NAME: (print!) ____

Section: _______ **E-Mail address:** _________________________

Solutions to MATH 251 (1-3,10), Dr. Z. , Final Exam , Fri., Dec. 18, 2009, SEC 111, 8:00-11:00am

WRITE YOUR FINAL ANSWER TO EACH PROBLEM IN THE INDICATED PLACE (right under the question)
Do not write below this line

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1. (out of 10)

2. (out of 10)

3. (out of 10)

4. (out of 10)

5. (out of 10)

6. (out of 10)

7. (out of 10)

8. (out of 10)

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20. (out of 10)

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tot. (out of 200)

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1. (10 points) Compute the line-integral

$$\int_C 7y\,dx + 3x\,dy \quad ,$$

where C is the closed path that consists from the line segment from $(0,0)$ to $(1,2)$ followed by the line segment from $(1,2)$ to $(2,0)$ followed by the line segment from $(2,0)$ back to $(0,0)$.

Ans.: 8

We use **Green's Theorem**:

$$\int_C P\,dx \, + \, Q\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad ,$$

where R is the **inside** of C, and C is traveled **counterclockwise**.
Here $P = 7y$ and $Q = 3x$ so we have

$$\int_C 7y\,dx \, + \, 3x\,dy = \int\int_R (3-7)\,dA \, = \, -4 \int\int_R dA = -4Area(R) \quad .$$

The region R is the triangle whose vertices are $(0,0)$, $(2,0)$, $(1,2)$. Its base has length 2, and its height is 2, so its area is $2 \cdot 2/2 = 2$, so this equals -8.

Finally, since the path goes from $(0,0)$ to $(1,2)$, then from $(1,2)$ to $(2,0)$ and then from $(2,0)$ back to $(0,0)$, if you draw it (or even in your head) you can see that the direction is **clockwise**. So we have to take the **minus** of that. So the **answer** is $-(-8) = 8$.

Comment: People who got the answer -8 (i.e. they didn't multiply by -1) got six points.

2. (10 points) Find an equation of the tangent plane to the surface

$$\Phi(u,v) = (u^2+v^2, 2uv, u^3) \quad ,$$

at the point $(5,4,1)$.

Ans.: $x-2y+2z=-1$

(or $x-2y+2z+1=0$ or $z=-1/2-x/2+y$).

$$\Phi_u = \langle\, 2u, 2v, 3u^2 \,\rangle \quad , \Phi_v = \langle\, 2v, 2u, 0 \,\rangle \quad .$$

Now it is time to find out what are the specific u and v at the point $(5,4,1)$. We have to solve

$$u^2+v^2=5 \quad , \quad 2uv=4 \quad , \quad u^3=1 \quad .$$

From the last equation we get $u=1$. Plugging $u=1$ into the second equation we get $2v=4$ so $v=2$. To make sure we plug-in $u=1$ and $v=2$ into the first equation and get $5=5$, so things are OK and indeed $u=1$ and $v=2$.
Plugging-in these values of u and v into Φ_u, Φ_v, we get

$$\Phi_u = \langle\, 2\cdot 1, 2\cdot 2, 3\cdot 1^2 \,\rangle \quad , \Phi_v = \langle\, 2\cdot 2, 2\cdot 1, 0 \,\rangle \quad .$$

So

$$\Phi_u = \langle\, 2,4,3 \,\rangle \quad , \Phi_v = \langle\, 4,2,0 \,\rangle \quad .$$

To get the **normal vector** we take the **cross-product** $\Phi_u \times \Phi_v$:

$$\mathbf{n} = \langle\, 2,4,3 \,\rangle \times \langle\, 4,2,0 \,\rangle = \langle -6, 12, -12 \rangle$$

(you do it!). So the equation of the tangent plane is

$$-6(x-5)+12(y-4)-12(z-1)=0 \quad .$$

Dividing by -6 we get:

$$(x-5)-2(y-4)+2(z-1)=0 \quad .$$

This is an OK answer. Simplifying, we get:

$$x-2y+2z=-1 \quad ,$$

that is the best answer.

