Complete to Solutions:
MATH 251 (1-3), Dr. Z. , Mid-Term #1, 10:20-11:40 , Thu., Oct. 12, 2006

1. Find an equation for the plane that passes through the points

$$(2,2,0) \quad , \quad (2,0,2) \quad , \quad (0,2,2) \quad .$$

Sol. to 1.

Let's call the points $P(2,2,0)$, $Q(2,0,2)$, $R(0,2,2)$. Compute The vectors:

$$\mathbf{PQ} = \langle 2-2, 0-2, 2-0 \rangle = \langle 0, -2, 2 \rangle$$

$$\mathbf{PR} = \langle 0-2, 2-2, 2-0 \rangle = \langle -2, 0, 2 \rangle \quad .$$

Now take the **cross-product**

$$\mathbf{PQ} \times \mathbf{PR} = \langle 0, -2, 2 \rangle \times \langle -2, 0, 2 \rangle$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 2 \\ -2 & 0 & 2 \end{vmatrix} \quad =$$

$$\mathbf{i} \begin{vmatrix} -2 & 2 \\ 0 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 0 & 2 \\ -2 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}$$

$$= \mathbf{i}((-2)\cdot(2) - (0)\cdot(2)) - \mathbf{j}(0 \cdot 2 - (-2)\cdot 2) + \mathbf{k}(0 \cdot 0 - (-2)\cdot(-2)) =$$

$$-4\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} = \langle -4, -4, -4 \rangle$$

This is the **normal vector** $\langle a, b, c \rangle$ to our plane. Taking as (x_0, y_0, z_0) any of the points P, Q, R, say P we have that the equation of the plane is

$$(-4)(x-2) + (-4)(y-2) + (-4)(z-0) = 0 \quad .$$

Simplifying (dividing by (-4)) we get that the **equation of our plane** is

$$x + y + z = 4 \quad .$$

Note: It is a good idea to **check** the answer by plugging-in the three points, and seeing whether it is correct. For P: $2+2+0=4$ (OK). For Q: $2+0+2=4$ (OK). For R: $0+2+2=4$.

Common mistakes: Careless handling of the signs. With these determinants you should be extra careful. That's why it is important to check the answer by plugging-in the points. If you would have gotten the wrong answer and realized it by plugging-in the points, I would have given you almost

all the points. People who did the method correctly but messed up the signs, without realizing that it was wrong, got 6 out of 10.

2. Find parametric equations for the line through the point $(1, 2, 0)$ that is parallel to the plane $x + y + z = -4$ and perpendicular to the line $x = 1 + t, y = 1 - t, z = t$.

Sol. of 2: Being **parallel** to the plane $x + y + z = -4$ is the same thing as being **perpendicular** to its normal $\langle 1, 1, 1\rangle$. It is also perpendicular to the line $x = 1+t, y = 1-t, z = t$, whose direction vector is $\langle 1, -1, 1\rangle$. So our desired line is perpendicular to both these vectors. So we need to take the **cross product**.

$$\langle 1,1,1\rangle \times \langle 1,-1,1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \quad =$$

$$\mathbf{i}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \mathbf{i}(1\cdot 1 - (-1)\cdot 1) - \mathbf{j}(1\cdot 1 - 1\cdot 1) + \mathbf{k}(1\cdot(-1) - 1\cdot 1) =$$

$$2\mathbf{i} - 0\mathbf{j} + (-2)\mathbf{k} = \langle 2, 0, -2\rangle$$

We know that our line passes through the point $(1, 2, 0)$, so the equation of our line in **vector-parametric form** is

$$\langle 1,2,0\rangle + t\langle 2,0,-2\rangle = \langle 1+2t, 2, -2t\rangle \quad ,$$

and in the **spelled-out** form

$$x = 1 + 2t \quad , \quad y = 2 \quad , z = -2t \quad .$$

this is the **answer**.

3. Find the curvature of the curve

$$\mathbf{r}(t) = \langle\, e^t \cos t \,,\, e^t \sin t \,, t \,\rangle \quad ,$$

at the point $(1, 0, 0)$.

