Dr. Z's Math251 Handout #12.5 [Equations of Lines and Planes]

By Doron Zeilberger

Problem Type 12.5a: Find an equation of the plane that passes through three given points

Example Problem 12.5a: Find an equation of the plane that passes through the points $(1,1,1)$, $(2,0,1)$, $(2,1,0)$.

Steps

1. Calling these points P, Q, R find the vectors $\mathbf{PQ}$ and $\mathbf{PR}$ by doing $Q-P$ and $R-P$.

2. Find a vector normal to the plane by computing the cross-product $\mathbf{PQ} \times \mathbf{PR}$.

Example

1. $P=(1,1,1)$, $Q=(2,0,1)$, $R=(2,1,0)$,

$$\mathbf{PQ} = Q-P = \langle 2-1, 0-1, 1-1 \rangle = \langle 1,-1,0 \rangle \quad ,$$

$$\mathbf{PR} = R-P = \langle 2-1, 1-1, 0-1 \rangle = \langle 1,0,-1 \rangle \quad .$$

2.

$$\mathbf{PQ} \times \mathbf{PR} = \langle 1,-1,0 \rangle \times \langle 1,0,-1 \rangle =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} =$$

$$\mathbf{i} \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= \mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 1,1,1 \rangle \quad .$$

This is the **normal vector** $\mathbf{n} = \langle a,b,c \rangle$.
So $\mathbf{n} = \langle a,b,c \rangle = \langle 1,1,1 \rangle$

3. Pick any of the three points (it does not matter which) as the refence point (x_0, y_0, z_0) and use the formula

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad .$$

3. Picking P we get $(x_0, y_0, z_0) = (1, 1, 1)$, since $a = 1, b = 1, c = 1$, we get that an equation is

$$1 \cdot (x - 1) + 1 \cdot (y - 1) + 1 \cdot (z - 1) = 0$$

and expanding we get

$$x + y + z = 3 \quad .$$

Ans.: An equation for the plane passing through P, Q and R is $x + y + z = 3$.

Check: Plug-in all the three points into the equation and make sure that they agree.

Problem Type 12.5b: Find direction numbers for the line of intersections of the planes $a_1 x + b_1 y + c_1 z = d_1$ and $a_2 x + b_2 y + c_2 z = d_2$.

Example Problem 12.5b: Find direction numbers for the line of intersections of the planes $2x + 3y + 4z = 2$ and $-3x + 2y + 3z = 1$.

Steps

1. By looking at the coeffs. of x, y, z extract the **normal vectors** $\mathbf{n_1} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{n_2} = \langle a_2, b_2, c_2 \rangle$.

Note: the numbers on the right sides (d_1, d_2) are not needed.

2. Take the cross-product $\mathbf{n_1} \times \mathbf{n_2}$. The components are the **direction numbers** of the line of intersection.

Example

1. $\mathbf{n_1} = \langle 2, 3, 4 \rangle$ and $\mathbf{n_2} = \langle -3, 2, 3 \rangle$.

2.

$$\mathbf{n_1} \times \mathbf{n_2} = \langle 2, 3, 4 \rangle \times \langle -3, 2, 3 \rangle = \langle 1, -18, 13 \rangle \quad .$$

(You do it!)

Ans.: The direction numbers are $\langle 1, -18, 13 \rangle$.