Dr. Z's Math251 Handout #15.8 [Triple Integrals in Cylindrical and Spherical Coordinates]

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Problem Type 15.8a: Evaluate

$$\int\int\int_E F(x,y,z)\,dV \quad ,$$

where E is a solid region descibed in terms of cylinders and other stuff.

Example Problem 15.8a: Evaluate

$$\int\int\int_E x^2\,dV \quad ,$$

where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.

Steps

1. Chances are that you are supposed to use *cylindrical coordinates*. Express E in the form

$$E = \{\,(r,\theta,z)\,|\,\alpha \le \theta \le \beta\,,$$

$$h_1(\theta) \le r \le h_2(\theta)\,,\, u_1(r,\theta) \le z \le u_2(r,\theta)\} \quad .$$

Example

1. Since $x^2+y^2=r^2$, the cone $z^2=4x^2+4y^2$ can be writen $z^2=4r^2$. The cylinder $x^2+y^2=1$ is really $r=1$, and this is the "base". $z^2=4r^2$ means that z ranges between $-2r$ and $2r$. **but** we are also told that our solid is **above** the plane $z=0$, so z ranges between 0 and $2r$. It turns out that

$$E = \{\,(r,\theta,z)\,|\,0 \le \theta \le 2\pi\,,\, 0 \le r \le 1\,,\, 0 \le z \le 2r\} \quad .$$

2. Express the integrand $F(x, y, z)$ in cylindrical coordinates, using the "dictionary"

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad x^2+y^2 = r^2 \quad .$$

Also $dV = rdrd\theta dz$. Then set up the volume integral as an iterated integral

$$\int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} F(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta \quad .$$

2.

$$\int_0^{2\pi} \int_0^1 \int_0^{2r} (r\cos\theta)^2 r\, dz\, dr\, d\theta \quad .$$

3. Evalute the integral from the inside to the outside.

3. The inside integral is:

$$\int_0^{2r} (r\cos\theta)^2 r\, dz = r^3 \cos^2\theta \int_0^{2r} dz = 2r^4 \cos^2\theta \quad .$$

The middle integral is

$$\int_0^1 \left[\int_{-2r}^{2r} (r\cos\theta)^2 r\, dz \right] dr$$

$$= \int_0^1 2r^4 \cos^2\theta\, dr = \cos^2\theta \int_0^1 2r^4\, dr = \frac{2}{5}\cos^2\theta \quad .$$

The outer integral is

$$\int_0^{2\pi} \left[\int_0^1 \int_{-2r}^{2r} (r\cos\theta)^2 r\, dz\, dr \right] d\theta$$

$$= \int_0^{2\pi} \frac{2}{5}\cos^2\theta \quad d\theta$$

$$= \frac{2}{5} \int_0^{2\pi} \frac{1+\cos 2\theta}{2} \quad d\theta$$

$$= \frac{2}{5} \cdot [\frac{\theta + (1/2)\sin 2\theta}{2}] \Big|_0^{2\pi} = \frac{2\pi}{5} \quad .$$

Ans.: $\frac{2\pi}{5}$.

Problem Type 15.8b: Evaluate

$$\int\int\int_E F(x, y, z)\, dV \quad ,$$

where E is bounded by the xz-plane and the hemispheres $y = \sqrt{r_1^2 - x^2 - z^2}$ and $y = \sqrt{r_2^2 - x^2 - z^2}$.

Example Problem 15.8b: Evaluate

$$\int\int\int_E x^2\, dV \quad ,$$

where E is bounded by the xz-plane and the hemispheres $y = \sqrt{1 - x^2 - z^2}$ and $y = \sqrt{4 - x^2 - z^2}$.

Steps

1. This is best handled with spherical coordinates. The hemispere $y = \sqrt{R^2 - x^2 - z^2}$ is half of the sphere $x^2 + y^2 + z^2 = R^2$ whose equation in spherical coordinates is really simple: $\rho = R$. Since $y > 0$ (the square-root is always positive), the range of θ is between 0 and π. ρ is between r_1 and r_2 and ϕ has its full range 0 to π. So

$$E = \{\, (\rho, \theta, \phi)\, |r_1 \le \rho \le r_2\, ,\, 0 \le \theta \le \pi\, ,\, 0 \le \phi \le \pi\, \} \quad .$$

2. Using the 'dictionary'

$$x = \rho \sin\phi \cos\theta\, ,\; y = \rho \sin\phi \sin\theta\, ,$$

$$z = \rho\cos\phi \quad ; \quad dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \quad .$$

Convert the volume integral into an iterated spherical integral (using the description of E in step 1).

$$\int_0^\pi \int_0^\pi \int_{r_1}^{r_2} F(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \quad .$$

Example

1. Here the radii are 1 and 2 so

$$E = \{\, (\rho, \theta, \phi)\, |1 \le \rho \le 2\, ,\, 0 \le \theta \le \pi\, ,\, 0 \le \phi \le \pi\, \} \quad .$$

2.

$$\int_0^\pi \int_0^\pi \int_1^2 (\rho \sin\phi\cos\theta)^2 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^\pi \int_0^\pi \int_1^2 \rho^4 \sin^3\phi \cos^2\theta\, d\rho\, d\theta\, d\phi \quad .$$

3. If you are lucky and all the limits of integration are numbers (do not involve ρ, ϕ or θ) and the integrand is a product of functions of a single variable, then the iterated integral is simply a product of three simple integrals. Express the big integral like that, and evaluate each single integral separately. Then multiply them together.

Warning: This is only possible if *all* the limits of integration are numbers and the integrand is *completely* separable as a product of functions of a single variable.

3.

$$\int_0^\pi \int_0^\pi \int_1^2 \rho^4 \sin^3\phi \cos^2\theta \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \sin^3\phi \, d\phi \int_0^\pi \cos^2\theta \, d\theta \int_1^2 \rho^4 \, d\rho \quad .$$

The first integral is ($u = \cos\phi$)

$$\int_0^\pi \sin^3\phi \, d\phi = \int_0^\pi \sin^2\phi \, d(-\cos\phi) =$$

$$\int_0^\pi (1-\cos^2\phi) \, d(-\cos\phi) = -\int_1^{-1} (1-u^2) \, du =$$

$$-u + \frac{u^3}{3}\Big|_1^{-1} = \frac{4}{3} \quad .$$

The second integral is

$$\int_0^\pi \cos^2\theta \, d\theta = \int_0^\pi \frac{1+\cos 2\theta}{2} \, d\theta =$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^\pi = \frac{\pi}{2} \quad .$$

The third integral is

$$\int_1^2 \rho^4 \, d\rho = \frac{\rho^5}{5}\Big|_1^2 = \frac{2^5 - 1^5}{5} = \frac{31}{5} \quad .$$

Multipluing these three single-integrals, we get that the original triple integral equals

$$\frac{4}{3} \cdot \frac{\pi}{2} \cdot \frac{31}{5} = \frac{62\pi}{15} \quad .$$

Ans.: $\frac{62\pi}{15}$.