3. (10 points) Find the absolute maximum value and the absolute minimum value of the function $f(x,y) = x^2 + y^2$ in the region

$$\{(x,y) \,|\, -1 \le x \le 1, -1 \le y \le 1\}.$$

Absolute minimum value: 0

Absolute maximum value: 2

We first find the **critical points** by solving $f_x = 0, f_y = 0$. Here $f_x = 2x$ and $f_y = 2y$. Solving $2x = 0, 2y = 0$, we get $x = 0, y = 0$ and so $(x,y) = (0,0)$. Since $(0,0)$ happens to be in our region, we keep it as a finalist. We also have **automatically**, as finalists, the four corners of the square $(-1,-1), (-1,1), (1,-1), (1,1)$. We also have to find potential candidates along the edges.
Left edge: $x = -1$: $f(-1,y) = 1 + y^2$. Since $(1+y^2)' = 2y$ we get $y = 0$ yielding the candidate $(-1,0)$
Right edge: $x = 1$: $f(1,y) = 1 + y^2$. Since $(1+y^2)' = 2y$ we get $y = 0$ yielding the candidate $(1,0)$
Bottom edge: $y = -1$: $f(x,-1) = x^2 + 1$. Since $(x^2+1)' = 2x$ we get $x = 0$ yielding the candidate $(0,-1)$
Top edge: $y = 1$: $f(x,1) = x^2 + 1$. Since $(x^2+1)' = 2x$ we get $x = 0$ yielding the candidate $(0,1)$.
So we have **nine** finalists, and for each we have to plug-in into $f(x,y) = x^2 + y^2$.

$$f(0,0) = 0 \quad , \quad f(\pm 1, 0) = 1 \quad , \quad f(0, \pm 1) = 1 \quad , \quad f(\pm 1, \pm 1) = 2 \quad .$$

The smallest of these values is 0, that is the **absolute minimum value**. The largest of these values is 2, that is the **absolute maximum value**.

Comment: Many people got the right answers but didn't get full credit, since they didn't examine the edges. In this problem it turned out that the candidates from the edges didn't matter, but in many other problems, they are the winners (or losers).

4. (10 points) Compute f_{xy} if

$$f(x,y) = \sin(x^2 + xy + y^2) \quad .$$

Ans.:

$$-(x+2y)(2x+y)\sin(x^2+xy+y^2) + \cos(x^2+xy+y^2) \quad .$$

By the chain rule applied to $f(x,y)$ as a function of x, and viewing y as as **stupid constant**:

$$f_x = \cos(x^2+xy+y^2)\cdot(2x+y) \quad .$$

By the product rule, viewing f_x as a function of y and treating x now as a stupid constant:

$$f_{xy} = (\cos(x^2+xy+y^2))'\cdot(2x+y) + \cos(x^2+xy+y^2)\cdot(2x+y)' \quad ,$$

where $'$ denotes differentiation with respect to y. Using the chain-rule for $(\cos(x^2+xy+y^2))'$ we get that it equals $-\sin(x^2+xy+y^2)\cdot(x+2y)$. This gives:

$$f_{xy} = -\sin(x^2+xy+y^2)\cdot(x+2y)\cdot(2x+y) + \cos(x^2+xy+y^2)\cdot 1 =$$

$$-(x+2y)(2x+y)\sin(x^2+xy+y^2) + \cos(x^2+xy+y^2) \quad .$$

5. (10 points) Find $\frac{\partial z}{\partial x}$ at the point $(1,1,1)$ if (x,y,z) are related by:

$$x^3y^3z^3 + 3xyz = 4$$

Ans.: -1 .

First let's make sure that our point $(1,1,1)$ indeed lies on our surface:

$$1^3 \cdot 1^3 \cdot 1^3 + 3 \cdot 1 \cdot 1 \cdot 1 = 4 \quad ,$$

is correct, so the question makes sense (if it didn't, you should refuse to do it).
Recall the formula

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad ,$$

for an implicitly-defined function of the form $F(x,y,z) = 0$. Here $F(x,y,z) = x^3y^3z^3 + 3xyz - 4$. So

$$\frac{\partial z}{\partial x} = -\frac{3x^2y^3z^3 + 3yz}{3x^3y^3z^2 + 3xy} \quad .$$

But we are interested only at what is going on at $(1,1,1)$. So plugging $x = 1, y = 1, z = 1$ we get

$$\frac{\partial z}{\partial x} = -\frac{3(1)^2(1)^3(1)^3 + 3(1)(1)}{3(1)^3(1)^3(1)^2 + 3(1)(1)} = \frac{-6}{6} = -1 \quad .$$

Comment: Students who forgot to plug-in $x = 1, y = 1, z = 1$ and gave as the answer $-\frac{3x^2y^3z^3+3yz}{3x^3y^3z^2+3xy}$ only got three points. It is very important to read and understand the full question, and in this problem the final answer is of type **number** not "multivariable function".

