Dr. Z's Math251 Handout #16.5 [Curl and Divergence]

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Problem Type 16.5a: Find (a) the curl and (b) the divergebce of the vector field

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\mathbf{k} \quad .$$

Example Problem 16.5a: Find (a) the curl and (b) the divergebce of the vector field

$$\mathbf{F}(x,y,z) = 2e^x \sin y\,\mathbf{i} + 3e^x \cos y\,\mathbf{j} + (4z^2 + x + y)\,\mathbf{k} \quad .$$

Steps

1. *curl* $\mathbf{F}$ equals

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad .$$

Set it up for the specific P, Q, R.

2. Evaluate the 'determinant'.

3. Set-up the formula for the divergence

$$div\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad .$$

Then compute it.

Example

1.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2e^x \sin y & 3e^x \cos y & 4z^2 + x + y \end{vmatrix} \quad .$$

2.

$$\mathbf{i}\left(\frac{\partial}{\partial y}(4z^2 + x + y) - \frac{\partial}{\partial z}(3e^x \cos y)\right)$$

$$-\mathbf{j}\left(\frac{\partial}{\partial x}(4z^2 + x + y) - \frac{\partial}{\partial z}(2e^x \sin y)\right)$$

$$+\mathbf{k}\left(\frac{\partial}{\partial x}(3e^x \cos y) - \frac{\partial}{\partial y}(2e^x \sin y)\right)$$

$$= \mathbf{i}\,(1-0) - \mathbf{j}\,(1-0) + \mathbf{k}\,(3e^x \cos y - 2e^x \cos y)$$

$$= \mathbf{i} - \mathbf{j} + e^x \cos y\,\mathbf{k} \quad .$$

Ans. to (a): *curl* $\mathbf{F} = \mathbf{i} - \mathbf{j} + e^x \cos y\,\mathbf{k}$.

3.

$$div\,\mathbf{F} = \frac{\partial}{\partial x}(2e^x \sin y) + \frac{\partial}{\partial y}(3e^x \cos y) + \frac{\partial}{\partial z}(4z^2 + x + y)$$

$$= 2e^x \sin y - 3e^x \sin y + 8z = -e^x \sin y + 8z \quad .$$

Ans. to (b): $div\,\mathbf{F} = -e^x \sin y + 8z$.

Problem Type 16.5b: Determine whether or not the vector field is conservative. If it is, find a function f such that $\mathbf{F} = \bigtriangledown f$.

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\mathbf{k} \quad .$$

Example Problem 16.5b: Determine whether or not the vector field is conservative. If it is, find a function f such that $\mathbf{F} = \bigtriangledown f$.

$$\mathbf{F}(x,y,z) = (y^2z + 2xyz)\,\mathbf{i} + (2xyz + x^2z)\,\mathbf{j} + (xy^2 + x^2y + 2z)\,\mathbf{k}$$

Steps

1. Compute *curl* $\mathbf{F}$

$$\bigtriangledown \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad .$$

If it is the **zero** vector (i.e. **all** compoents are zero) then the vector field $\mathbf{F}$ is conservative. Otherwise not. If it is not, end of story. If it is, go on.

Example

1.

$$\bigtriangledown \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z + 2xyz & 2xyz + x^2z & xy^2 + x^2y + 2z \end{vmatrix} \quad .$$

This equals

$$\mathbf{i}\left(\frac{\partial}{\partial y}(xy^2 + x^2y + 2z) - \frac{\partial}{\partial z}(2xyz + x^2z)\right)$$

$$-\mathbf{j}\left(\frac{\partial}{\partial x}(xy^2 + x^2y + 2z) - \frac{\partial}{\partial z}(y^2z + 2xyz)\right)$$

$$+\mathbf{k}\left(\frac{\partial}{\partial x}(2xyz + x^2z) - \frac{\partial}{\partial y}(y^2z + 2xyz)\right)$$

$$= \mathbf{i}\,(2xy + x^2 - 2xy - x^2) - \mathbf{j}\,(y^2 + 2xy - y^2 - 2xy)$$

$$+\mathbf{k}\,(2yz + 2xz - 2yz - 2xz)$$

$$= 0\,\mathbf{i} - 0\,\mathbf{j} + 0\mathbf{k} = \mathbf{0} \quad .$$

Since the curl of $\mathbf{F}$ is $\mathbf{0}$, the vector field $\mathbf{F}$ is conservative, and we must go on.

2. Find a function $f(x, y, z)$ such that $\bigtriangledown F = f$, in other words

$$f_x = P \quad , \quad f_y = Q \quad , \quad f_z = R \quad .$$

You first integrate P w.r.t. to x getting that f equals something **plus** a function $g(y, z)$. Then you plug that expression for f and use it in $f_y = Q$ getting that $g(y, z)$ equals something explicit **plus** a function $h(z)$. Plug-it back into f, and use $f_z = Q$ to get what $h(z)$ is, and plug it back into f.

2. $f_x = y^2z + 2xyz$, means that

$$f = \int (y^2z+2xyz)\,dx = xy^2z+x^2yz+g(y,z) \quad .$$

$f_y = 2xyz + x^2z$ means that

$$2xyz + x^2z + g_y = 2xyz + x^2z \quad ,$$

so $g_y = 0$ and $g(y, z) = h(z)$, for some function, $h(z)$, of z. So now

$$f = xy^2z + x^2yz + h(z)$$

$f_z = xy^2 + x^2y + 2z$ means that

$$xy^2 + x^2y + h'(z) = xy^2 + x^2y + 2z$$

So $h'(z) = 2z$ and $h(z) = z^2$. It follows that

$$f = xy^2z + x^2yz + z^2 \quad .$$

Ans.: **F** is conservative, and the potential function f such that $\bigtriangledown f = \mathbf{F}$ is $f = xy^2z + x^2yz + z^2$.