Dr. Z's Math251 Handout #16.6 [Parametric Surfaces and Their Areas]

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Problem Type 16.6a: Find an equation of the tangent plane to the given parametric surface at the specified point.

$$x = x(u,v) \quad , \quad y = y(u,v) \quad , \quad z = z(u,v) \quad ; \quad u = 1, v = 1 \quad .$$

Example Problem 16.6a: Find an equation of the tangent plane to the given parametric surface at the specified point.

$$x = u^2 \quad , \quad y = v^2 \quad , \quad z = uv \quad ; \quad u = 1, v = 1 \quad .$$

Steps

1. Set-up

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \quad ,$$

and compute the partial derivatives w.r.t. u and w.r.t. v :

$$\mathbf{r}_u = x_u\,\mathbf{i} + y_u\,\mathbf{j} + z_u\,\mathbf{k} \quad ,$$

$$\mathbf{r}_v = x_v\,\mathbf{i} + y_v\,\mathbf{j} + z_v\,\mathbf{k} \quad .$$

Then **plug-in** the given values of u and v.

Example

1. In this problem

$$\mathbf{r} = u^2\,\mathbf{i} + v^2\,\mathbf{j} + uv\,\mathbf{k} \quad ,$$

We have

$$\mathbf{r}_u = 2u\,\mathbf{i} + 0\,\mathbf{j} + v\,\mathbf{k} \quad ,$$

$$\mathbf{r}_v = 0\,\mathbf{i} + 2v\,\mathbf{j} + u\,\mathbf{k} \quad .$$

Now **plug-in** $u = 1, v = 1$ to get **numerical vectors**.

$$\mathbf{r}_u(1,1) = 2\,\mathbf{i} + 0\,\mathbf{j} + 1\,\mathbf{k} \quad ,$$

$$\mathbf{r}_v(1,1) = 0\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k} \quad .$$

So at this point, $r_u = \langle 2, 0, 1 \rangle$, $r_v = \langle 0, 2, 1 \rangle$.

2. Find the cross-product $\mathbf{r}_u \times \mathbf{r}_v$. This is a vector **normal** to the tangent plane.

2.

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= -2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k} \quad .$$

Or in $\langle\rangle$ notation

$$\mathbf{N} = \langle -2, -2, 4 \rangle \quad .$$

3. Find the **point** (x_0, y_0, z_0) by plugging into x, y, z the specific values of u and v given in the problem The desired equation of the tangent plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad .$$

where $N = \langle a, b, c \rangle$ and the point is (x_0, y_0, z_0).

3. The point is $(1^2, 1^2, 1 \cdot 1) = (1, 1, 1)$. The desired equation of the tangent plane is

$$(-2)(x - 1) - 2(y - 1) + 4(z - 1) = 0 \quad .$$

Or, in expanded form

$$-2x - 2y + 4z = 0 \quad .$$

Dividing by -2 (to make it nicer), we get:

Ans.: $x + y - 2z = 0$.