

Dr. Z's Math251 Handout #14.5 [The Chain Rule]

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Problem Type 14.5a: Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$:

$$z = f(x, y) \quad , \quad x = h_1(s, t) \quad , \quad y = h_2(s, t) \quad .$$

Example Problem 14.5b: Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$:

$$z = x^3 + 2xy + y^2 \quad , \quad x = s + 2t \quad , \quad y = s^2t \quad .$$

Steps

1. Set-up the chain-rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad ,$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \quad .$$

2. For the specific functions given compute all the necessary quantities that show up on the right sides, namely: $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$, $\frac{\partial x}{\partial t}$, $\frac{\partial y}{\partial t}$.

3. Incorporate them into the formulas of step 1.

Example

1. Ditto.

2.

$$\frac{\partial z}{\partial x} = 3x^2 + 2y \quad , \quad \frac{\partial z}{\partial y} = 2x + 2y \quad .$$

$$\frac{\partial x}{\partial s} = 1 \quad , \quad \frac{\partial y}{\partial s} = 2st \quad .$$

$$\frac{\partial x}{\partial t} = 2 \quad , \quad \frac{\partial y}{\partial t} = s^2 \quad .$$

3.

$$\frac{\partial z}{\partial s} = (3x^2 + 2y) \cdot 1 + (2x + 2y)(2st) = 3x^2 + 2y + 4xst + 4yst \quad .$$

$$\frac{\partial z}{\partial t} = (3x^2 + 2y) \cdot 2 + (2x + 2y)s^2 = 6x^2 + 4y + 2xs^2 + 2ys^2 \quad .$$

Ans.: $\frac{\partial z}{\partial s} = 3x^2 + 2y + 4xst + 4yst$, $\frac{\partial z}{\partial t} = 6x^2 + 4y + 2xs^2 + 2ys^2$.

Note: It is acceptable to have x, y featuring in the answers as well as t, s . If you are asked to express your answers **only** in terms of the “basic” variables t, s , then you would have to do an extra step: substituting everywhere you see x or y their expressions in terms of t and s . So in that case, the answers would be

$$\frac{\partial z}{\partial s} = 3(s + 2t)^2 + 2s^2t + 4(s + 2t)st + 4(s^2t)st \quad ,$$

$$\frac{\partial z}{\partial t} = 6(s + 2t)^2 + 4s^2t + 2(s + 2t)s^2 + 2(s^2t)s^2 \quad .$$

Do not bother to expand (unless specifically asked to).