

## Dr. Z's Math251 Handout #16.9 [The Divergence Theorem]

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**Problem Type 16.9a:** Use the Divergence Theorem to calculate the surface integral  $\int \int_S \mathbf{F} \cdot d\mathbf{S}$ , where

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \quad ,$$

where  $S$  is a surface bounding some solid region that can be expressed in some coordinate system (Cartesian, cylindrical, or spherical).

**Example Problem 16.9a:** Use the Divergence Theorem to calculate the surface integral  $\int \int_S \mathbf{F} \cdot d\mathbf{S}$ , where

$$\mathbf{F}(x, y, z) = e^{2x} \sin 2y \mathbf{i} + e^{2x} \cos 2y \mathbf{j} + y^2 z^2 \mathbf{k} \quad ,$$

where  $S$  is the surface of the box bounded by the planes  $x = 0$ ,  $x = 2$ ,  $y = 0$ ,  $y = 1$ ,  $z = 0$ ,  $z = 3$ .

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### Steps

1. Compute the divergence of  $\mathbf{F}$ :

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad .$$

2. Write the solid region  $E$  in 'iterated form' in either Cartesian, Cylindrical, or Spherical coordinates.

### Example

1.

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(e^{2x} \sin 2y) + \frac{\partial}{\partial y}(e^{2x} \cos 2y) + \frac{\partial}{\partial z}(y^2 z^2) \\ &= 2e^{2x} \sin 2y - 2e^{2x} \sin 2y + 2y^2 z = 2y^2 z \quad . \end{aligned}$$

2. The solid region  $E$  bounding our surface is clearly the box

$$\{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 3\} \quad .$$

**3.** Set-up the Divergence Theorem, and evaluate the triple integral.

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int \int_E \operatorname{div} \mathbf{F} dV \quad .$$

**3.**

$$\begin{aligned} \int \int_S \mathbf{F} \cdot d\mathbf{S} &= \int \int \int_E \operatorname{div} \mathbf{F} dV \quad . \\ &= \int_0^2 \int_0^1 \int_0^3 2y^2 z dz dy dx \quad . \\ &= 2 \left( \int_0^2 dx \right) \left( \int_0^1 y^2 dy \right) \left( \int_0^3 z dz \right) \\ &= 2(2-0) \cdot \frac{1^3 - 0^3}{3} \cdot \frac{3^2 - 0^2}{2} = 2 \cdot 3 = 6 \quad . \end{aligned}$$

**Ans.:** 6.