Solutions to the "QUIZ" for Nov.2, 2006

1. Evaluate

$$\int\int\int_E (x^2+y^2+z^2)\,dV \quad ,$$

where E is bounded by the xy-plane and the hemispheres $z=\sqrt{1-x^2-y^2}$ and $z=\sqrt{4-x^2-y^2}$.

Solutuion: Since z is positive, and in the dictionary

$$z=\rho\cos\phi \quad ,$$

it means that $\cos\phi$ must be positive, which means that ϕ is between 0 and $\pi/2$ (recall that $\cos\phi$ is *negative* for ϕ between $\pi/2$ and π). On the other hand x and y have no restriction, so θ has the *full* (default) range $0 \le \theta \le 2\pi$.

So the description of E in **spherical coordinates** is

$$E=\{(\rho,\theta,\phi)\Big|1\le\rho\le 2, 0\le\theta\le 2\pi, 0\le\phi\le\pi/2\}.$$

Now remember that the "phrase" $x^2+y^2+z^2$ becomes ρ^2 in "spherical" language, and $dV = \rho^2\sin\phi\,d\phi\,d\theta\,d\rho$. So our integral is

$$\int_1^2\int_0^{2\pi}\int_0^{\pi/2}(\rho^2)\cdot\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$

$$=\int_1^2\int_0^{2\pi}\int_0^{\pi/2}\rho^4\sin\phi\,d\phi\,d\theta\,d\rho$$

By the **separation trick** (**Note:** it only works when all the *limits of integration* are **numbers** and the integrand is a *product* of functions that only depend on **one variable at a time**).

$$=\left(\int_1^2\rho^4\,d\rho\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi/2}\sin\phi\,d\phi\right)$$

$$=\left(\frac{\rho^5}{5}\Big|_1^2\right)(2\pi)\left(-\cos\phi\Big|_0^{\pi/2}\right)$$

$$=(\frac{31}{5})(2\pi)(0--1)=\frac{62\pi}{5} \quad .$$

Ans.: $\frac{62\pi}{5}$.

Common Mistakes:

1. Mess-up the **limits of integration** due to rusty trig.

2. Even though I YELLED so many times that the "phrase" $x^2+y^2+z^2$ **is** ρ^2 some people just literally consulted the full dictionary and it did the "long way" and of course never finished it.

3. Some people replaced dV by $d\phi\,d\theta\,d\rho$ **instead** of the correct: $dV=\rho^2\sin\phi\,d\phi\,d\theta\,d\rho$.