Sol. to 3:

$$\mathbf{r}'(t) = \langle\, (e^t \cos t)' \,,\, (e^t \sin t)' \,, t' \,\rangle$$

$$= \langle\, e^t(\cos t)' + (e^t)'(\cos t) \,,\, e^t(\sin t)' + (e^t)'(\sin t), , 1 \,\rangle$$

$$= \langle\, e^t(-\sin t) + (e^t)(\cos t) \,,\, e^t \cos t + e^t(\sin t), , 1 \,\rangle$$

$$= \langle\, e^t(\cos t - \sin t), \, e^t(\cos t + \sin t) \,,\, 1 \,\rangle$$

$$\mathbf{r}''(t) =$$

$$\langle\, (e^t)'(\cos t - \sin t) + (e^t)(\cos t - \sin t)',\ (e^t)'(\cos t + \sin t) + (e^t)(\cos t + \sin t)', 0\,\rangle$$

$$= \langle\, (e^t)(\cos t - \sin t) + (e^t)(-\sin t - \cos t),\ (e^t)(\cos t + \sin t) + (e^t)(-\sin t + \cos t), 0\,\rangle$$

$$= \langle\, -2e^t \sin t,\ 2e^t \cos t, 0\,\rangle \qquad .$$

Now comes a **crucial** step. Plug-in $t = 0$. Why $t = 0$ and not something else? Since the point of interest is $(1, 0, 0)$ and setting this equal to $\langle e^t \cos 5, e^t \sin t, t\rangle$, and solving for t (start with the last component) gives $t = 0$.

Watch out: Many people were stupmed here, since they didn't know what to plug-in. Many people tried to plug-in $(1, 0, 0)$. This makes no sense! $\mathbf{r}(t)$ is a function of the single variable t, and we can't plug-in a point in xyz-space.

Watch out: If you wait for later to plug-in $t = 0$, it is not a mistake, but it will make life much harder. Since $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ are complicated vectors featuring t, but $\mathbf{r}'(0)$ and $\mathbf{r}''(0)$ are numerical vectors, much easier to handle.

Going back to the problem,

$$\mathbf{r}'(0) = \langle\, e^0(\cos 0 - \sin 0),\ e^0(\cos 0 + \sin 0)\,,\ 1\,\rangle = \langle\, 1,\ 1\,,\ 1\,\rangle \quad .$$

$$\mathbf{r}''(0) = \langle -2e^0 \sin 0,\ 2e^0 \cos 0, 0\,\rangle = \langle 0\, 2, 0\,\rangle \quad .$$

Now use the formula for the curvature

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3}$$

$$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle\, 1, 1, 1\,\rangle \times \langle\, 0, 2, 0\,\rangle = \langle\, -2, 0, 2\,\rangle \quad .$$

(you do it!). So

$$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2 + 0^2 + (2)^2} = \sqrt{8} \quad .$$

Also

$$|\mathbf{r}'(0)| = |\langle 1, 1, 1\rangle| = \sqrt{3} \quad ,$$

and

$$\kappa(0) = \frac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{||\mathbf{r}'(0)|^3} = \frac{\sqrt{8}}{(\sqrt{3})^3} = \frac{2\sqrt{2}}{3\sqrt{3}} \quad .$$

Ans.: $\frac{2\sqrt{2}}{3\sqrt{3}}$.

Common mistakes: Not plugging in $t = 0$ as soon as you have $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. Some people (surprisngly) got it right, but most people messed up. Even those who got it right wasted a lot of examination time that should have been spent elsewhere.

Conceptual Note: If you have to find the curvature at a **specified** place (where either $t = a$ for some given a or you have to figure out the right a like above, the answer is always a **number**, and should not have t in it! If it does, it is a conceptual mistake. If however, no point is speficied, then the answer should (in general) involve t (in some rare cases it happens to be a constant).

