Complete Solutions to :
MATH 251 (4-6), Dr. Z. , Mid-Term #1, 12:0-1:20 , Thu., Oct. 12, 2006

[Version of Nov. 27, 2006, (typo in No. 9, corrected thanks to Christine L.)]

1. Find an equation for the plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes

$$x + y - z = 2 \quad , \quad 2x - y + 3z = 1 \quad .$$

Sol. to 1: Most people did the **wrong problem**. Instead of "contains the line" they understood it to be "perpendicular to the line". Giving the right answer to the wrong question can't do me much good. So let's answer the *right* question.

We all know how to find an equation of the plane passing through three given points P, Q, R. We already have **one** point at our disposal, namely $(-1, 2, 1)$, let's call it P. We still have to manufacture two others. These two points lie on the line of intersection of $x+y-z = 2$, $2x-y+3z = 1$. Let's decide that one of our points would have $x = 0$, then plugging-in $x = 0$ into these equations would give $y - z = 2$ and $-y + 3z = 1$. Solving these two equations with two unknowns we get $y = 7/2$ and $z = 3/2$ (you do it!). So we have the point $Q(0, 7/2, 3/2)$. We still have to find another point, let's call it R. Deciding for the sake of convenience to have $z = 0$ (you could also decide to take $y = 0$), we have, plugging-in $z = 0$ into $x + y - z = 2$, $2x - y + 3z = 1$, the two equations $x + y = 2$, and $2x - y = 1$. Solving them gives $x = 1, y = 1$ (you do it!). Together with the $z = 0$, this manufactures the point $(1, 1, 0)$.

Hooray! Now we have **three** points

$$P = (-1, 2, 1) \quad Q = (0, 7/2, 3/2) \quad R(1, 1, 0) \quad .$$

Now we find

$$\mathbf{PQ} = \langle 1, 3/2, 1/2 \rangle \quad \mathbf{PR} = \langle 2, -1, -1 \rangle$$

Taking the **cross-product**, we get

$$\mathbf{PQ} \times \mathbf{PR} = \langle -1, 2, -4 \rangle$$

(you do it!). That's our normal vector $\langle a, b, c \rangle$. Now pick any of the point P, Q, R as your reference point, (x_0, y_0, z_0), say $P(-1, 2, 1)$, using

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad ,$$

getting

$$(-1)(x + 1) + 2(y - 2) + (-4)(z + 1) = 0 \quad ,$$

dividing by -1 and simplifying, finally yields the **answer**:

$$x - 2y + 4z = -1 \quad .$$

2. Determine whether the planes are parallel, perpendicular or neither. If neither find the angle between them.

$$x + 4y - 3z = 1 \quad , \quad -3x + 6y + 7z = 4 \quad .$$

Sol. to 2.: I admit that Number 1 was hard! But this one is a **piece of cake**. The **normals** to the planes are (extract the coefficients of x, y, z in each): $\mathbf{n_1} = \langle 1, 4, -3 \rangle$, $\mathbf{n_2} = \langle -3, 6, 7 \rangle$.

Since these two vectors are not a multiple of each other (e.g. $(-3)/1$ is **not** $6/4$) they are obviously not parallel to each other.

Now take the **dot-product** $\mathbf{n_1} \cdot \mathbf{n_2} = (1)(-3) + (4)(6) + (-3)(7) = -3 + 24 - 21 = 0$. Since it is 0, the planes are **perpendicular**. That's it!

Ans.: The planes are perpendicular since the dot product of their normal vectors is 0.

3. Find the arclength of the curve

$$\mathbf{r}(t) = \langle\, 2\sqrt{2}\, t \,,\, e^{2t} \,,\, e^{-2t} \,\rangle \quad , \quad 0 \leq t \leq 1 \quad .$$

Sol. of 3: The arclength formula is

$$\int_0^1 |\mathbf{r}'(t)| \, dt \quad .$$

Let's first find $\mathbf{r}'(t)$.

$$\mathbf{r}'(t) = \langle\, 2\sqrt{2} \,,\, 2e^{2t} \,,\, -2e^{-2t} \,\rangle \quad .$$

$$|\mathbf{r}'(t)| = \sqrt{(2\sqrt{2})^2 + (2e^{2t})^2 + (-2e^{-2t})^2} = \sqrt{8 + 4e^{4t} + 4e^{-4t}} \quad .$$

Watch out: Check the algebra and don't do crazy things like $(e^{2t})^2 = e^{(2t)^2}$ this is **WRONG!**.

