Dr. Z's Math251 Handout #15.7 [Triple Integrals]

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Problem Type 15.7a: Evaluate the iterated integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\, dz\, dy\, dx \quad .$$

Example Problem 15.7a: Evaluate the iterated integral

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz\, dz\, dy\, dx \quad .$$

Steps

1. You go from **inside to outside**. First isolate and compute the **inner** integral

$$\int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\, dz \quad ,$$

which in this problem is a z-integral, since in $dz\, dy\, dx$, dz comes **first**. The answer should not have z in it, but in general depends on x and y.

Example

1.

$$\int_0^y 2xyz\, dz = xyz^2 \Big|_0^y = xy \cdot y^2 - 0 = xy^3 \quad .$$

2. Compute the **middle integral**

$$\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \quad ,$$

using what you already know from Step 1. This should be an expression in x.

2.

$$\int_x^{2x} \int_0^y 2xyz\,dz\,dy$$

$$= \int_x^{2x} \left[\int_0^y 2xyz\,dz \right] dy$$

from step 1, this equals:

$$= \int_x^{2x} xy^3\,dy \quad .$$

This integral equals:

$$x\frac{y^4}{4}\Big|_x^{2x} = x\frac{(2x)^4}{4} - x\frac{(x)^4}{4} = \frac{15}{4}x^5 \quad .$$

3. Do the outside integral, by viewing it as

$$\int_a^b \left[\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \right] dx \quad .$$

and using the result from step 2. The final answer should not depend on anything involving x, y, z (i.e. is a pure number or a constant).

3.

$$\int_0^1 \int_x^{2x} \int_0^y 2xyz\,dz\,dy\,dx$$

$$= \int_0^1 \left[\int_x^{2x} \int_0^y 2xyz\,dz\,dy \right] dx \quad .$$

By step 2, this is

$$= \int_0^1 \frac{15}{4}x^5\,dx \quad .$$

This is a Calc I integral

$$= \frac{15}{4}\int_0^1 x^5\,dx \quad = \frac{15}{4}\frac{x^6}{6}\Big|_0^1 = \frac{5}{8} \quad .$$

Ans.: $\frac{5}{8}$.

Problem Type 15.7b: Evaluate the triple integral

$$\int\int\int_E F(x,y,z)\,dV \quad ,$$

where

$$E = \{(x,y,z) \,|\, a \le x \le b\,,\; f_1(x) \le y \le f_2(x)\,,\; g_1(x,y) \le z \le g_2(x,y)\,\} \quad .$$

Example Problem 15.7b: Evaluate the triple integral

$$\int\int\int_E yz\cos(x^5)\,dV \quad ,$$

where

$$E = \{(x,y,z)\,|\,0 \le x \le 1\,,\,0 \le y \le x\,,\,x \le z \le 2x\,\} \quad .$$

Steps

1. Convert the volume integral into a triple iterated interal by transcribing the limits from left to right.

$$\int\int\int_E F(x,y,z)dV$$

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad .$$

2. Compute the iterated integral like we did in 15.7a. First the inner intgeral, then the middle, then the outer.

Example

1. Our volume integral equals the iterated integral

$$\int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx \quad .$$

2. The inner integral is:

$$\int_x^{2x} yz\cos(x^5)\,dz = y\cos(x^5)\int_x^{2x} z\,dz$$

$$= y\cos(x^5)\frac{z^2}{2}\Big|_x^{2x} = y\cos(x^5)\frac{(2x)^2 - x^2}{2}$$

$$= \frac{3}{2}yx^2\cos(x^5) \quad .$$

3. Compute the middle integral

$$\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \quad ,$$

by writing it as

$$\int_{f_1(x)}^{f_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz \right] dy \quad ,$$

and using the result from the previous step.

3.

$$\int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy = \int_0^x \left[\int_x^{2x} yz\cos(x^5)\,dz \right] dy$$

By step 2, this equals

$$\int_0^x \frac{3}{2} yx^2 \cos(x^5)\,dy = \frac{3}{2} x^2 \cos(x^5) \int_0^x y\,dy$$

$$= \frac{3}{2} x^2 \cos(x^5) \frac{y^2}{2} \Big|_0^x$$

$$= \frac{3}{2} x^2 \cos(x^5) \frac{x^2 - 0^2}{2} = \frac{3}{4} x^4 \cos(x^5) \quad .$$

4. Compute the outside integral

$$\int_a^b \int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad ,$$

by writing it as

$$\int_a^b \left[\int_{f_1(x)}^{f_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} F(x,y,z)\,dz\,dy \right] dx \quad ,$$

and using the result from the previous step.