4. What force is required so that a particle of mass 100 kg has the position function

$$\mathbf{r}(t) = \langle t^4, \sin t, \cos 3t \rangle \quad .$$

Sol. of 4: You use Newton's Second Law

$$\mathbf{F} = m\mathbf{r}''(t) \quad ,$$

Here $m = 100$

$$\mathbf{r}'(t) = \langle 4t^3, \cos t, -3\sin 3t \rangle \quad ,$$

$$\mathbf{r}''(t) = \langle 12t^2, -\sin t, -9\cos 3t \rangle \quad .$$

Finally

$$\mathbf{F} = 100\mathbf{r}''(t) = 100\langle 12t^2, -\sin t, -9\cos 3t \rangle$$

$$= \langle 1200t^2, -100\sin t, -900\cos 3t \rangle \quad .$$

This is the **ans.**

Common mistakes: Most people got it right, but a few people messed up the differentiation. Recall $(\cos at)' = -a\sin at$, $(\sin at)' = -a\cos at$, for any number a.

5. Find the following limit, if it exists, or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + 5y^4}{(x^2 + 3y^2)^2} \quad .$$

Sol. to 5. Let's try to prove that it does not exist.

First way : When the function approaces $(0, 0)$ along the x axis (where $y = 0$) it tends to

$$\lim_{x\to 0} \frac{x^4 + 50^4}{(x^2 + 30^2)^2} = \lim_{x\to 0} \frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1$$

When the function approaces $(0, 0)$ along the y axis (where $x = 0$) it tends to

$$\lim_{y\to 0} \frac{0^4 + 5y^4}{(0^2 + 3y^2)^2} = \lim_{y\to 0} \frac{5y^4}{(3y^2)^2} = \lim_{y\to 0} \frac{5y^4}{9y^4} = \frac{5}{9} \quad .$$

Since the function approaces different values along the x-axis and the y-axis, there is no way that there is a global limit, so this means that the **limit does not exist**.

Second way : More generally, let's find what the function tends to when it approaches $(0,0)$ along a line of the form $y = cx$.

$$\lim_{x \to 0} \frac{x^4 + 5(cx)^4}{(x^2 + 3(cx)^2)^2} = \lim_{x \to 0} \frac{x^4 + 5c^4x^4}{(x^2 + 3c^2x^2)^2}$$

$$= \lim_{x \to 0} \frac{x^4 + 5c^4x^4}{(x^2(1 + 3c^2))^2} = \lim_{x \to 0} \frac{x^4(1 + 5c^4)}{x^4(1 + 3c^2)^2} = \lim_{x \to 0} \frac{(1 + 5c^4)}{(1 + 3c^2)^2}$$

$$= \frac{(1 + 5c^4)}{(1 + 3c^2)^2} \quad ,$$

since this **depends** on c, it means that the limit **does not exist**.

Note: If you get a pure number then the limit **may** exist, and you should try to convert to polar coordinates.

Watch out: A limit of a function $f(x, y)$ at a designated point (if it exists) is always a pure **number** that can't have x and/or y in it. Of course it may have c in it, but **not** x or y. If I ever see you leave as answer x and y I'll be very angry! It is a huge conceptual mistake that shows that you have no clue what is a limit.

6. Find the linear approximation to the function

$$f(x, y, z) = \frac{1}{\sqrt{x + y + z}}$$

at the point $(1, 2, 1)$, and use it to approximate $f(1, 1.9, 0.9)$.

Sol. to 6: First write $f(x, y, z)$ in **power notation** (and don't mess up, many people did!)

$$f(x, y, z) = (x + y + z)^{-1/2} \quad .$$

(Be very careful!, make sure that it is to the power $-1/2$ **not** $1/2$ (in our case).

Now take partial derivatives

$$f_x(x, y, z) = (-1/2)(x + y + z)^{-3/2} = \frac{-1}{2(x + y + z)^{3/2}}$$

$$= \frac{-1}{2(\sqrt{x + y + z})^3} \quad .$$

Similarly

$$f_y(x, y, z) = \frac{-1}{2(\sqrt{x + y + z})^3} \quad .$$

$$f_z(x,y,z) = \frac{-1}{2(\sqrt{x+y+z})^3} \quad .$$

Now plug-in the specfic point $(1,2,1)$

$$f_x(1,2,1) = \frac{-1}{2(\sqrt{1+2+1})^3} = \frac{-1}{2(\sqrt{4})^3} = \frac{-1}{2(2)^3} = \frac{-1}{16} \quad .$$