6. (10 points) Find an equation for the plane that contains both the line

$$x = 1 + t\,,\; y = 2 + t\,,\; z = 3 + t \quad (-\infty < t < \infty)$$

and the line

$$x = -t\,,\; y = 1 + t\,,\; z = 2 + t \quad (-\infty < t < \infty)$$

Ans.: $y - z = -1$.

In vector notation, the two lines are

$$\langle 1, 2, 3\rangle + \langle 1, 1, 1\rangle t \quad , \quad \langle -1, 1, 2\rangle + \langle -1, 1, 1\rangle t \quad ,$$

So the **direction vectors** are $\langle 1, 1, 1\rangle$ and $\langle -1, 1, 1\rangle$. To get the **normal vector** to the plane, we take the **cross-prodcut**:

$$\mathbf{n} = \langle 1, 1, 1\rangle \times \langle -1, 1, 1\rangle = \langle 0, -2, 2\rangle \quad .$$

(You do it!). We also need a **point** $(x_0, y_0.z_0)$. Plugging-in $t = 0$ (for example, you can plug-in any other number, but 0 is the easiest) into the first equation you get that one point on our plane is $(x_0, y_0, z_0) = (1, 2, 3)$. In general the equation of a plane through a point (x_0, y_0, z_0), with normal vector $\mathbf{n}$ is:

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \mathbf{n} = 0 \quad .$$

So, in this problem it is:

$$\langle x - 1, y - 2, z - 3\rangle \cdot \langle 0, -2, 2\rangle = 0 \quad ,$$

that spell out to:

$$(x - 1)(0) + (y - 2)(-2) + (z - 3)(2) = 0 \quad .$$

Dividing by -2 we get:

$$(y - 2) - (z - 3) = 0 \quad ,$$

that simplifies to $y - z = -1$.

Comment: There are many choices for (x_0, y_0, z_0) but at the end of the day you would get the same answer for the equation of the plane (after simplification).

7. (10 points) A certain particle has law of motion

$$\mathbf{r}(t) = \langle\, 2\sin t\,,\, 2\cos 2t\,,\, 2\sin 3t\,\rangle \quad .$$

Find its acceleration at $t = \pi/6$.

Ans.: $\langle -1, -4, -18\rangle \quad .$

First we need the **velocity**:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle\, 2\cos t\,,\, -4\sin 2t\,,\, 6\cos 3t\,\rangle \quad .$$

The acceleration is: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$:

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle\, -2\sin t\,,\, -8\cos 2t\,,\, -18\sin 3t\,\rangle \quad .$$

This is the **acceleration** for **general** t. But we are only interested at what is going on when $t = \pi/6$, getting:

$$\mathbf{a}(\pi/6) = \langle\, -2\sin \pi/6\,,\, -8\cos \pi/3\,,\, -18\sin \pi/2\,\rangle \quad .$$

Doing the trig-function evaluations (that you either memorize, or put in the formula sheet, I told you many times that you are allowed to do that), we get

$$\mathbf{a}(\pi/6) = \langle\, -2\cdot(1/2)\,,\, -8\cdot(1/2)\,,\, -18\cdot(1)\,\rangle = \langle\, -1\,,\, -4\,,\, -18\,\rangle \quad .$$

Comment: People who left it unevaluated (i.e. as $\langle\, -2\sin \pi/6\,,\, -8\cos \pi/3\,,\, -18\sin \pi/2\,\rangle$) got six points only. People who didn't plug-in $t = \pi/6$ (i.e. left it as: $\langle\, -2\sin t\,,\, -8\cos 2t\,,\, -18\sin 3t\,\rangle$) only got three points. Read the question!

8. (10 points) Compute the (scalar-function) line-integral

$$\int_C y\,ds$$

where the curve C is given by the parametric equation:

$$\mathbf{r}(t) = \langle\, 1+\sin 5t\,,\, 1+\cos 5t\,,\, \sqrt{75}t\,\rangle \quad , \quad 0 \le t \le 1 \quad .$$

Ans.: $10 + 2\sin 5$.

