

Dr. Z's Math251 Handout #12.4 [The Cross Product]

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Problem Type 12.4a: Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both \mathbf{a} and \mathbf{b} .

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \quad , \quad \mathbf{b} = \langle b_1, b_2, b_3 \rangle \quad .$$

Example Problem 12.4a: Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both \mathbf{a} and \mathbf{b} .

$$\mathbf{a} = \langle 4, 5, -3 \rangle \quad , \quad \mathbf{b} = \langle 2, 1, -3 \rangle \quad .$$

Steps

1. Form the **determinant** with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the first row, the vector \mathbf{a} in the second row and the vector \mathbf{b} in the third row.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad .$$

2. Evaluate the determinant step-by-step:

$$\begin{aligned} & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \\ & \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ & = \mathbf{i}(a_2 b_3 - b_2 a_3) - \mathbf{j}(a_1 b_3 - b_1 a_3) + \mathbf{k}(a_1 b_2 - b_1 a_2) \quad . \end{aligned}$$

Example

1.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -3 \\ 2 & 1 & -3 \end{vmatrix} \quad .$$

2.

$$\begin{aligned} & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 5 & -3 \\ 2 & 1 & -3 \end{vmatrix} = \\ & \mathbf{i} \begin{vmatrix} 5 & -3 \\ 1 & -3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & -3 \\ 2 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} \\ & = \mathbf{i}(5 \cdot (-3) - (1) \cdot (-3)) - \mathbf{j}(4 \cdot (-3) - (2) \cdot (-3)) + \mathbf{k}(4 \cdot (1) - (2) \cdot (5)) = \\ & \quad \quad \quad -12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} \end{aligned}$$

First Part of the Answer:

$$-12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} \text{ or } \langle -12, 6, -6 \rangle .$$

3. Take the dot product of this vector with **a** and **b** and verify that they are both 0. If you get something else, something went wrong.

$$\mathbf{3.} \quad \langle -12, 6, -6 \rangle \cdot \langle 4, 5, -3 \rangle = (-12)(4) + (6)(5) + (-6)(-3) = -48 + 30 + 18 = 0$$

and

$$\langle -12, 6, -6 \rangle \cdot \langle 2, 1, -3 \rangle = (-12)(2) + (6)(1) + (-6)(-3) = -24 + 6 + 18 = 0.$$

Second Part of Ans.: Both are zero as expected.

Problem Type 12.4b: (a) Find a vector orthogonal to the plane through the points $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$, $R = (r_1, r_2, r_3)$. (b) Find the area of the triangle PQR .

Example Problem 12.4b: (a) Find a vector orthogonal to the plane through the points $P = (2, 1, 5)$, $Q = (-1, 3, 4)$, $R = (3, 0, 6)$. (b) Find the area of the triangle PQR .

Steps

1. Compute the vector **PQ** by subtracting Q from P and the vector **PR** by subtracting R from P .

Example

1.

$$\mathbf{PQ} = (-1, 3, 4) - (2, 1, 5) = \langle -3, 2, -1 \rangle \quad ,$$

$$\mathbf{PR} = (3, 0, 6) - (2, 1, 5) = \langle 1, -1, 1 \rangle \quad .$$

2. Take the cross product of **PQ** and **PR**.

$$\begin{aligned} \mathbf{PQ} \times \mathbf{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} \quad . \\ &= \mathbf{i} \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} \\ &= \mathbf{i} + 2\mathbf{j} + \mathbf{k} = \langle 1, 2, 1 \rangle \quad . \end{aligned}$$

Ans. to First Part: A vector orthogonal to the plane of the triangle PQR is $\langle 1, 2, 1 \rangle$.

3. Take the magnitude of the vector you found in step 2. This is the **area of the parralelogram**. To get the **area of the triangle**, divide it by 2.

3.

$$|\langle 1, 2, 1 \rangle| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6} \quad .$$

So the area of the parallelogram formed by PQ and PR is $\sqrt{6}$ and the area of triangle PQR is half of that, $\sqrt{6}/2$.

Ans. to Second Part: The area of triangle PQR is $\sqrt{6}/2$.

Problem Type 12.4c: Find the volume of the parallelopiped with adjacent edges PQ, PR , and PS , where P, Q, R, S are given points.

Example Problem 12.4c: Find the volume of the parallelopiped with adjacent edges PQ, PR , and PS , where $P(0, 1, 2), Q(2, 4, 5), R(-1, 0, 1), S(6, -1, 4)$.

Steps

1. Compute the vectors **PQ, PR, PS** by the appropriate subtractions.

Example

1.

$$\mathbf{PQ} = Q - P = (2, 4, 5) - (0, 1, 2) = \langle 2, 3, 3 \rangle \quad ,$$

$$\mathbf{PR} = R - P = (-1, 0, 1) - (0, 1, 2) = \langle -1, -1, -1 \rangle \quad ,$$

$$\mathbf{PS} = S - P = (6, -1, 4) - (0, 1, 2) = \langle 6, -2, 2 \rangle \quad .$$

2. Compute the cross product $\mathbf{PQ} \times \mathbf{PR}$.

2.

$$\begin{aligned}\mathbf{PQ} \times \mathbf{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 3 \\ -1 & -1 & -1 \end{vmatrix} . \\ &= \mathbf{i} \begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} \\ &= 0 \cdot \mathbf{i} - (1) \cdot \mathbf{j} + (1) \cdot \mathbf{k} = -\mathbf{j} + \mathbf{k} = \langle 0, -1, 1 \rangle\end{aligned}$$

3. Compute the dot product of what you found in step 2 with the vector \mathbf{PS} .

3.

$$\begin{aligned}(\mathbf{PQ} \times \mathbf{PR}) \cdot \mathbf{PS} &= \langle 0, -1, 1 \rangle \cdot \langle 6, -2, 2 \rangle = \\ &= (0) \cdot (6) + (-1) \cdot (-2) + (1) \cdot (2) = 4 .\end{aligned}$$

Ans.: The volume of the parallelepiped is 4.