Dr. Z's Math251 Handout #13.3 [Arc Length and Curvature]

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Problem Type 13.3a: Find the length of the curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $t_0 \leq t \leq t_1$.

Example Problem 13.3a: Find the length of the curve $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}$, $1 \leq t \leq e$.

Steps

1. Compute the derivative

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} \quad .$$

2. Find the **magnitude** of $\mathbf{r}'(t)$,

$$|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \quad ,$$

and use algebra and/or trig to simplify as much as you can.

3. Integrate the expression that you got in step 2 from t_0 to t_1.

$$\int_{t_0}^{t_1} |\mathbf{r}'(t)|\, dt$$

Example

1.

$$\mathbf{r}'(t) = (t^2)'\,\mathbf{i} + (2t)'\,\mathbf{j} + (\ln t)'\,\mathbf{k}$$

$$= 2t\,\mathbf{i} + 2\,\mathbf{j} + \frac{1}{t}\,\mathbf{k} \quad .$$

2.

$$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 2^2 + \frac{1}{t^2}}$$

$$= \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{(2t^2+1)}{t} = 2t + \frac{1}{t} \quad .$$

3.

$$\int_1^e |\mathbf{r}'(t)|\, dt = \int_1^e [2t + \frac{1}{t}]\, dt == t^2 + \ln t\Big|_1^e =$$

$$e^2 + \ln e - (1^2 + \ln 1) = e^2 + 1 - (1+0) = e^2 \quad .$$

Ans.: The arc length of that curve is e^2.

Problem Type 13.3b: Reparametrize the curve with respect to arc length measured from the point when $t = t_0$ in the direction of increasing t.

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \quad .$$

Example Problem 13.3b: Reparametrize the curve with respect to arc length measured from

the point when $t = 0$ in the direction of increasing t.

$$\mathbf{r}(t) = 5\sin t\,\mathbf{i} + 3\,\mathbf{j} + 5\cos t\,\mathbf{k} \quad .$$

Steps

1. Compute $\mathbf{r}'(t)$, and then take its magnitude $|\mathbf{r}'(t)|$.

2. Integrate it from t_0 to t_1. Get an expression in terms of t_1 and call it s. Now change t_1 into t. Now solve for t in terms of s.

3. Go back to the original $\mathbf{r}(t)$ and replace t by the expression in s that you found in step 2.

Example

1.

$$\mathbf{r}'(t) = (5\sin t)'\,\mathbf{i} + 3'\,\mathbf{j} + (5\cos t)'\,\mathbf{k} \quad .$$

$$(5\cos t)\,\mathbf{i} - (5\sin t)\,\mathbf{k} \quad .$$

So

$$|\mathbf{r}'(t)| = \sqrt{(5\cos t)^2 + (-5\sin t)^2} = \sqrt{25(\cos^2 t + \sin^2 t)} = 5 \quad .$$

2.

$$s = \int_0^{t_1} 5\,dt = 5t_1 \quad .$$

Changing the t_1 into t we get

$$s = 5t$$

and expressing t in terms of s, we get

$$t = s/5 \quad .$$

3.

$$\mathbf{r}(t) = 5\sin t\,\mathbf{i} + 3\,\mathbf{j} + 5\cos t\,\mathbf{k} \quad .$$

becomes

$$5\sin(s/5)\,\mathbf{i} + 3\,\mathbf{j} + 5\cos(s/5)\,\mathbf{k} \quad .$$

This is the **Ans.**.

Problem Type 13.3c: Find the curvature for

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \quad .$$

Example Problem 13.3c: Find the curvature for

$$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k} \quad .$$

Steps	Example
1. Compute $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.	**1.** $\mathbf{r}'(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k} \quad .$ $\mathbf{r}''(t) = 2\,\mathbf{k} \quad .$
2. Compute the **cross product** $\mathbf{r}'(t) \times \mathbf{r}''(t)$.	**2.** $\mathbf{r}'(t) \times \mathbf{r}''(t)$ equals $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix} =$ $\mathbf{i}\begin{vmatrix} 2 & 2t \\ 0 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}$ $= 4\,\mathbf{i} - 2\,\mathbf{j}$
3. Find the magnitude of the vector that you found in step 2 (namely $\mathbf{r}'(t)\times\mathbf{r}''(t)$). Also find the magnitude of $\mathbf{r}'(t)$, and finally use the formula for the curvature $\kappa(t) = \dfrac{\lvert\mathbf{r}'(t) \times \mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert^3}$	**3.** $\lvert\mathbf{r}'(t) \times \mathbf{r}''(t)\rvert = \sqrt{4^2 + 2^2 + 0^2} = \sqrt{20} \quad .$ $\lvert\mathbf{r}'(t)\rvert = \lvert\mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}\rvert$ $= \sqrt{1^2 + 2^2 + (2t)^2} = \sqrt{5 + 4t^2} \quad .$ Finally, $\kappa(t) = \dfrac{\sqrt{20}}{(\sqrt{5 + 4t^2})^3}$ This is the **Ans.**.