Combinatorics and Algebras From A to Z, Monday July 26, 2021

Growth rates of grids and merges of permutation classes

Vince Vatter (U Florida)

with
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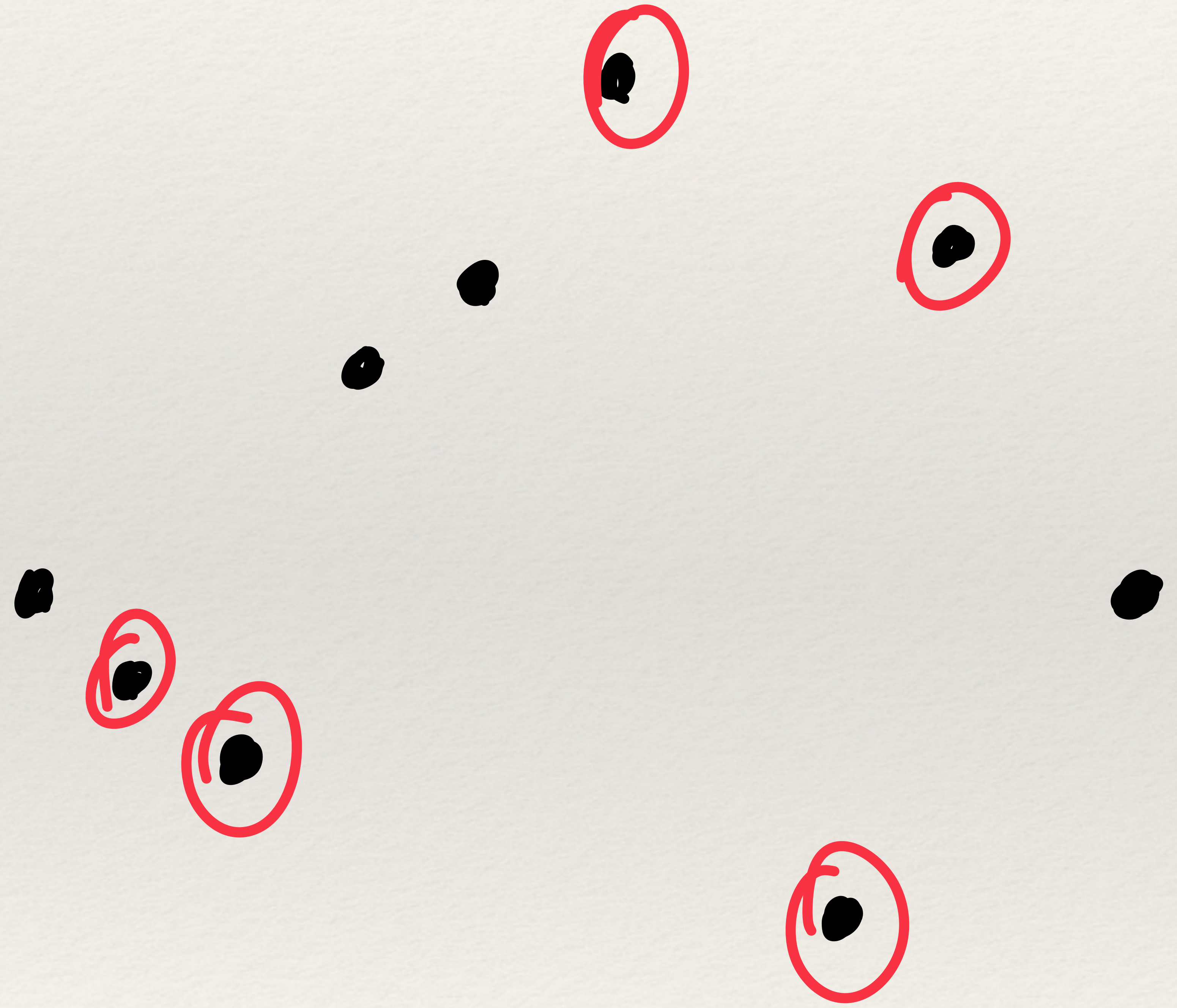
Permutation patterns

4 3 2 6 7 9 1 8 5

Permutation patterns



432679185

$\geq$

32514

Permutation classes

This is a partial order on the set of all (finite) permutations.

