Dr. Z's Math251 Handout #15.6 [Surface Area]

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Problem Type 15.6a:Find the area of the surface $z = F(x, y)$ that lies above a given region.

Example Problem 15.6a: Find the area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$,$(0, 1)$,$(2, 1)$.

Steps

1. Write the region either as a type I region

$$D = \{(x, y) \,|\, a \leq x \leq b \,, f_1(x) \leq y \leq f_2(x)\,\} \ ,$$

or type II region

$$D = \{(x, y) \,|\, c \leq y \leq d \,, g_1(y) \leq x \leq g_2(y)\,\} \ ,$$

whatever is convenient.

2. Write the surface as $z = F(x, y)$ (if not already in that form) and set-up the integral for the surface area

$$\int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad .$$

Compute $\frac{\partial z}{\partial x} = F_x(x, y)$ and $\frac{\partial z}{\partial y} = F_y(x, y)$ and plug them in. Then use the description of D to convert it into an iterated integral.

Example

1. The triangle with vertices $(0, 0)$,$(0, 1)$,$(2, 1)$ is most conveniently written as a type II region

$$D = \{(x, y) \,|\, 0 \leq y \leq 1 \,, 0 \leq x \leq 2y\,\} \quad .$$

2. In this problem, $F(x, y) = 1+3x+2y^2$. So $F_x = 3$ and $F_y = 4y$ and the integral is

$$\int\int_D \sqrt{1 + 3^2 + (4y)^2}\, dA$$

$$= \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy \quad .$$

3. Evaluate that iterated integral by first doing the inner integral and then the outer integral.

3. The inner integral is

$$\int_0^{2y} \sqrt{10+16y^2}\,dx = \sqrt{10+16y^2}\int_0^{2y} dx$$

$$= 2y\sqrt{10+16y^2} \quad ,$$

and the outer integral is

$$\int_0^1 \int_0^{2y} \sqrt{10+16y^2}\,dx\,dy$$

$$= \int_0^1 \left[\int_0^{2y} \sqrt{10+16y^2}\,dx\right] dy$$

$$= \int_0^1 2y(10+16y^2)^{1/2}\,dy = \frac{1}{16}\frac{(10+16y^2)^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{1}{24}\left[(10+16\cdot 1^2)^{3/2} - (10+16\cdot 0^2)^{3/2}\right]$$

$$= \frac{26^{3/2}-10^{3/2}}{24} \quad .$$

Ans.: $\frac{26^{3/2}-10^{3/2}}{24}$.