**Dr. Z's Math251 Handout #15.9 [Change of Variables in Multiple Integrals]**

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**Problem Type 15.9a**: Find the Jacobian of the transformation

$$x = g(u,v,w) \quad , \quad y = h(u,v,w) \quad , \quad z = k(u,v,w) .$$

**Example Problem 15.9a**: Find the Jacobian of the transformation

$$x = u^2 v \quad , \quad y = v^2 w \quad , \quad z = w^2 u .$$

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**Steps**

**1.** Compute all the entries in the Jacobian matrix

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad .$$

**2.** Evaluate the determinant:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= (\frac{\partial x}{\partial u}) \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - (\frac{\partial x}{\partial v}) \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} + (\frac{\partial x}{\partial w}) \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

**Example**

**1.**

$$\begin{vmatrix} 2uv & u^2 & 0 \\ 0 & 2vw & v^2 \\ w^2 & 0 & 2uw \end{vmatrix} \quad .$$

**2.**

$$= 2uv \begin{vmatrix} 2vw & v^2 \\ 0 & 2wu \end{vmatrix} - u^2 \begin{vmatrix} 0 & v^2 \\ w^2 & 2wu \end{vmatrix}$$

$$+0 \cdot \begin{vmatrix} 0 & 2vw \\ w^2 & 0 \end{vmatrix}$$

$$= 2uv[(2vw)(2uw)-0]-u^2[0-(v^2)(w^2)]+0$$

$$= 9u^2v^2w^2 \quad .$$

**Ans.**: $9u^2v^2w^2$.

**Problem Type 15.9b**: Use the given transformation to evaluate the integral

$$\int\int_R F(x,y)\, dA \quad ,$$

where $R$ is the triangular region with vertices $(p_1,p_2)$,$(q_1,q_2)$, $(r_1,r_2)$; $x = au + bv$, $y = cu + dv$.

**Example Problem 15.9b**: Use the given transformation to evaluate the integral

$$\int\int_R (x+y)\,dA \quad ,$$

where $R$ is the triangular region with vertices $(0,0)$,$(2,1)$, $(1,2)$; $x = 2u + v$, $y = u + 2v$.

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**Steps**

**1.** Figure out the region in the $uv$-plane that gets transformed. Since a triangle goes to a triangle, we need to find the 3 vertices. Solve for $u$, $v$ in terms of $x, y$ and find the three points. Call the new triangle $R'$.

**2.** Find the Jacobian of the transformation. In this case of a so-called linear transformation, the Jacobian is simply $ad-bc$. Also express $F(x,y)$ in terms of $(u,v)$ using the transformation.

$$\int\int_R F(x,y)\,dA =$$

$$\int\int_{R'} F(au+bv, cu+dv)(ad-bc)\,dA \quad .$$

**Example**

**1.** Since $x = 2u + v$, $y = u + 2v$, when $(x,y) = (0,0)$ $u = 0, v = 0$ so the point $(0,0)$ goes to the point $(0,0)$. When $(x,y) = (1,2)$, we have to solve the system $1 = 2u+v, 2 = u+2v$ giving us $u = 0, v = 1$ so $(1,2)$ goes to $(0,1)$. Similarly, $(2,1)$ goes to $(1,0)$. So the region in the $uv$-plane is the far simpler triangle whose vertices are $(0,0),(1,0),(0,1)$. Let's call this region $R'$.

**2.** The Jacobian is $(2)(2)-(1)(1) = 3$, so

$$\int\int_R (x+y)\,dA = \int\int_{R'} (2u+v+u+2v)\cdot 3\,dA =$$

$$9\int\int_{R'} (u+v)\,dA \quad .$$

**3.** Draw the region (in this case triangle) in the $uv$- plane and express it as a type I (or type II) region. Then set-up the appropriate iterated integral, by deciding on the **main road** and the **side streets**.

**3.** The region is the triangle bounded by the axes and the line $u+v=1$. It can be written as

$$\{(u,v)\,|\,0 \le u \le 1\,,\, 0 \le v \le 1-u\,\} \quad .$$

Our area-integral is thus equal to the iterated integral

$$9\int_0^1 \int_0^{1-u} (u+v)\,dv\,du \quad .$$

The inner integral is

$$\int_0^{1-u} (u+v)\,dv = uv + \frac{v^2}{2}\Big|_0^{1-u}$$

$$= u(1-u) + \frac{(1-u)^2}{2} = (1-u^2)/2 \quad ,$$

and the whole integral is

$$\frac{9}{2}\int_0^1 (1-u^2)\,du = \frac{9}{2}\left[u - \frac{u^3}{3}\right]\Big|_0^1 = \frac{9}{2}\cdot\frac{2}{3} = 3 \quad .$$

**Ans.**: 3.