Dr. Z's Math251 Handout #15.4 [Double Integrals in Polar Coordinates]

By Doron Zeilberger

Problem Type 15.4a: Evaluate the integral

$$\int\int_D F(x,y)\,dA \quad ,$$

where D is a region best described in polar coordinates,

$$D = \{\,(r,\theta)\,|\,\alpha \le \theta \le \beta\,,\,h_1(\theta) \le r \le h_2(\theta)\,\} \quad .$$

Example Problem 15.4a: Evaluate the integral

$$\int\int_D e^{-x^2-y^2}\,dA \quad ,$$

where D is the region bounded by the semi-circle $x = \sqrt{25-y^2}$ and the y-axis.

Steps

1. Draw the region and express it, if possible and convenient, as

$$D =$$

$$\{\,(r,\theta)\,|\,\alpha \le \theta \le \beta\,,\,h_1(\theta) \le r \le h_2(\theta)\,\} \quad .$$

Of course, in many problems, the $h_1(\theta)$ and/or $h_2(\theta)$ may be plain numbers (i.e. not involve θ).

Example

1. This is a **semi**-circle, i.e. **half** a circle, center origin, radius 5, and since it is bounded by the y-axis, and $x \ge 0$, it is the **right** half

[Had it been $x = -\sqrt{25-y^2}$ it would have been the left-half. Had it been $y = \sqrt{25-x^2}$ it would have been the upper-half. Had it been $y = -\sqrt{25-x^2}$ it would have been the lower-half.]

Since it is the right-half, θ ranges from $\theta = -\pi/2$ (the downwards direction) to $\theta = \pi/2$ (the upwards direction). For each ray $\theta = \theta_0$, r, the distance from the origin, ranges from $r = 0$ to $r = 5$ (and indeed does not depend on θ in this problem). So our region phrased in **polar coordinates** is:

$$D = \{\,(r,\theta)\,|\,-\pi/2 \le \theta \le \pi/2\,,\,0 \le r \le 5\,\} \quad .$$

2. Rewrite the area integral

$$\int\int_D F(x,y)\,dA \quad ,$$

in **polar** coordinates by replacing

x by $r\cos\theta$, y by $r\sin\theta$, dA by $r\,dr\,d\theta$.

[**shortcut:** Whenever you see x^2+y^2 you can replace it by r^2.]

Write it as an iterated integral

$$\int\int_D F(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} F(r\cos\theta\,,\, r\sin\theta)\, r\,dr\,d\theta \quad ,$$

with the θ-integral being at the **outside** and the r-integral being in the **inside**.

2.

$$\int\int_D e^{-x^2-y^2}\,dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2}\, rdr\,d\theta \quad .$$

3. Evaluate this iterated integral by first doing the inner-integral (possibly getting an expression in θ, or just a number), and then the outer integral.

3. The inside integral is (do the change-of-variable $u=-r^2$):

$$\int_0^5 e^{-r^2}\, rdr \;=\; (-1/2)e^{-r^2}\Big|_0^5 = (1-e^{-25})/2 \quad ,$$

and the whole double-integral is

$$\int_{-\pi/2}^{\pi/2} \int_0^5 e^{-r^2}\, rdr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\int_0^5 e^{-r^2}\, rdr\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} [(1-e^{-25})/2]\;\; d\theta = (1-e^{-25})/2 \int_{-\pi/2}^{\pi/2} d\theta =$$

$$[(1-e^{-25})/2][\pi/2-(-\pi/2)] = \pi(1-e^{-25})/2 \quad .$$

Ans.: $\pi(1-e^{-25})/2$.

Problem Type 15.4b: Find the volume of the solid above the surface $z = f(x, y)$ and below the surface $z = g(x, y)$.

Example Problem 15.4b: Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$.

Steps

1. Find the "floor", let's call it D, by setting $f(x, y) = g(x, y)$ (or if convenient already convert to polar coordinates).

Example

1. In polar coordinates, the two surfaces are $z = r$ and $z = \sqrt{2 - r^2}$. Setting them equal gives $r = \sqrt{2 - r^2}$. Squaring both sides gives $r^2 = 2 - r^2$, which gives $2r^2 = 2$, which gives $r^2 = 1$ and so $r = \pm 1$. But r is never negative, so $r = -1$ is nonsense. Hence the "floor", D, is the region bounded by the circle $r = 1$, or, if you wish, the disk $r \leq 1$.

So

$$D = \{(r, \theta) | 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\} \quad .$$

2. The volume is the area integral of TOP-BOTTOM

$$\int\int_D [f(x, y) - g(x, y)] \, dA \quad .$$

Set it up. Then convert it to polar-coordinates.