A downset in this order is called a permutation class.

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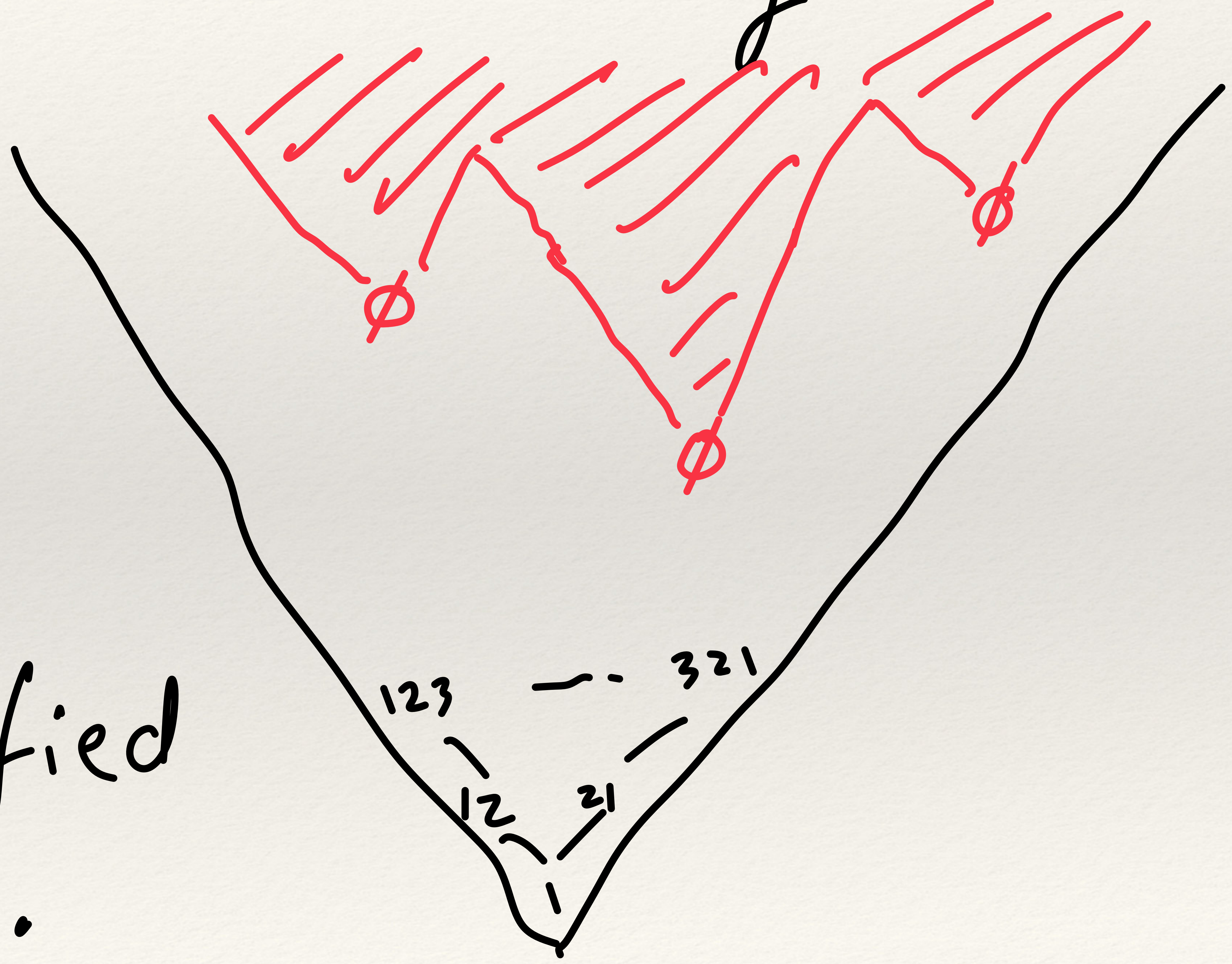
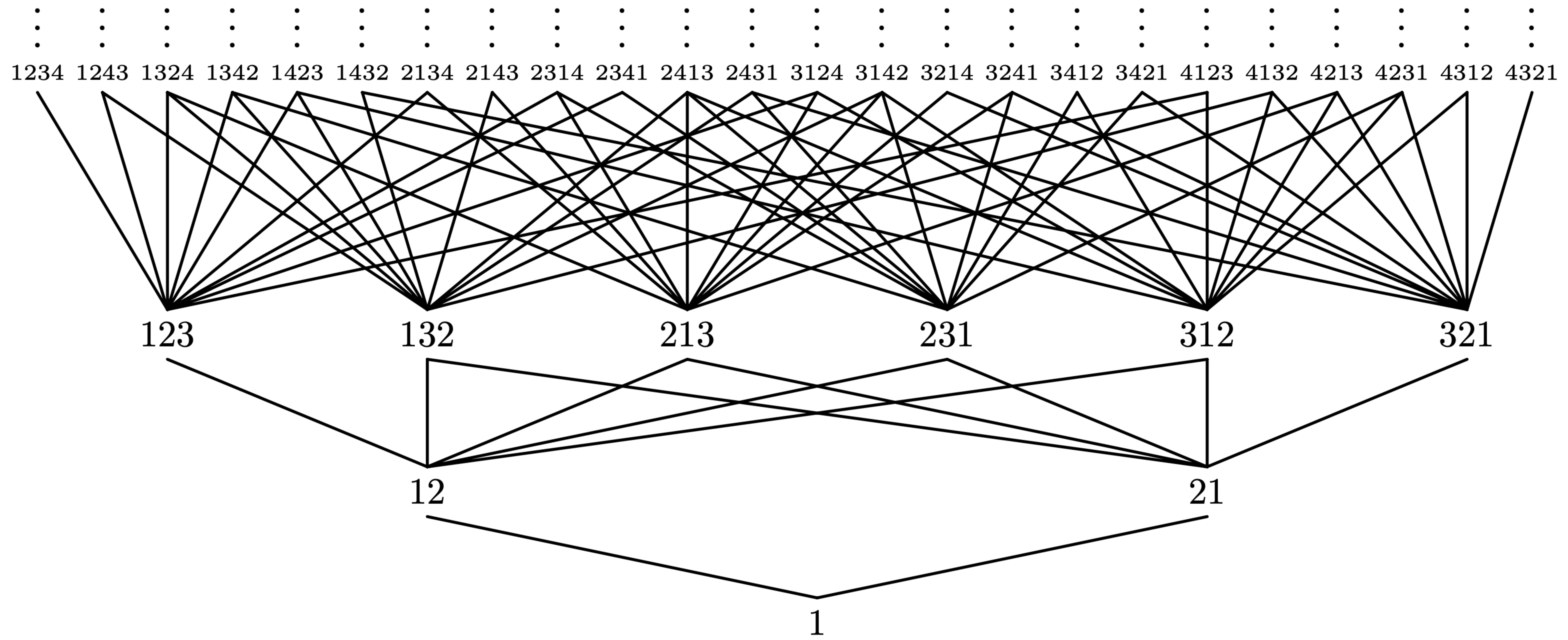