Similary,

$$f_y(1,2,1) = \frac{-1}{16} \quad .$$

$$f_z(1,2,1) = \frac{-1}{16} \quad .$$

We also need

$$f(1,2,1) = (1+2+1)^{-1/2} = \frac{1}{\sqrt{2}} \quad .$$

We use the formula for the linearization

$$L(x,y,z) =$$

$$f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) \quad ,$$

getting:

$$L(x,y,z) = \frac{1}{2} + \frac{-1}{16}(x-1) + \frac{-1}{16}(y-2) + \frac{-1}{16}(z-1) \quad .$$

Note: Leave it like that! Do not "simplify" any further (it is not a mistake, but it is not necessary).

Finally, to get the approximation, $f(1,1.9,0.9)$ is roughly equal to

$$L(1,1.9,0.9) = \frac{1}{2} + \frac{-1}{16}(1-1) + \frac{-1}{16}(1.9-2) + \frac{-1}{16}(0.9-1)$$

$$\frac{1}{2} + \frac{-1}{16}(-0.2) = \frac{1}{2} + \frac{1}{80} = \frac{41}{80} \quad .$$

Tip: Some people have trouble juggling stuff like $4^{-3/2}$. Just do it step-by-step.

$$4^{-3/2} = \frac{1}{4^{3/2}} = \frac{1}{(4^{1/2})^3} = \frac{1}{2^3} = \frac{1}{8} \quad .$$

7. Use the chain rule to find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$, if

$$w = x^2 + y^2 + z^2 \quad , \quad x = st \quad , \quad y = s\cos t \quad , \quad z = s\sin t \quad ,$$

when $s = 1$ and $t = 0$.

Note: Whenever the problem specifies "at $s = Something$ and $t = SomethingElse$" then the **final answer** is a **NUMBER not** involving any variables! Many people messed up by leaving the

answer in terms of x, y, z. The trick is that you have also to find the numerical values of x, y, z at the given values of s, t.

Sol. to 7: By the chain rule

$$w_s = (w_x)(x_s) + (w_y)(y_s) + (w_z)(z_s)$$

$$w_t = (w_x)(x_t) + (w_y)(y_t) + (w_z)(z_t)$$

We have:

$$w_x = 2x \quad , \quad w_y = 2y \quad , \quad w_z = 2z \quad .$$

$$x_s = t \quad y_s = \cos t \quad z_s = \sin t$$

$$x_t = s \quad y_t = -s \sin t \quad z_t = s \cos t$$

Now, when $s = 1, t = 0$, $x = 0, y = 1, z = 0$, and we have

$$w_x = 0 \quad , \quad w_y = 2 \quad , \quad w_z = 0 \quad .$$

$$x_s = 0 \quad y_s = 1 \quad z_s = 0$$

$$x_t = 1 \quad y_t = 0 \quad z_t = 1$$

Finally

$$w_s = (w_x)(x_s) + (w_y)(y_s) + (w_z)(z_s) = (0)(0) + (2)(1) + (0)(0) = 2 \quad .$$

$$w_t = (w_x)(x_t) + (w_y)(y_t) + (w_z)(z_t) = (0)(0) + (2)(0) + (0)(1) = 0$$

Ans.: At $s = 1, t = 0$, $w_s = 2$ and $w_t = 0$.

Common mistake: Leaving your answer in terms of x, y, z (or worse, also featuring s, t). On the other hand, if no specific values of s, t are given, then of course you leave your answer in terms of x, y, z, s, t (or just s, t if asked to do so, by substituting for x, y, z their expressions in terms of s, t given by the problem.)

8. Find the maximum rate of change of $f(x, y, z) = x^2 y^3 z^4$ at the point $(1, 1, 1)$, and the direction in which it occurs.