2. The bottom is $z = \sqrt{x^2 + y^2}$, and in polar $z = r$, and the top is $x^2 + y^2 + z^2 = 2$ which is $z = \sqrt{2 - x^2 - y^2}$ and in polar $z = \sqrt{2 - r^2}$. So the volume in polar coordinates is

$$\int_0^{2\pi} \int_0^1 [\sqrt{2 - r^2} - r] \, r \, dr \, d\theta \quad .$$

3. Evaluate the iterated integral. First do the inner integral (w.r.t. to r) getting an expression in θ (or just a number), and then do the outer integral.

3. The inner integral is

$$\int_0^1 [\sqrt{2-r^2} - r]\, r\, dr = \int_0^1 [r\sqrt{2-r^2} - r^2]\, dr$$

$$= \int_0^1 r(2-r^2)^{1/2}\, dr - \int_0^1 r^2\, dr$$

$$= -(1/3)(2-r^2)^{3/2}\Big|_0^1 - r^3/3\Big|_0^1$$

$$= -(1/3)(2-r^2)^{3/2}\Big|_0^1 - r^3/3\Big|_0^1$$

$$= -(1/3)[(2-1^2)^{3/2} - (2-0^2)^{3/2}] - 1/3$$

$$= [2^{3/2} - 2]/3 = (2\sqrt{2}-1)/3 \quad .$$

The whole integral is thus:

$$\int_0^{2\pi} \int_0^1 [\sqrt{2-r^2} - r]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\int_0^1 [\sqrt{2-r^2} - r]\, r\, dr \right] d\theta$$

$$= \int_0^{2\pi} (2\sqrt{2}-1)/3\, d\theta$$

$$= 2\pi(2\sqrt{2}-1)/3 \quad .$$

Ans.: The volume is $2\pi(2\sqrt{2}-1)/3$.

Problem Type 15.4c: Evaluate the iterated integral by converting to polar coordinates.

$$\int_a^b \int_{f_1(y)}^{f_2(y)} F(x,y)\, dx\, dy$$

Example Problem 15.4c: Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y\, dx\, dy$$

Steps

Example

1. By looking at the limits of integration of the outer and inner integral signs, figure out the region D.

$$D = \{(x, y) \,|\, a \le y \le b\,,\ f_1(y) \le x \le f_2(y)\,\} \quad .$$

Draw this region, and express it in polar coordinates

$$D = \{(r, \theta) \,|\, \alpha \le \theta \le \beta\,,\ g_1(\theta) \le r \le g_2(\theta)\,\}$$

1. Our region is:

$$D = \{(x, y) \,|\, 0 \le y \le 3\,,\ -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}\,\} \quad .$$

Drawing it (do it!), we see that this is the upper-half of the circle whose center is the origin and whose radius is 3. In polar coordiantes it is:

$$D = \{(r, \theta) \,|\, 0 \le \theta \le \pi\,,\ 0 \le r \le 3\,\} \quad .$$

2. Write the iterated integral as an area integral, then convert it to an iterated integral in polar coordinates. Use the "disctionary" $x = r\cos\theta$ $y = r\sin\theta$ $dx\,dy = r\,dr\,d\theta$.

2.

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2 y\,dx\,dy$$

$$= \int_0^{\pi} \int_0^3 (r\cos\theta)^2 (r\sin\theta) r\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^3 r^4 \sin\theta \cos^2\theta\,dr\,d\theta \quad .$$

3. Evaluate that iterated integral by doing the inner integral first, and then the outer integral.

3. The inner integral is

$$\int_0^3 r^4 \sin\theta \cos^2\theta \, dr = \sin\theta\cos^2\theta \int_0^3 r^4 \, dr$$

$$= \sin\theta\cos^2\theta \left[\frac{r^5}{5} \Big|_0^3 \right]$$

$$= \frac{243}{5} \sin\theta\cos^2\theta \quad .$$

The outer integral is:

$$\int_0^\pi \int_0^3 r^4 \sin\theta\cos^2\theta \, dr \, d\theta$$

$$= \int_0^\pi \left[\int_0^3 r^4 \sin\theta\cos^2\theta \, dr \right] d\theta$$

$$= \int_0^\pi \frac{243}{5} \cos^2\theta \sin\theta \, d\theta = \frac{243}{5} \int_0^\pi \cos^2\theta\sin\theta \, d\theta$$

$$= \frac{243}{5} \cdot \left[\frac{-\cos^3\theta}{3} \Big|_0^\pi \right] = \frac{81}{5} \cdot \left(-\cos^3(\pi) - -\cos^3(0) \right) = \frac{162}{5} \quad .$$

Ans.: $\frac{162}{5}$.