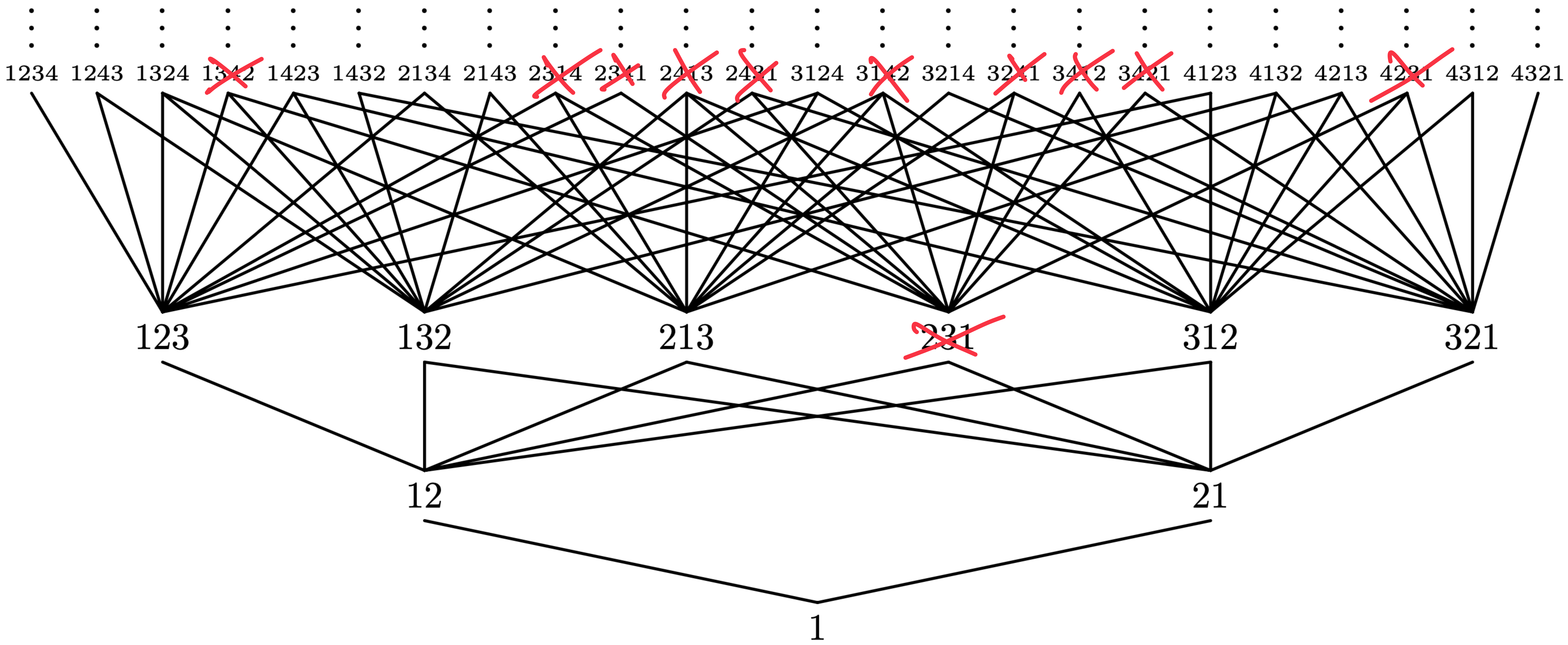
Every class can be specified by avoidance — Av(B).