So the desired arclength is the definite integral

$$\int_0^1 \sqrt{8 + 4e^{4t} + 4e^{-4t}} \, dt \quad .$$

Now most people got stumped. Those few honest souls who set it up correctly as above, and admitted ignorance got 6 out of 10 partial credit. But many people committed horrendous algebra and calculus mistakes, and got penalized. First the following is **wrong**!

$$\int_0^1 \sqrt{8 + 4e^{4t} + 4e^{-4t}} \, dt = \int_0^1 (8 + 4e^{4t} + 4e^{-4t})^{1/2} \, dt =$$

$$\frac{(8 + 4e^{4t} + 4e^{-4t})^{3/2}}{3/2} \Big|_0^1$$

wrong, wrong, wrong!. It is true that

$$\int t^{1/2} \, dt = t^{3/2}/(3/2) \quad ,$$

but this is not true for
$$\int COMPLICATED^{1/2}\,dt \quad .$$
. Sometimes it is possible to perform a **change of variable**, but not in this case!

In this problem we had to use an **algebra trick**.

$$8+4e^{4t}+4e^{-4t}=4(2+e^{4t}+e^{-4t})=$$

$$4(e^{4t}+2+e^{-4t})=4((e^{2t})^2+2e^{2t}e^{-2t}+(e^{-2t})^2) \quad ,$$

and take advantage of the **famous** factorization

$$a^2+2ab+b^2=(a+b)^2 \quad ,$$

with $a=e^{2t}$ and $b=e^{-2t}$. So moving right along we get

$$\int_0^1 \sqrt{8+4e^{4t}+4e^{-4t}}\,dt=\int_0^1 \sqrt{4(e^{2t}+e^{-2t})^2)}\,dt=\int_0^1 2e^{2t}+2e^{-2t}\,dt=$$

$$=e^{2t}-e^{-2t}\Big|_0^1=e^2-e^{-2} \quad .$$

Ans.: The arclength is e^2-e^{-2}.

4. A particle of mass 100 kg is moving thanks to a force

$$\mathbf{F}=\langle 100,200,100\rangle \quad .$$

At $t=0$, it is at the point $(1,2,3)$ moving at a velocity $\langle\, 1\,,\,2\,,\,-1\rangle$. Find its position at $t=5$.

Sol. to 4.

By Newton's Second Law $F=m\mathbf{r}''(t)$, so

$$\langle 100,200,100\rangle=100\mathbf{r}''(t) \quad ,$$

and

$$\mathbf{r}''(t)=\langle 1,2,1\rangle \quad .$$

We first find the **velocity**, by integrating

$$\mathbf{v}(t)=\mathbf{r}'(t)=\int\langle 1,2,1\rangle\,dt=\langle t,2t,t\rangle+\mathbf{C} \quad ,$$

To find $\mathbf{C}$, we plug $t=0$ and use the data $\mathbf{v}(0)=\langle 1,2,-1\rangle$ getting $\mathbf{C}=\langle 1,2,-1\rangle$. So

$$\mathbf{v}(t)=\langle t,2t,t\rangle+\langle 1,2,-1\rangle=\langle t+1,2t+2,t-1\rangle \quad .$$

To get the position at time t, $\mathbf{r}(t)$, we integrate $\mathbf{v}(t)$:

$$\mathbf{r}(t) = \int \langle t+1, 2t+2, t-1 \rangle \, dt = \langle \frac{t^2}{2} + t, t^2 + 2t, \frac{t^2}{2} - t \rangle \; + \mathbf{C} \quad .$$