First we compute:

$$\mathbf{r}'(t) = \langle\, 5\cos 5t\,,\, -5\sin 5t\,,\, \sqrt{75}\,\rangle \quad .$$

Next we take the **magnitude**:

$$|\mathbf{r}'(t)| = |\langle\, 5\cos 5t\,,\, -5\sin 5t\,,\, \sqrt{75}\,\rangle| = \sqrt{(5\cos 5t)^2 + (-5\sin 5t)^2 + (\sqrt{75})^2} =$$

$$\sqrt{25\cos^2 5t + 25\sin^2 5t + 75} = \sqrt{25(\cos^2 5t + \sin^2 5t) + 75} = \sqrt{100} = 10 \quad .$$

So $ds = 10\,dt$, and our line-integral is:

$$\int_0^1 (1+\cos 5t)10\,dt = 10(t + \frac{\sin 5t}{5})\Big|_0^1 = (10t + 2\sin 5t)\Big|_0^1 = 10 + 2\sin 5 - 0 = 10 + 2\sin 5 \quad .$$

9. (10 points)

If
$$\lim_{(x,y,z)\to(1,2,3)} f(x,y,z) = 1 \quad , \quad \lim_{(x,y,z)\to(1,2,3)} g(x,y,z) = \pi/6$$
compute
$$\lim_{(x,y,z)\to(1,2,3)} (f(x,y,z) + \cos(2g(x,y,z)))^2 \sin(g(x,y,z))$$

Ans.: $\frac{9}{8}$.

By the limit laws:

$$\lim_{(x,y,z)\to(1,2,3)} (f(x,y,z) + \cos(2g(x,y,z)))^2 \sin(g(x,y,z)) = (1+\cos(2\cdot\pi/6))^2 \sin(\pi/6) =$$

$$(1+\frac{1}{2})^2 \cdot \frac{1}{2} = (\frac{3}{2})^2 \cdot \frac{1}{2} = \frac{9}{8} \quad .$$

Comment: People who left is as $(1+\cos(2\cdot\pi/6))^2 \sin(\pi/6)$ got six points.

10. Compute

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} \quad ,$$

where

$$\mathbf{F} = \langle x^2 + \sin(y+z)\,,\, y^2 + xz^3\,,\, z^2 + e^{xy}\,\rangle$$

and where S is the boundary (consisting of all six faces) of the cube

$$\{(x,y,z)\,|\,0 \le x,y,z \le 1\}$$

with the normal pointing **outward**.

Ans.: 3 .

Since the (vector-field) surface-integral is over a **closed** surface (the whole boundary of the cube), we use the **divergence theorem**.

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} = \int\int\int_E (div\mathbf{F})\,dV \quad ,$$

where E is the inside of the cube, i.e. the region

$$E = \{(x,y,z)\,|\,0 \le x,y,z \le 1\} \quad .$$

We must first compute $div\mathbf{F}$:

$$div\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \frac{\partial(x^2+\sin(y+z))}{\partial x} + \frac{\partial(y^2+xz^3)}{\partial y} + \frac{\partial(z^2+e^{xy}}{\partial z} = 2x+2y+2z \quad .$$

So we have to compute:

$$\int_0^1 \int_0^1 \int_0^1 (2x+2y+2z)\,dz\,dy,\,dx \quad .$$

The **inside integral** is:

$$\int_0^1 (2x+2y+2z)\,dz = (2x+2y)z + z^2\Big|_0^1 = (2x+2y)+1-0 = 2x+2y+1 \quad .$$

The **middle integral is**

$$\int_0^1 (2x+2y+1)\,dy = (2xy+y^2+y)\Big|_0^1 = 2x+1^2+1-0 = 2x+2 \quad ,$$

and finally, the **outside** integral is:

$$\int_0^1 (2x+2)\,dx = x^2+2x\Big|_0^1 = 1^2+2\cdot 1 = 3 \quad .$$

11. By finding a function f such that $\mathbf{F} = \bigtriangledown f$, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x,y,z) = 2e^{2x+3y+4z}\,\mathbf{i} + 3e^{2x+3y+4z}\mathbf{j} + 4e^{2x+3y+4z}\,\mathbf{k} \quad ,$$

$$C : x = t^2 \quad , \quad y = t^4 \quad , \quad z = -t^9 \quad , \quad 0 \le t \le 1 \quad .$$

Ans: $e - 1$.