4.

$$\int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy\,dx$$

$$= \int_0^1 \left[\int_0^x \int_x^{2x} yz\cos(x^5)\,dz\,dy \right] dx \quad .$$

By step 3, this equals

$$= \int_0^1 \frac{3}{4} x^4 \cos(x^5)\,dx = \frac{3}{4} \int_0^1 x^4 \cos(x^5)\,dx \quad .$$

Making the u-substitution $u = x^5$, gives that this equals

$$= \frac{3}{4} \cdot \frac{1}{5} \sin(x^5) \Big|_0^1 = \frac{3}{20} (\sin(1) - \sin(0)) = \frac{3\sin(1)}{20} \quad .$$

Ans.: $\frac{3\sin(1)}{20}$.

Problem Type 15.7c: Evaluate the triple integral

$$\int\int\int_E F(x,y,z)\,dV \quad ,$$

where E is bounded by the surfaces $z = g_1(x,y)$ $z = g_2(x,y)$ and possibly some planes of the form $x = a$.

Example Problem 15.7c: Evaluate the triple integral

$$\int\int\int_E x^2 y^2\,dV \quad ,$$

where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0$, $x = 1$ and $x = -1$.

Steps

1. Express the solid region E in the form

$$E = \{(x, y, z)|(x, y) \in D\,,$$
$$u_1(x, y) \le z \le u_2(x, y)\,\} \quad ,$$

where D is the projection of E on the xy-plane. To find D, set the two surfaces equal to each other i.e. solve $g_1(x, y) = g_2(x, y)$ and add to it the other restictions only involving x and/or y. After you figure out D, express E in the form

$$E = \{(x, y, z)|a \le x \le b\,, g_1(x) \le y \le g_2(x)\,,$$
$$u_1(x, y) \le z \le u_2(x, y)\,\} \quad .$$

Example

1. To get the projection D solve $1 - y^2 = 0$, which gives $y = -1$ and $y = 1$. Together with $x = 1$ and $x = -1$, the region D is

$$D = \{\,(x, y)\,|-1 \le x \le 1\,, -1 \le y \le 1\,\} \quad .$$

And so

$$E = \{(x, y, z)|(x, y) \in D\,,\, 0 \le z \le 1 - y^2\,\} \quad ,$$

and finally (combining it with D), we get

$$E = \{(x, y, z)|-1 \le x \le 1\,,\, -1 \le y \le 1\,,\, 0 \le z \le 1 - y^2\,\} \quad .$$

2. Convert the triple integral into an iterated integral.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} F(x,y,z)\,dz\,dy\,dx \quad .$$

Now evaluate that iterated integral by going from the inside to the outside.

2.

$$\int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1-y^2} x^2y^2\,dz\,dy\,dx \quad .$$

The inner integral is:

$$\int_0^{1-y^2} x^2y^2\,dz = x^2y^2 \int_0^{1-y^2} dz = x^2y^2(1-y^2) = x^2(y^2-y^4)$$

The middle integral is:

$$\int_{-1}^{1} \int_0^{1-y^2} x^2y^2\,dz\,dy = \int_{-1}^{1} \left[\int_0^{1-y^2} x^2y^2\,dz \right] dy$$

$$= \int_{-1}^{1} x^2(y^2-y^4)\,dy = x^2\left(\frac{y^3}{3}-\frac{y^5}{5}\right)\Big|_{-1}^{1} = \frac{4}{15}x^2 \quad .$$

The outer integral (i.e. the whole thing) is

$$\int_{-1}^{1} \int_{-1}^{1} \int_{0}^{1-y^2} x^2y^2\,dz\,dy\,dx$$

$$= \int_{-1}^{1} \left[\int_{-1}^{1} \int_0^{1-y^2} x^2y^2\,dz\,dy \right] dx \quad .$$

By the previous result, this is:

$$= \int_{-1}^{1} \frac{4}{15}x^2\,dx \quad = \frac{4}{15}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{4}{15}\frac{2}{3} = \frac{8}{45} \quad .$$

Ans.: $\frac{8}{45}$.