Dr. Z's Math251 Handout #16.8 [Stokes' Theorem]

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Problem Type 16.8a: Use Stokes' Theorem to evaluate $\int\int_S curl\,\mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad ,$$

and S is some surface with a given orientation (that should boil down to either outwards or inwards).

Example Problem 16.8a: Use Stokes' Theorem to evaluate $\int\int_S curl\,\mathbf{F}\cdot d\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = x^2e^{2yz}\,\mathbf{i} + y^2e^{3xz}\,\mathbf{j} + z^2e^{4xy}\,\mathbf{k} \quad ,$$

and S is the hemisphere $x^2+y^2+z^2=9$, $z \geq 0$, oriented upwards.

Steps **Example**

1. You are going to use Stokes' Theorem

$$\int\int curl\,\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\,\mathbf{r} \quad .$$

The challenge is to find the bounding curve C.

For hemispheres $x^2+y^2+z^2 = R^2$, $z \geq 0$, like in this problem, the bounding curve is simply the circle $x^2+y^2 = R^2$, on the xy-plane $z=0$ and the parametric representation is

$$x = R\cos t\,,\; y = R\sin t\,,\; z = 0 \quad .$$

If it is the hemisphere $z \leq 0$ then it is the same but in the negative direction . If it is the hemisphere $x^2+y^2+z^2 = R^2$, $y \leq 0$, then the parametric representation is

$$x = R\cos t\,,\; z = R\sin t\,,\; y = 0 \quad ,$$

etc.

If it is the part of a surface $z = g(x,y)$ that lies above a plane $z = a$, oriented outwards, then C is obtained by solving $g(x,y) = a$ and representing $g(x,y) = a$ in parametric notation and adding to it $z = a$. The orientation of C is such as to obey the right-hand rule.

At the end you need to represent C in parametric form

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle\,,\; a \leq t \leq b \quad ,$$

for some expressions $x(t), y(t), z(t)$ of t and some numbers a and b.

1. Here C is simply the circle $x^2+y^2 = 3^2$ that lives in the xy-plane, and its parametric representation is

$$x = 3\cos t\,,\; y = 3\sin t\,,\; z = 0 \quad .$$

So

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 0\,\mathbf{k} \quad ,$$

$$0 \leq t \leq 2\pi \quad .$$

2. Plug-in the expressions for x, y, z in terms of t in $\mathbf{F}$, in order to express it in terms of the parameter t. Also figure out $d\mathbf{r} = \mathbf{r}'(t)\,dt$.

2.

$$\mathbf{F}(x,y,z) = (3\cos t)^2 e^0\,\mathbf{i} + (3\sin t)^2 e^0\,\mathbf{j} + 0\,\mathbf{k}$$

$$= 9\cos^2 t\,\mathbf{i} + 9\sin^2 t\,\mathbf{j} + 0\,\mathbf{k} \quad ,$$

$$d\mathbf{r}(t) = (-3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 0\,\mathbf{k})\,dt \quad .$$

3. Find the dot product $\mathbf{F}\cdot d\mathbf{r}$ and integrate from $t = a$ to $t = b$.

3.

$$\mathbf{F}\cdot d\mathbf{r} = ((9\cos^2 t)\cdot(-3\sin t) + (9\sin^2 t)\cdot(3\cos t) + 0\cdot 0)\,dt =$$

$$(-27\cos^2 t\sin t + 27\sin^2 t\cos t)\,dt \quad .$$

Finally, integrating from $t = 0$ to $t = 2\pi$, we get

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (-27\cos^2 t\sin t + 27\sin^2 t\cos t)\,dt$$

$$= 9\cos^3 t + 9\sin^3 t\Big|_0^{2\pi}$$

$$= (9\cos^3(2\pi) + 9\sin^3(2\pi)) - (9\cos^3(0) + 9\sin^3(0))$$

$$= 9 - 9 = 0 \quad .$$

Ans.: 0.

Problem Type 16.8b: Use Stokes' Theorem to evaluate $\int\int_S curl\,\mathbf{F}\cdot d\,\mathbf{S}$, if

$$\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad ,$$

and S consists of the top and four sides (but not the bottom) of the cube with vertices $(\pm A, \pm A, \pm A)$.

Example Problem 16.8b: Use Stokes' Theorem to evaluate $\int\int_S curl\,\mathbf{F}\cdot d\,\mathbf{S}$, if

$$\mathbf{F} = x^2y^2\,\mathbf{i} + y^2z\,\mathbf{j} + yz^2\,\mathbf{k} \quad ,$$

and S consists of the top and four sides (but not the bottom) of the cube with vertices $(\pm 2, \pm 2, \pm 2)$.

Steps

Example

1. In this problem, it is possible to find the bounding curve C, and use Stokes's Theorem directly, **but**, in this case C is a square with four sides and we would have to do four integrals, and it is a pain. We will use Stokes's theorem **indirectly** by finding **another** surface with the same bounding curve. Naturally for a box in which the given surface consists of the top and the four walls, the bottom is such a surface.