It is easy enough in this problem to realize that $\mathbf{F}$ is a **conservative** vector-field, and, by inspection that the **potential function**, f equales $f(x,y,z) = e^{2x+3y+4z}$ (of course you are welcome to do it the long way, it is not that long, but in this problem it is acceptable to do it by inspection).
The **starting** point is (plug-in $t = 0$ into $x = t^2 \quad , \quad y = t^4 \quad , \quad z = -t^9$, we get $(0,0,0)$. The **end** point is (plug-in $t = 1$), is $(1,1,-1)$. So, by the so-called **fundamental theorem of vector-field line-integrals**:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \bigtriangledown f(x,y,z) \cdot d\mathbf{r} = f(EndPoint) - f(StartingPoint) =$$

$$f(1,1,-1) - f(0,0,0) = e^{2\cdot 1+3\cdot 1+4\cdot(-1)} - e^{2\cdot 0+3\cdot 0+4\cdot(0)} = e^1 - e^0 = e - 1 \quad .$$

Comment: People who left it as $e - e^0$ got eight points. People who did it correctly, but the long way, not using the potential function as specifically asked by the question only got two points. Follow instructions!

12. Evaluate the line integral

$$\int_C 5y\,dx + 5x\,dy + 6z\,dz \quad ,$$

where $C : x = t^2\,,\ y = t\,,\ z = t^2\,,\ 0 \leq t \leq 1$.

Ans.: 8 .

Here you are not told what method to use, so you can do it whichever way you want, as long as it is correct.

First Way: We can see that the vector-field of the question $\mathbf{F} = \langle 5y, 5x, 6z \rangle$ is conservative, and by insection or the offical way, the potential function is: $f(x, y, z) = 5xy + 3z^2$. The starting point is $(0, 0, 0)$ and the end-point is $(1, 1, 1)$, so the answer is $f(1, 1, 1) - f(0, 0, 0) = 8 - 0 = 8$.

Second Way:

$$\int_0^1 (5t)(2t)\,dt + (5t^2)\,dt + (6t^2)(2t)\,dt = \int_0^1 (10t^2 + 5t^2 + 12t^3)\,dt =$$

$$\int_0^1 (15t^2 + 12t^3)\,dt = (5t^3 + 3t^4)\Big|_0^1 = (5 \cdot 1^3 + 3 \cdot 1^4) - (5 \cdot 0^3 + 3 \cdot 0^4) = 5 + 3 - 0 - 0 = 8 \quad .$$

13. Evaluate

$$\int\int\int_E \frac{1}{x^2+y^2+z^2}\, dV \quad ,$$

where E is the portion of the ball $x^2+y^2+z^2 \leq 4$ that lies in the octant

$$\{(x,y,z)|x \geq 0\,,\, y \geq 0\,,\, z \geq 0\} \quad .$$

Ans.: π

We convert to **spherical coordinates**. Since $z \geq 0$, we have $0 \leq \phi \leq \pi/2$. Since $x \geq 0, y \geq 0$ we have $0 \leq \theta \leq \pi/2$. Since the radius of the sphere is 2, we have $0 \leq \rho \leq 2$. Recall that $x^2+y^2+z^2=\rho^2$ and $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\,\phi$. Our integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \frac{1}{\rho^2}\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \, = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \sin\phi\, d\rho\, d\theta\, d\phi \, =$$

$$(\int_0^{\pi/2} \sin\phi\, d\phi)(\int_0^{\pi/2} d\theta)(\int_0^2 d\rho\,)$$

$$= \left((-\cos\phi)\Big|_0^{\pi/2} \right)(\pi/2)(2) = (-cos(\pi/2) - -cos(0))(\pi/2)(2) = (\pi/2)(1)(2) = \pi \quad .$$

Comment: People who got the right answer (by luck) but set-up the wrong limits-of-integration in the triple-spherical-integral (but did everything else correctly) got five points.

14. Evaluate the triple integral

$$\int\int\int_E 10yz\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,0 \le z \le 1\,,\,0 \le y \le z\,,\,y+z \le x \le y+2z\,\} \quad .$$

Ans.: 1 .

