Dr. Z's Math251 Handout #15.2 [Iterated Integrals]

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Problem Type 15.2a: Calculate the iterated integral

$$\int_a^b \int_c^d f(x,y)\,dx\,dy \quad .$$

Example Problem 15.2a: Calculate the iterated integral

$$\int_1^4 \int_0^2 (x+\sqrt{y})\,dx\,dy \quad .$$

Steps

1. First evaluate the **inner integral**.

$$\int_c^d f(x,y)\,dx \quad .$$

Since it is an x-integral, integrate with respect to x, and treat y as a constant. The answer to this step should be an expression in y. Of course, once in awhile it could be just a number.

2. Do the **outer integral** by integrating w.r.t. y the expression that you got in step 1.

Example

1. The inner integral is

$$\int_0^2 (x+\sqrt{y})\,dx$$

The anti-derivative is $x^2/2 + x\sqrt{y}$, so

$$\int_0^2 (x+\sqrt{y})\,dx = \frac{x^2}{2} + x\sqrt{y} \quad \Big|_0^2$$

$$= \frac{2^2-0^2}{2} + (2-0)\sqrt{y} = 2 + 2y^{1/2} \quad .$$

2.

$$\int_1^4 \int_0^2 (x+\sqrt{y})\,dx\,dy$$

$$= \int_1^4 \left[\int_0^2 (x+\sqrt{y})\,dx\right]\,dy$$

$$= \int_1^4 [2+2y^{1/2}]\,dy = 2y + 2\frac{y^{3/2}}{3/2}\Big|_1^4$$

$$= 2y + \frac{4}{3}(\sqrt{y})^3\Big|_1^4 = 2\cdot(4-1) + \frac{4}{3}[(\sqrt{4})^3 - (\sqrt{1})^3]$$

$$= 6 + \frac{4}{3}\cdot 7 = \frac{46}{3} \quad .$$

Ans.: 46/3.

Problem Type 15.2b: Calculate the double integral

$$\int\int_R f(x,y)\,dA \quad ,$$

$$R = \{(x,y)\,|\,a \le x \le b\,,\; c \le y \le d\,\} \quad .$$

Example Problem 15.2b: Calculate the double integral

$$\int\int_R \frac{xy^2}{x^2+1}\,dA \quad ,$$

$$R = \{(x,y)\,|\,0 \le x \le 1\,,\; -3 \le y \le 3\,\} \quad .$$

Steps

1. Convert it to an iterated integral:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \quad ,$$

or, if you wish

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \quad .$$

Both are correct.

Example

1.

$$\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx \quad .$$

2. Evaluate the iterated intergal like in 15.2a .

2.
$$\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\,dx$$
$$= \int_0^1 \left[\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy\right]\,dx \quad .$$
The inner integral is:
$$\int_{-3}^3 \frac{xy^2}{x^2+1}\,dy = \frac{x}{x^2+1}\int_{-3}^3 y^2\,dy = \left(\frac{x}{x^2+1}\right)\frac{y^3}{3}\Big|_{-3}^3$$
$$= \left(\frac{x}{x^2+1}\right)18 = \frac{18x}{x^2+1} \quad .$$
Hence the iterated integral is:
$$\int_0^1 \frac{18x}{(x^2+1)}\,dx = 9\ln(x^2+1)\Big|_0^1$$
$$= 9[\ln(1^2+1) - \ln(0^2+1)] = 9\ln 2 \quad .$$
Ans.: $9\ln 2$.

Problem Type 15.2c: Find the volume of the solid that lies under the plane $ax+by+cz=d$ and above the rectangle
$$R = \{\,(x,y)\,|A_1 \le x \le A_2\,,\, B_1 \le y \le B_2\} \quad .$$

Example Problem 15.2c: Find the volume of the solid that lies under the plane $2x+3y+z=10$ and above the rectangle
$$R = \{\,(x,y)\,|0 \le x \le 2\,,\, 0 \le y \le 1\} \quad .$$

Steps

1. Solve for z putting it in the form $z = f(x,y)$. Technically, you would have to check that $f(x,y)$ is always positive above the given region R but you can trust the problem. Set up the integral
$$\int_R f(x,y)\,dA \quad .$$

Example

1. Solving for z in $2x+3y+z=10$ gives $z = 10-2x-3y$, so $f(x,y) = 10-2x-3y$ and the desired volume is
$$\int_R (10-2x-3y)\,dA \quad .$$

2. Convert the area-integral to an iterated integral either way.

2.

$$\int_R (10-2x-3y)\,dA = \int_0^1 \int_0^2 (10-2x-3y)\,dx\,dy \quad .$$

Or, if you wish

$$\int_R (10-2x-3y)\,dA = \int_0^2 \int_0^1 (10-2x-3y)\,dy\,dx \quad .$$

3. Compute (one of those) iterated integral(s), like we did in 15.2a.

3.

$$\int_0^1 \int_0^2 (10-2x-3y)\,dx\,dy = \int_0^1 \left[\int_0^2 (10-2x-3y)\,dx \right] dy$$

$$= \int_0^1 \left[10x - x^2 - 3xy \Big|_0^2 \right] dy = \int_0^1 \left[10 \cdot 2 - 2^2 - 3 \cdot 2y - 0 \right] dy$$

$$= \int_0^1 [16 - 6y]\,dy$$

$$= 16y - 3y^2 \Big|_0^1 = 16 - 3 = 13 \quad .$$

Ans.: The volume is 13.

Problem Type 15.2d: Find the volume of the solid in the first octant bounded by the cylinder $z = r^2 - x^2$ and the plane $y = a$.

Example Problem 15.2d: Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 3$.

Steps

Example

1. Unlike the previous problem, *now* we also need to figure out the "floor plan" R. If the surface is $z = f(x, y)$ then set $f(x, y) = 0$ getting $x = \pm r$ and remember that *positive* octant means $x \geq 0$, $y \geq 0$. It follows that

$$R = \{(x, y) | 0 \leq x \leq r, 0 \leq y \leq a\} \quad .$$

1. Setting $z = 0$ gives $16 - x^2 = 0$ so $x = \pm 4$ but in the *positive orthant* we have $x \geq 0$ so the boundary of R are $x = 0, x = 4, y = 0, y = 3$. So the boundary is:

$$R = \{(x, y) | 0 \leq x \leq 4, 0 \leq y \leq 3\} \quad .$$

2. Set up the integral

$$\int_R f(x, y)\, dA \quad ,$$

and convert it to an iterated integral.

2.

$$\int_R (16-x^2)\, dA = \int_0^3 \int_0^4 (16-x^2)\, dx\, dy \quad .$$

3. Evaluate this iterated integral.

3.

$$\int_0^3 \int_0^4 (16-x^2)\, dx\, dy = \int_0^3 \left[\int_0^4 (16 - x^2)\, dx \right] dy$$

$$= \int_0^3 \left[16x - \frac{x^3}{3} \Big|_0^4 \right] dy = \int_0^3 \frac{128}{3}\, dy$$

$$= \frac{128}{3} y \Big|_0^3 = 128 \quad .$$

Ans.: The volume is 128.