Sol. to 8 You first find the gradient:

$$\nabla f = \langle f_x, f_y, f_z \rangle \quad .$$

$$f_x = 2xy^3z^4 \quad , \quad f_y = 3x^2y^2z^4 \quad , \quad f_z = 4x^2y^3z^3 \quad .$$

So

$$\nabla f = \langle 2xy^3z^4 \, , \, 3x^2y^2z^4 \, , \, 4x^2y^3z^3 \rangle \quad .$$

At our point $(1,1,1)$ it is:

$$\bigtriangledown f(1,1,1) = \langle 2,3,4 \rangle$$

The **maximum rate of change** is simply the **magnitude** of $\langle 2,3,4\rangle$, $|\langle 2,3,4\rangle| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$, and the **direction where it occurs** is simply that gradient vector itself!, namely $\langle 2,3,4\rangle$, and if you wish to give a **unit** direction-vector, simply divide by the magnitude ($\sqrt{29}$ in our case), getting: $\langle 2/\sqrt{29}, 3/\sqrt{29}, 4/\sqrt{29}\rangle$.

Ans. The maximum rate of change is $\sqrt{29}$ and it occurs in the direction of $\langle 2,3,4\rangle$ (or, if you wish $\langle 2/\sqrt{29}, 3/\sqrt{29}, 4/\sqrt{29}\rangle$).

Common Mistakes: Some people confuse the **maximum rate of change** and the **direction**. The former is a **number** while the latter is a **vector**. Also, don't forget to plug-in at the designated point.

9. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$xz = \ln(y+z) \quad .$$

Sol. to 9.

First Way:

Take derivaive w.r.t. to x:

$$(xz)' = (\ln(y+z))'$$

$$x'z + xz' = \frac{0+z'}{y+z} \quad ,$$

$$z + xz' = \frac{0+z'}{y+z} \quad ,$$

Now solve for z'

$$z'(x - \frac{1}{y+z}) = -z$$

$$z_x = \frac{-z}{x - \frac{1}{y+z}} = \frac{z(y+z)}{1-xy-xz} \quad .$$

Now differentiate w.r.t. to y

$$(xz)' = (\ln(y+z))'$$

$$xz' = \frac{1+z'}{y+z}$$

Now solve for z'

$$z'(x - \frac{1}{y+z}) = \frac{1}{y+z} \quad .$$

So:

$$z_y = \frac{\frac{1}{y+z}}{x - \frac{1}{y+z}} = \frac{1}{xy + xz - 1}$$

Common Mistakes: Mess-up the differentiation by not applying the product rule or the (usual) chain rule correctly, and messing up the subsequent algebra.

Second Way: Write the relationsip as

$$xz - \ln(y+z) = 0$$

and use the formulas that if $F(x,y,z) = C$ then

$$z_x = -F_x/F_z \quad z_y = -F_y/F_z \quad .$$

Here $F = xz - \ln(y+z)$. $F_z = x - 1/(y+z)$, $F_x = z$, $F_y = -1/(y+z)$, and you get the same answers as above.

10. Find an equation of the tangent plane to the surface

$$z = e^{x^3+y^3}$$

at the point $(1,-1,1)$.

Sol. to 10 The relevant formula is

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \quad .$$

Here $f(x,y) = e^{x^3+y^3}$, $f_x = (3x^2)e^{x^3+y^3}$,$f_y = (3y^2)e^{x^3+y^3}$. Here $x_0 = 1, y_0 = -1, z_0 = 1$, so $f_x(1,-1) = (3 \cdot 1^2)e^{1^3+(-1)^3} = 3$, $f_x(1,-1) = (3 \cdot (-1)^2)e^{1^3+(-1)^3} = 3$. And we get

$$z - 1 = 3(x-1) + 3(y-(-1)) \quad ,$$

that simplifies to:

$$3x + 3y - z = -1 \quad .$$

Common Mistakes: Messing up the derivatives of $e^{x^3+y^3}$. Some people forgot about the y^3 and said that $f_x = 3x^2e^{x^3}$ this is **wrong!**. Some other creative spirits thought that it was e^{3x^2}. Please watch out.

Note: The desired answer is a tangent plane. A tangent plane is a **plane**! So make sure that **it looks like a plane**. For example, if you forgot to plug-in x=1, y=-1 you would come something complicated-looking that can't be a plane.