We first, set-up the **volume-integral** as a **triple-iterated-integral**.

$$\int_0^1 \int_0^z \int_{y+z}^{y+2z} 10yz\,dx\,dy\,dz \quad .$$

The **inner-integral** is:

$$\int_{y+z}^{y+2z} 10yz\,dx \;=\; 10yz\left(\int_{y+z}^{y+2z} 1\,dx\right) = 10yz\left(x\Big|_{y+z}^{y+2z}\right) = 10yz(y+2z-(y+z)) = 10yz(z) = 10yz^2 \quad .$$

The **middle-integral** is:

$$\int_0^z 10yz^2\,dy = 10z^2\left(\int_0^z y\,dy\right) = 10z^2\left(\frac{y^2}{2}\Big|_0^z\right) = 10z^2\left(\frac{z^2}{2} - \frac{0^2}{2}\right) = 5z^4 \quad .$$

The **outside-integral** is

$$\int_0^1 5z^4\,dz = z^5\Big|_0^1 = 1^5 - 0^5 = 1 - 0 = 1 \quad .$$

15. Find the Jacobian of the transformation from (u, v, w)-space to (x, y, z)-space.

$$x = 2u + 3v + w^2 \quad , \quad y = -u + 2v \quad , \quad z = v + 3w,$$

at the point $(u, v, w) = (1, 1, 0)$.

Ans.: 21 .

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2w \\ -1 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} \quad .$$

At the point $(u, v, w) = (1, 1, 0)$ this equals

$$\begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} \quad - 3\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} \quad + 0\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 2[(2)(3) - (0)(1)] - 3[(-1)(3) - (0)(0)] + 0((-1)(1) - (2)(0)) = 2 \cdot 6 + 9 = 21 \quad .$$

16. Evaluate the integral

$$\int\int_D \frac{8e^8}{\pi} e^{-2x^2-2y^2}\, dA \quad ,$$

where D is the region

$$\{(x,y)\,|\,x^2+y^2 \leq 4, x \geq 0, y \leq 0\}$$

Ans.: $e^8 - 1$

We convert this to **polar coordinates**. Since x is **positive** and y is **negative**, this is the lower-right quarter-circle (living in the fourth quadrant) and the limits-of-integration in the θ variable are $-\pi/2 \leq \theta \leq 0$ or $3\pi/2 \leq \theta 2\pi$ (both are equally correct). The limits of integration in the r variable are $0 \leq r \leq 2$, since the radius of the (quarter)-circle is 2. Recall that $x^2+y^2 = r^2$ and $dA = r\,dr\,d\theta$. So we have

$$\int\int_D \frac{8e^8}{\pi} e^{-2x^2-2y^2}\, dA = \frac{8e^8}{\pi} \int_{-\pi/2}^{0} \int_0^2 e^{-2r^2}\, r\, dr\, d\theta = \frac{8e^8}{\pi} \left(\int_{-\pi/2}^{0} d\theta \right) \left(\int_0^2 e^{-2r^2}\, r\, dr \right)$$

$$\frac{8e^8}{\pi} \frac{\pi}{2} \left(\frac{e^{-2r^2}}{-4} \Big|_0^2 \right) = \frac{8e^8}{\pi} \frac{\pi}{2} \left(\frac{e^{-8}}{-4} - \frac{e^0}{-4}) \right) = e^8 - 1 \quad .$$

17.. Calculate the iterated integral

$$\int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\, dx\, dy\, dz \quad .$$

Ans.: 3

This is the same as the second part of problem #10. The **inside integral** is:

$$\int_0^1 (2x + 2y + 2z)\, dz = (2x + 2y)z + z^2 \Big|_0^1 = (2x + 2y) + 1 - 0 = 2x + 2y + 1 \quad .$$

The **middle integral is**

$$\int_0^1 (2x + 2y + 1)\, dy = (2xy + y^2 + y) \Big|_0^1 = 2x + 1^2 + 1 - 0 = 2x + 2 \quad ,$$

and finally, the **outside** integral is:

$$\int_0^1 (2x + 2)\, dx = x^2 + 2x \Big|_0^1 = 1^2 + 2 \cdot 1 = 3 \quad .$$

18. Find the maximum value of the function

$$f(x,y) = 3x + 4y \quad ,$$

subject to the constraint $x^2 + y^2 = 25$.

