

**Dr. Z's Math251 Handout #16.7 [Surface Integrals]**

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**Problem Type 16.7a:** Evaluate the surface integral

$$\int \int_S F(x, y, z) \, dS \quad ,$$

where  $S$  is a given surface that is either given parametrically in terms of  $u \, v$ , or can be made so, or can be written in the form  $z = f(x, y)$ ,  $\{(x, y) | (x, y) \in D\}$  for some set  $D$  in the  $xy$ -plane.

**Example Problem 16.7a:** Evaluate the surface integral

$$\int \int_S 2x^2 z^2 \, dS \quad ,$$

where  $S$  is the part of the cone  $z^2 = x^2 + y^2$  that lies between the planes  $z = 1$  and  $z = 3$ .

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**Steps**

**Example**

1. If the surface can be described parametrically as

$$\{(x(u, v), y(u, v), z(u, v)) | (u, v) \in D\} \quad ,$$

for some set  $D$  in  $uv$ -plane, set-up  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , and compute  $r_u, r_v$ , then their cross product  $r_u \times r_v$ , and finally its length  $|r_u \times r_v|$ . Put

$$dS = |r_u \times r_v| du dv \quad .$$

On the other hand if the surface is given as  $z = f(x, y)$ , with some description where it lives, figure out the “floor” (projection on the  $xy$ -plane), and put

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

**Note:** The first method also works in the second case, just take  $x, y$  as the parameters and put  $x = x, y = y, z = f(x, y)$ .

1. In this problem  $z^2 = x^2 + y^2$  means  $z = \sqrt{x^2 + y^2}$ , so it is possible to do it the second way. But it is a bit easier to use cylindrical coordinates and then the surface is  $z = r$ , and the parametric representation is

$$x = r \cos \theta, y = r \sin \theta, z = r \quad ,$$

where the parameters are  $r$  and  $\theta$ . The region on the  $xy$ -plane below the surface is

$$\{(r, \theta) | 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\} \quad .$$

So

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k} \quad .$$

$$\mathbf{r}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \mathbf{k} \quad .$$

$$\mathbf{r}_\theta = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} + 0 \mathbf{k} \quad .$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = -r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} + r \mathbf{k} \quad ,$$

(you do it!). Also

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r \cos \theta)^2 + (-r \sin \theta)^2 + r^2} = \sqrt{2}r \quad ,$$

$$\text{so } dS = \sqrt{2}r dr d\theta.$$

2. Set-up the integral

$$\int \int_S F(x, y, z) dS \quad ,$$

with the  $x, y, z$  replaced by their expressions in terms of the parameters, and  $S$  replaced by its description in terms of  $u, v$ , and  $dS$  replaced by what you found in step 1.

3. Evaluate the iterated integral.

2.

$$\begin{aligned} & \int \int_S 2x^2 z^2 dS \\ &= \int \int_D 2(r \cos \theta)^2 r^2 \sqrt{2} r dr d\theta \quad , \end{aligned}$$

where  $D$  is the region  $1 \leq r \leq 3, 0 \leq \theta \leq 2\pi$ . Converting it to an iterated integral, we get

$$= 2\sqrt{2} \int_0^{2\pi} \int_1^3 r^5 \cos^2 \theta dr d\theta \quad .$$

3.

$$\begin{aligned} &= 2\sqrt{2} \int_0^{2\pi} \int_1^3 r^5 \cos^2 \theta dr d\theta \quad . \\ &= \sqrt{2} \int_0^{2\pi} \left[ \int_1^3 r^5 2 \cos^2 \theta dr \right] d\theta \quad . \\ &= \sqrt{2} \int_0^{2\pi} 2 \cos^2 \theta \left[ \frac{r^6}{6} \right]_1^3 d\theta \quad . \\ &= \sqrt{2} \left( \frac{3^6 - 1^6}{6} \right) \cdot \int_0^{2\pi} 2 \cos^2 \theta d\theta \\ &= \sqrt{2} \frac{364}{3} \cdot \int_0^{2\pi} (1 + \cos(2\theta)) d\theta \\ &= \sqrt{2} \left( \frac{364}{3} \right) \cdot \left( \theta + \frac{\sin(2\theta)}{2} \right) \Big|_0^{2\pi} = \sqrt{2} \frac{364}{3} \cdot 2\pi = \frac{728\sqrt{2}\pi}{3} \quad . \end{aligned}$$

**Ans.:**  $\frac{728\sqrt{2}\pi}{3}$ .

**Problem Type 16.7b:** Evaluate the surface integral  $\int \int_S \mathbf{F} \cdot d\mathbf{S}$  for the given vector field  $\mathbf{F}$  and oriented surface  $S$ .

$$\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k} \quad ,$$

where  $S$  is the part of the surface  $z = g(x, y)$  that lies above some region  $D$ , in the  $xy$ -plane and has upward orientation.

**Example Problem 16.7b:** Evaluate the surface integral  $\int \int_S \mathbf{F} \cdot d\mathbf{S}$  for the given vector field  $\mathbf{F}$

and oriented surface  $S$ .

$$\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k} \quad ,$$

$S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$  that lies above the square  $0 \leq x \leq 1, 0 \leq y \leq 1$  and has upward orientation.

## Steps

1. Set-up the formula

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int \int_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA$$

Be also sure to replace  $z$  by  $g(x, y)$ .

## Example

1. Here  $g = 4 - x^2 - y^2$ ,

$$P = xy \quad , \quad Q = yz \quad , \quad R = zx \quad .$$

Plugging everything in, our surface integral

$$= \int \int_D -xy(-2x) - yz(-2y) + xz \, dA$$

$$= \int \int_D (2x^2y + (2y^2 + x)z) \, dA \quad ,$$

but since  $z = 4 - x^2 - y^2$ , this equals

$$= \int \int_D (2x^2y + (2y^2 + x)(4 - x^2 - y^2)) \, dA \quad ,$$

$$= \int \int_D (2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2) \, dA \quad ,$$

where  $D$  is the square

$$\{ (x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \} \quad .$$

2. Looking at  $D$  convert it into an iterated integral.

2.

$$= \int_0^1 \int_0^1 (2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2) \, dx \, dy \quad .$$

$$= \int_0^1 \int_0^1 (-x^3 + x^2(-2y^2 + 2y) + x(-y^2 + 4) - 2y^4 + 8y^2) \, dx \, dy \quad .$$

**3.** Evaluate the iterated integral.

**3.** You do it! (it is rather tedious). The answer turns out to be  $\frac{713}{180}$ .