function $f(x,y) = x^2y^4$ subject to the constraint $x^2 + 2y^2 = 6$.

Ans.: 8 .

The **goal function** is $f(x,y) = x^2y^4$, and the **constraint function** is $g(x,y) = x^2 + 2y^2 - 6$. We have to solve the Lagrange-multiplier equation

$$\bigtriangledown f = \lambda \bigtriangledown g \quad ,$$

that spells out, in this problem, to be:

$$\langle 2xy^4, 4x^2y^3 \rangle = \lambda \langle 2x, 4y \rangle \quad ,$$

That yields the two equations:

$$2xy^4 = 2\lambda x \quad , 4x^2y^3 = 4\lambda y \quad .$$

It would be nice to get rid of the annoying λ by dividing the first equation by the second, but we must first consider the case when:

$x = 0$ (and using $x^2 + 2y^2 = 6$ gives $0^2 + 2y^2 = 6$, so $y = \pm\sqrt{3}$, giving the two candidates $(0, \pm\sqrt{3})$).

$y = 0$ (and using $x^2 + 2y^2 = 6$ gives $x^2 + 2 \cdot 0^2 = 6$, so $x = \pm\sqrt{6}$, giving the two candidates $(\pm\sqrt{6}, 0)$).

Now that we have taken care of the cases of "division by 0", we can safely divide the first equation by the second, getting

$$\frac{2xy^4}{4x^2y^3} = \frac{2\lambda x}{4\lambda y} \quad ,$$

that simplifies to:

$$\frac{y}{2x} = \frac{x}{2y} \quad ,$$

and cross-multiplying gives $x^2 - y^2 = 0$. Factorizing, we get $(x-y)(x+y)$, so we have two remaining options $y = x$ and $y = -x$. Plugging this into the constant equation we get

$$x^2 + 2(\pm x)^2 = 6 \quad ,$$

yielding $3x^2 = 6$, so $x^2 = 2$ and $x = \pm\sqrt{2}$. Since $y = \pm x$, this gives the four additional candidates $(\pm\sqrt{2}, \pm\sqrt{2})$.

Now it is time for the **final contest**:

$$f(0, \pm\sqrt{3}) = 0 \quad , \quad f(\pm\sqrt{6}, 0) = 0, f(\pm\sqrt{2}, \pm\sqrt{2}) = 2 \cdot 4 = 8 \quad .$$

The largest value amongst these is 8, so this is the answer.

Comment: Many students got the right answer, but forgot to consider the finalists $(0, \pm\sqrt{3}), (\pm\sqrt{6}, 0)$. These people only got (very generously!) six points. If the problem would have been to find the absolute minimum value, they would have gotten the wrong answer.

20. (10 points) Find the local maximum and minimum **values**, and saddle point(s) of the function $f(x, y) = e^x - xe^y$.

Local maximum value(s): None

Local minimum value(s): None

saddle point(s): $(0, 0)$

$$f_x = e^x - e^y \quad , \quad f_y = -xe^y \quad .$$

For future reference:

$$f_{xx} = e^x \quad , \quad f_{xy} = -e^y \quad , f_{yy} = -xe^y \quad .$$

To get the critical points, we solve

$$e^x - e^y = 0 \quad , \quad -xe^y = 0 \quad .$$

It is easier to tackle the second equation first. Since e^y is **never** zero, it is safe to divide by it, getting as the only option, $x = 0$. Plugging it into the first equation gives $e^0 - e^y = 0$, so $e^y = 1$ yielding $y = 0$. So the only critical point is $(0, 0)$.

It now remains to find its **nature**. When $x = 0, y = 0$ we have

$$f_{xx}(0, 0) = e^0 = 1 \quad , \quad f_{xy}(0, 0) = -e^0 = -1 \quad , \quad f_{yy}(0, 0) = 0 \quad .$$

so the **discriminant** $D = f_{xx}f_{yy} - f_{xy}^2$ equals $(1)(0) - (-1)^2 = 0 - 1 = -1$. Since this is **negative** it means that $(0, 0)$ is a saddle point.