**Dr. Z's Math251 Handout #14.8 [Lagrange Multipliers]**

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**Problem Type 14.8a**: Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given conditions.

$$f(x,y,z) = Expression_(x,y,z) \quad ; \quad g(x,y,z) = k \quad .$$

**Example Problem 14.8a**: Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given conditions.

$$f(x,y,z) = xyz \quad ; \quad 3x^2 + 2y^2 + z^2 = 6 \quad .$$

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**Steps**

**1.** Find the gradients of $f(x,y,z)$ and $g(x,y,z)$.

**2.** Introduce another variable, $\lambda$, and set up the equations implied by

$$\nabla f = \lambda \nabla g \quad .$$

To them add, the equation $g(x,y,z) = k$.

**Example**

**1.** $f_x = \frac{\partial}{\partial x} xyz = yz$, $f_y = \frac{\partial}{\partial y} xyz = xz$, $f_z = \frac{\partial}{\partial z} xyz = xy$. So,

$$\nabla f = \langle yz, xz, xy \rangle \quad .$$

$g_x = \frac{\partial}{\partial x}(3x^2 + 2y^2 + z^2) = 6x$, $g_y = \frac{\partial}{\partial y}(3x^2 + 2y^2 + z^2) = 4y$, $g_z = \frac{\partial}{\partial z}(3x^2 + 2y^2 + z^2) = 2z$. So

$$\nabla g = \langle 6x, 4y, 2z \rangle \quad .$$

**2.** $\nabla f = \lambda \nabla g$ means

$$\langle yz, xz, xy \rangle = \lambda \langle 6x, 4y, 2z \rangle \quad ,$$

that spells-out to the set of equations

$$yz = 6\lambda x \quad , \quad xz = 4\lambda y \quad ,$$

$$xy = 2\lambda z \quad , \quad 3x^2 + 2y^2 + z^2 = 6 \quad .$$

**3.** Use algebra to solve the system of four equations and four unknowns.

**3.** Multiplying the first three equations, we get $(xyz)^2 = 48\lambda^3 xyz$ so $xyz = 48\lambda^3$, and hence $yz = 48\lambda^3/x$ and we get from the first equation $48\lambda^3/x = 6\lambda x$. This means $8\lambda^2 = x^2$ and so $x = \sqrt{8}\lambda$.

From $xyz = 48\lambda^3$, we also get $xz = 48\lambda^3/y$ and we get from the second equation $48\lambda^3/y = 4\lambda y$. This means $12\lambda^2 = y^2$ and so $y = \sqrt{12}\lambda$.

From $xyz = 48\lambda^3$, we also get $xy = 48\lambda^3/z$ and we get from the third equation $48\lambda^3/z = 2\lambda z$. This means $24\lambda^2 = z^2$ and so $z = \sqrt{24}\lambda$.

Plugging these expressions in $\lambda$ into the last equation $3x^2 + 2y^2 + z^2 = 6$ we get

$$3(\sqrt{8}\lambda)^2 + 2(\sqrt{12}\lambda)^2 + (\sqrt{24}\lambda)^2 = 6 \quad .$$

So,

$$(3 \cdot 8 + 2 \cdot 12 + 24)\lambda^2 = 6 \quad ,$$

$$\lambda^2 = \frac{1}{12} \quad ,$$

and so

$$\lambda = \pm\frac{1}{\sqrt{12}} \quad .$$

We get **two** solutions. The first one is

$$\lambda = \frac{1}{\sqrt{12}} \quad , \quad x = \sqrt{8} \cdot \frac{1}{\sqrt{12}} = \sqrt{2/3} \quad ,$$

$$y = \sqrt{12} \cdot \frac{1}{\sqrt{12}} = 1 \quad , \quad z = \sqrt{24} \cdot \frac{1}{\sqrt{12}} = \sqrt{2} \quad ,$$

which means the point $(\sqrt{2/3}, 1, \sqrt{2})$.

And the second is

$$\lambda = \frac{-1}{\sqrt{12}} \quad , \quad x = \sqrt{8} \cdot \frac{-1}{\sqrt{12}} = -\sqrt{2/3} \quad ,$$

$$y = \sqrt{12} \cdot \frac{-1}{\sqrt{12}} = -1 \quad , z = \sqrt{24} \cdot \frac{-1}{\sqrt{12}} = -\sqrt{2} \quad ,$$

which means the point $(-\sqrt{2/3}, -1, -\sqrt{2})$.

**4.** Now you can forget about the $\lambda$ and plug-in these point(s) into $f$ and see who gives the largest value, that is the **maximum value** and who is the smallest, that is the **minimum value**.

**4.**

$$f(\sqrt{2/3}, 1, \sqrt{2}) = 2/\sqrt{3} \quad ,$$

$$f(-\sqrt{2/3}, -1, -\sqrt{2}) = -2/\sqrt{3} \quad .$$

**Ans.:** The maximum value is $2/\sqrt{3}$ and the minimum value is $-2/\sqrt{3}$.