Plugging-in $t = 0$ yields $\mathbf{C} = \langle 1, 2, 3 \rangle$, so

$$\mathbf{r}(t) = \langle \frac{t^2}{2} + t, t^2 + 2t, \frac{t^2}{2} - t \rangle + \langle 1, 2, 3 \rangle = \langle \frac{t^2}{2} + t + 1, t^2 + 2t + 2, \frac{t^2}{2} - t + 3 \rangle$$

Finally: you plug-in $t = 5$ and get

$$\mathbf{r}(5) = \langle \frac{5^2}{2} + 5 + 1, 5^2 + 2(5) + 2, \frac{5^2}{2} - 5 + 3 \rangle = \langle \frac{37}{2}, 37, \frac{21}{2} \rangle \quad .$$

Ans.: $\langle \frac{37}{2}, 37, \frac{21}{2} \rangle$.

5. Find the following limit, if it exists, or show that it does not exist:

$$\lim_{(x,y) \to (0,0)} \frac{x^5 + y^5}{(x^2 + y^2)^2} \quad .$$

Sol. to 5: You first try to prove that the limit does not exist, by investigating what happens to the function upon approaching the point $(0, 0)$ along the line $y = cx$. Along this line the function equals

$$\frac{x^5 + (cx)^5}{(x^2 + (cx)^2)^2} = \frac{x^5 + c^5 x^5}{(x^2 + (c^2 x^2)^2} = \frac{x^5(1 + c^5)}{(x^2(1 + c^2))^2} = \frac{x^5(1 + c^5)}{x^4(1 + c^2)^2} = \frac{(1 + c^5)}{(1 + c^2)^2} x$$

As $x \to 0$, this goes to 0. So the limit along the line $y = cx$ does **not** depend on c, which means that the limit **probably** exists (and equals 0).

In this case, where the limit probably exists, to prove that it indeed exists, we must convert to polar. Using $x = r \cos \theta$, $y = r \sin \theta$ as well as $x^2 + y^2 = r^2$, our function, in polar is,

$$\frac{r^5 \cos^5 \theta + r^5 \sin^5 \theta}{r^4} = r(\cos^5 \theta + \sin^5 \theta) \quad .$$

Now taking the limit as (r, θ) goes to $(0, 0)$, is easy, just plug-in $r = 0, \theta = 0$. Now it is a simple plug-in (not $0/0$ as before), and we get 0. This is the **answer**.

6. Find an equation for the tangent plane of the surface

$$z = \frac{1}{\sqrt{x + y}}$$

at $(2, 2, 1/2)$.

Sol. to 6

The equation to use is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad .$$

Here $f(x,y) = \frac{1}{\sqrt{x+y}}$. Writing it in **power notation**, this is

$$f(x,y) = (x+y)^{-1/2}$$

(**Watch out** that reciprocal become negative powers). Now:

$$f_x(x,y) = (-1/2)(x+y)^{-3/2} \quad ,$$

$$f_y(x,y) = (-1/2)(x+y)^{-3/2} \quad ,$$

In this problem $x_0 = 2, y_0 = 2, z_0 = 1/2$. So

$$f_x(2,2) = (-1/2)(2+2)^{-3/2} = (-1/2)4^{-3/2} \quad ,$$

$$f_y(2,2) = (-1/2)(2+2)^{-3/2} = (-1/2)4^{-3/2} \quad .$$

Now **simplify** (step-by-step) $4^{-3/2}$.

$$4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{(4^{1/2})^3} = \frac{1}{(\sqrt{4})^3} = \frac{1}{(2)^3} = \frac{1}{8} \quad .$$