Ans.: 25

This calls for **Lagrange multipliers**. The **constraint** is $x^2 + y^2 - 25 = 0$ so $f(x,y) = 3x + 4y$ and $g(x,y) = x^2 + y^2 - 25$. The Lagrange multiplier equation

$$\triangledown f = \lambda \triangledown g \quad ,$$

becomes, in this problem

$$\langle 3, 4 \rangle = \lambda \langle 2x, 2y \rangle \quad ,$$

that spells out to the equations

$$3 = 2\lambda x \quad , \quad 4 = 2\lambda y \quad .$$

Since x can't be zero (if it was you would get nonsense $3 = 0$), we can divide the second equation by the first, getting:

$$\frac{4}{3} = \frac{y}{x} \quad ,$$

that means that $y = \frac{4}{3}x$. Plugging-this into the constraint equation, gives

$$x^2 + (\frac{4}{3}x)^2 = 25 \quad ,$$

that means

$$x^2(1 + (\frac{4}{3})^2) = 25 \quad ,$$

so:

$$x^2 \frac{25}{9} = 25 \quad .$$

This gives $x^2 = 9$ and we have two solutions $x = 3$ and $x = -3$. By doing **back substitution** we get that when $x = 3, y = \frac{4}{3} \cdot 3 = 4$ and when $x = -3, y = \frac{4}{3} \cdot (-3) = -4$. So we have two **finalists** $(3, 4)$ and $(-3, -4)$. Plugging-in the function $f(x, y)$, we have

$$f(3,4) = 3 \cdot 3 + 4 \cdot 4 = 25$$

$$f(-3,-4) = 3 \cdot (-3) + 4 \cdot (-4) = -25 \quad .$$

The larger value of these is 25, so 25 is the **maximum value**. The minimum value happens to be -25, but I didn't ask for it, so don't give it. (Also the location where the maximum value is attained is the point $(3, 4)$ but I didn't ask for that either, so don't tell me what I didn't ask for.)

19. Find the local maximum and minimum **values** and saddle point(s) of the function $f(x, y) = x^3 + y^3 - 3xy$

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local maximum value(s): None

local minimum value(s): -1 (at $(1, 1)$, but I didn't ask for the location).

saddle point(s): $(0, 0)$

$f_x = 3x^2 - 3y$, $f_y = 3y^3 - 3x$. For future reference $f_{xx} = 6x, f_{xy} = -3, f_{yy} = 6y$. We need to solve $f_x = 0, f_y = 0$, so we have to solve

$$3x^2 - 3y = 0 \quad , 3y^2 - 3x = 0 \quad .$$

So $y = x^2$ and $x = y^2$. Plugging the first equation into the second, we get $x = (x^2)^2 = x^4$ so $x - x^4 = 0$, so $x(1 - x^3) = 0$ that has **two** solutions $x = 0$ and $x = 1$. By back-substitution, when $x = 0$, $y = 0$ and when $x = 1$, $y = 1$.
So we found two **critical points**: $(0, 0)$ and $(1, 1)$.
Now we need to compute the **discriminant** $D = f_{xx}f_{yy} - f_{xy}^2$ at each of these points.

When $(x, y) = (0, 0)$, we have $f_{xx} = 0, f_{xy} = -3, f_{yy} = 0$, so $D = (0)(0) - (-3)^2 = -9 < 0$. This means that $(0, 0)$ is a saddle point.

When $(x, y) = (1, 1)$, we have $f_{xx} = 6, f_{xy} = -3, f_{yy} = 6$, so $D = (6)(6) - (-3)^2 = 36 - 9 = 25 > 0$. This means that $(1, 1)$ is a local max. or local min. Since $f_{xx} > 0$, it is a **local min**, and the value there is $f(1, 1) = 1^3 + 1^3 - 3 \cdot 1 \cdot 1 = -1$.

Comment: Some people missed the point $(0, 0)$ (since they solved $x^4 = x$ by dividing both sides by x, that is only legitimate when $x \neq 0$, thereby missing the option of $x = 0$). These people got five points.

20. Sketch the region of integration and change the order of integration.

$$\int_0^1 \int_0^x F(x,y)\,dy\,dx \;+\; \int_1^2 \int_0^{2-x} F(x,y)\,dy\,dx$$

Ans.: $\int_0^1 \int_y^{2-y} F(x,y)\ dx\,dy$

The region is the triangle whose vertices are $(0,0)$, $(2,0)$, and $(1,1)$, that is a simple type II region (horizontally simple):

$$R = \{(x,y)\,|\,0 \le y \le 1\,,\; y \le x \le 2-y\,\} \quad .$$

yielding $\int_0^1 \int_y^{2-y} F(x,y)\ dx\,dy$.

Comment: Some people treated each of the two double integrals separately, getting as answer:

$$\int_0^1 \int_y^1 F(x,y)\ dx\,dy + \int_0^1 \int_1^{2-y} F(x,y)\ dx\,dy \quad .$$

These people got nine points, since it is a correct answer, but not as simple as the one above, that only involves one double-integral.