1. The bottom face is

$$-2 \le x, y \le 2 \quad , \quad z = -2 \quad .$$

2. Find $curl\,\mathbf{F}$.

2.

$$curl\,\mathbf{F} = (z^2 - y^2)\,\mathbf{i} + 0\,\mathbf{j} - 2x^2y\,\mathbf{k}$$

(You do it!)

3. Plug-in $z = -A$ and note that $dS = d\,x\,d\,y\,\mathbf{k}$, and the region of inegration is

$$\{(x, y) \,|\, -A \le x \le A, -A \le y \le A\} \quad .$$

Do the integration

3. When $z = -2$,

$$curl\,\mathbf{F} = (4 - y^2)\,\mathbf{i} + 0\,\mathbf{j} - 2x^2y\,\mathbf{k} \quad .$$

So

$$curl\,\mathbf{F} \cdot d\mathbf{S} = ((4 - y^2)\,\mathbf{i} + 0\,\mathbf{j} - 2x^2y\,\mathbf{k}) \cdot \mathbf{k} = -2x^2y \quad .$$

Finally,

$$\int\int_S curl\,\mathbf{F} \cdot d\mathbf{S} = \int_{-2}^{2}\int_{-2}^{2} -2x^2y\,dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-2}^{2} -2x^2y\,dx\right]dy = \int_{-2}^{2}(-2y)\left[\frac{x^3}{3}\Big|_{-2}^{2}\right]dy$$

$$= \int_{-2}^{2}(-2y)\frac{16}{3}\,dy$$

$$= \frac{-32}{3}\int_{-2}^{2} y\,dy = \frac{-32}{3} \cdot \frac{y^2}{2}\Big|_{-2}^{2} = \frac{-32}{3} \cdot 0 = 0 \quad .$$

Ans.: 0 .

Problem Type 16.8c: Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\,\mathbf{r}$, where

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad ,$$

and C is a curve that bounds some surface (that you have to figure out!), $z = g(x,y)$, above the region $\{(x,y)|(x,y) \in D\}$ (that you have to find!). C is oriented counterclockwise as viewed from above.

Example Problem 16.8c: Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\,\mathbf{r}$, where

$$\mathbf{F}(x,y,z) = (2x + y^2)\,\mathbf{i} + (2y + z^2)\,\mathbf{j} + (3z + x^2)\,\mathbf{k} \quad ,$$

and C is the triangle with vertices $(2,0,0)$,$(0,2,0)$, $(0,0,2)$, and is oriented counterclockwise as viewed from above.

Steps

1. Find a convenient surface that our curve bounds, and express it in terms of $z = g(x,y)$. Also figure out its projection on the xy-plane.

2. Find $curl\,\mathbf{F}$.

Example

1. The three vertices of our triangle lie on the plane $x + y + z = 2$ (**you do it!**), so $z = 2 - x - y$, and $g(x,y) = 2 - x - y$. Also the projection of the triangle on the xy plane is bounded by the line $x+y = 2$ and the axes, so it is the type I region

$$D = \{(x,y)\,|\,0 \le x \le 2\,, 0 \le y \le 2-x\,\} \quad .$$

2.

$$curl\,\mathbf{F} = -2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k} \quad .$$

(You do it!)

3. You have to use Stokes' Theorem

$$\int_C \mathbf{F}\cdot d\,\mathbf{r} = \int_S curl\,\mathbf{F}\cdot d\,\mathbf{S} \quad .$$

Convert the surface integral into an area integral using the formula from 16.7

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\,dA$$

Be also sure to replace z by $g(x,y)$. **Note:** The $\mathbf{F}$ from this formula is **not** the same as our $\mathbf{F}$, it is rather our $curl\,\mathbf{F}$, so use $\mathbf{F}$ as a *local variable.*

3.

$$\int_S curl\mathbf{F}\cdot d\,\mathbf{S} = \int_S (-2z\,\mathbf{i} - 2x\,\mathbf{j} - 2y\,\mathbf{k})\cdot d\,\mathbf{S}$$

where the surface is the one from step 1, i.e. $z = 2-x-y$ over D given above. Here we have $P = -2z$, $Q = -2x$, $R = -2y$, $g = 2-x-y$, so

$$\int\int_D (-(-2z)(-1) - (-2x)(-1) + (-2y))\,dA \quad .$$

Now we have to replace z by $2-x-y$, so this equals

$$\int\int_D (-2(2-x-y) - 2x - 2y)\,dA = \int\int_D -4\,dA$$

$$= -4\int\int_D dA \quad ,$$

where D is the triangle

$$D = \{(x,y)\,|\,0 \le x \le 2\,,0 \le y \le 2-x\,\} \quad .$$

In this case the area integral is simply the area of the triangle $(2\cdot 2/2 = 2)$ times -4, so the answer is -8, but of course you are welcome to to do it without the shortcut:

$$\int\int_D (-4)\,dA = \int_0^2 \int_0^{2-x} (-4)\,dx\,dy = -8 \quad .$$

Ans.: -8.