So

$$f_x(2,2) = -(\frac{1}{2})\frac{1}{8} = -\frac{1}{16}$$

$$f_y(2,2) = -(\frac{1}{2})\frac{1}{8} = -\frac{1}{16}$$

Plugging it all above, we get

$$z - \frac{1}{2} = -\frac{1}{16}(x-2) - \frac{1}{16}(y-2) \quad ,$$

and simplifying we get

$$z = -\frac{1}{16}x - \frac{1}{16}y + \frac{3}{4} \quad ,$$

This is the **ans.**. Also correct is

$$x + y + 16z = 12 \quad .$$

7. Use the chain rule to find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$, if

$$w = x^3y^2 \quad , \quad x = s^2t + 1 \quad , \quad y = t^2s + 3 \quad .$$

Sol. to 7.: The chain rule says:

$$w_s = (w_x)(x_s) + (w_y)(y_s)$$

$$w_t = (w_x)(x_t) + (w_y)(y_t)$$

Now: $w_x = 3x^2y^2$, $w_y = 2x^3y$, $x_s = 2st$, $y_s = t^2$. $x_t = s^2$, $y_t = 2st$. So

$$w_s = (3x^2y^2)(2st) + (2x^3y)(t^2) = 6x^2y^2st + 2x^3yt^2 \quad .$$

$$w_t = (3x^2y^2)(s^2) + (2x^3y)(2st) = 3x^2y^2s^2 + 4x^3yst$$

Ans.: $w_s = 6x^2y^2st + 2x^3yt^2$, $w_t = 3x^2y^2s^2 + 4x^3yst$.

Note: These are acceptable answers since you were not told 'express your answers in terms of s and t'. But, if you were then you would have to replace x and y by their expressions in terms of s, t.

Ans. in terms of s, t: $w_s = 6(s^2t+1)^2(t^2s+3)^2st + 2(s^2t+1)^3(t^2s+3)t^2$, $w_t = 3(s^2t+1)^2(t^2s+3)^2s^2 + 4(s^2t+1)^3(t^2s+3)st$. **Do not simplify!**.

8. Find the directional derivative of the function

$$g(x, y, z) = (x + 2y + 3z)^{3/2}$$

at the point $(-1, 1, 1)$, in the direction of the vector $\langle 1, 2, 2\rangle$.

Watch out: *Directional Derivative* of a function at a point (like in this problem) is a **number** *not* a vector. Do not confuse with the gradient, that is needed as an intermediate step, but is not the final answer.

Sol. to 8:

$$g_x = (3/2)(x + 2y + 3z)^{1/2} \cdot 1 = (3/2)(x + 2y + 3z)^{1/2}$$

$$g_y = (3/2)(x + 2y + 3z)^{1/2} \cdot 2 = 3(x + 2y + 3z)^{1/2}$$

$$g_z = (3/2)(x + 2y + 3z)^{1/2} \cdot 3 = (9/2)(x + 2y + 3z)^{1/2}$$

So

$$\bigtriangledown g = \langle (3/2)(x + 2y + 3z)^{1/2}, 3(x + 2y + 3z)^{1/2}, (9/2)(x + 2y + 3z)^{1/2}\rangle$$

Now is the time to **plug-in**! the **point** $(-1, 1, 1)$ (**not** the vector!)

$$\bigtriangledown g(-1, 1, 1) = \langle (3/2)(-1 + 2(1) + 3(1))^{1/2}, 3((-1) + 2(1) + 3(1))^{1/2}, (9/2)(-1 + 2(1) + 3(1))^{1/2}\rangle$$

$$= \langle (3/2)(4)^{1/2}, 3(4)^{1/2}, (9/2)(4)^{1/2}\rangle \quad .$$

But $4^{1/2} = \sqrt{4} = 2$, so:

$$\bigtriangledown g(-1, 1, 1) = \langle (3/2)(2), 3 \cdot 2, (9/2)(2)\rangle = \langle 3, 6, 9\rangle \quad .$$

The direction that is given is $\mathbf{v} = \langle 1, 2, 2\rangle$. We have to find the **unit** direction vector. So we take the **magnitude**

$$|\langle 1, 2, 2\rangle| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3 \quad .$$

To get $\mathbf{u}$, we **divide v** by its magnitude, getting

$$\mathbf{u} = \frac{1}{3}\langle 1,2,2\rangle = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle \quad .$$

Finally, the **directional derivative** is simply $(\bigtriangledown g) \cdot \mathbf{u}$ (at that point).

$$D_{\mathbf{u}} = \langle 3,6,9\rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle = (3)(\frac{1}{3}) + (6)(\frac{2}{3}) + (9)(\frac{2}{3}) = 11 \quad .$$

Ans.: 11. (Note that it is a **number!**, not a vector.)

9. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$x^2 + y^2 + z^2 = 3xyz \quad .$$

Sol. to 9

Bring it to the form $F(x,y,z) = 0$ by moving everything to the left side, and leaving 0 on the right side.

$$x^2 + y^2 + z^2 - 3xyz = 0 \quad .$$

This means that our $F(x,y,z)$ is

$$F(x,y,z) = x^2 + y^2 + z^2 - 3xyz \quad .$$

Now use

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad .$$

We have: $F_z = 2z - 3xy$, $F_x = 2x - 3yz$, $F_y = 2y - 3xz$. So:

$$\frac{\partial z}{\partial x} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy} \quad .$$

Ans: $\frac{\partial z}{\partial x} = \frac{3yz-2x}{2z-3xy}$, $\frac{\partial z}{\partial y} = \frac{3xz-2y}{2z-3xy}$

10. Find the linearization, $L(x,y)$, of

$$f(x,y) = x\cos(3x - 2y)$$

at the point $(2,3)$.

Sol. of 10: The formula to use is

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \quad .$$

In this problem $f(x,y) = x\cos(3x-2y)$, $x_0 = 2, y_0 = 3$. We have (now x is the boss!), by the **product rule**

$$f_x = x' \cos(3x-2y) + x(\cos(3x-2y))' \quad .$$

Since $x' = 1$, and by the **chain rule**,

$$(\cos(3x-2y))' = (-\sin(3x-2y))(3x)' = (-\sin(3x-2y))(3) = -3\sin(3x-2y)$$

so going back to f_x,

$$f_x = \cos(3x-2y) - 3x\sin(3x-2y) \quad .$$

To find f_y, y is the boss! (and x is *just* a constant, so we can take it out of the differentiation

$$f_y = x(\cos(3x-2y))' = x(-\sin(3x-2y))(-2y)' = x(-\sin(3x-2y))(-2) = 2x\sin(3x-2y) \quad .$$

Watch out: Surprisingly many people messed up one or both of these differentiations. Just do it step-by-step applying the product rule and/or the chain rule as the case maybe, remembering who is the current boss and treating the other variable(s) as mere constants! This does not mean that you can ignore them completely! For example one student did $\frac{\partial}{\partial x}(x+y+z)^{3/2} = (3/2)x^{1/2}$. This is wrong! y and z are constants and $((x+3)^{3/2})' = (3/2)(x+3)^{1/2}$ by the chain rule, **not** $(3/2)x^{1/2}$.

Back to out problem, **now** is the time to **plug-in**

$$f_x(2,3) = \cos(3(2)-2(3)) - 3(2)\sin(3(2)-2(3)) = \cos 0 - 6\sin 0 = 1 \quad .$$

$$f_y(2,3) = 2(2)\sin(3(2)-2(3)) = 4\sin 0 = 0 \quad .$$

Also we need $f(2,3) = (2)\cos(3(2)-2(3)) = 2\cos 0 = 2$. Putting all these numbers together, we get

$$L(x,y) = 2 + 1(x-2) + 0(y-3) = 2 + x - 2 = x \quad .$$

Ans.: The required linearization is x.

Careful: A *linearization* should be **linear** which means that it can have x and/or y by themselves but never squared (x^2) or cubed, or inside cos or any other function. Some people forgot to plug-in $x = 2$ $y = 3$ into f_x f_y and as a result got something horrendously complicated. This should ring a warning signal. A linearization of a function at a point has the format $ax + by + c$ where a, b, c are **plain numbers**. In the case of a function of x, y, z it has the format $ax + by + cz + d$ where a, b, c, d are **plain numbers**. Of course some of these numbers may be 0, in which case you might get something featuring only x (like in our problem) or only y. Sometimes none of them shows up not even the free number d in which case the linearization is 0 which is a possible scenario. (For example the linearization of $\cos(x+y+z)$ at $(0,0,0)$ is 0.