1234 1243 1324 1342 1423 1432 2134 2143 2314 2341 2413 2431 3124 3142 3214 3241 3412 3421 4123 4132 4213 4231 4312 4321
123 132 213 231 312 321
12 21
1

Av(231)


1234 1243 1324 ~~1342~~ 1423 1432 2134 2143 ~~2314~~ ~~2341~~ ~~2413~~ ~~2431~~ 3124 ~~3142~~ 3214 ~~3241~~ ~~3412~~ ~~3421~~ 4123 4132 4213 ~~4231~~ 4312 4321
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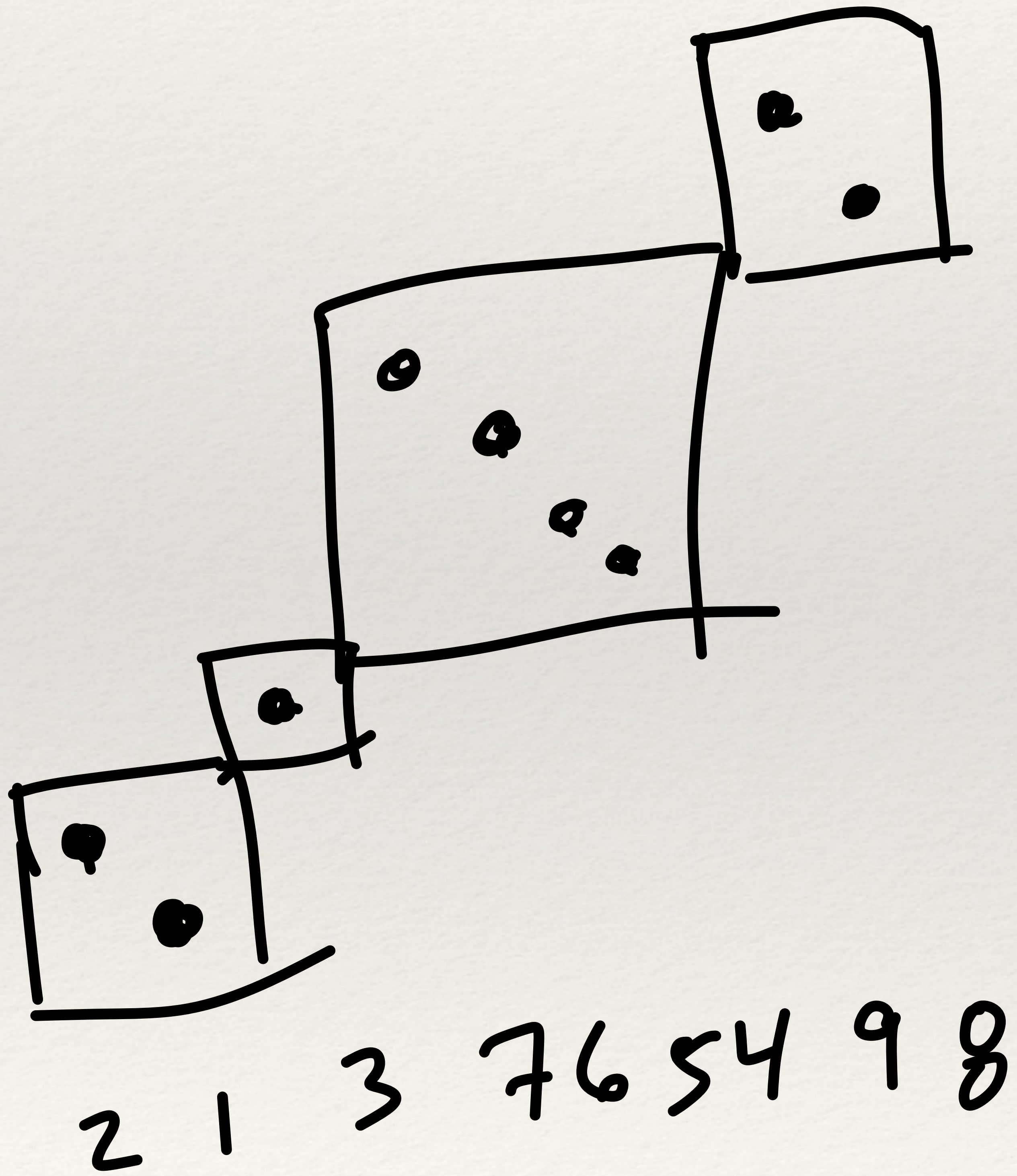
1

Example: layered permutations



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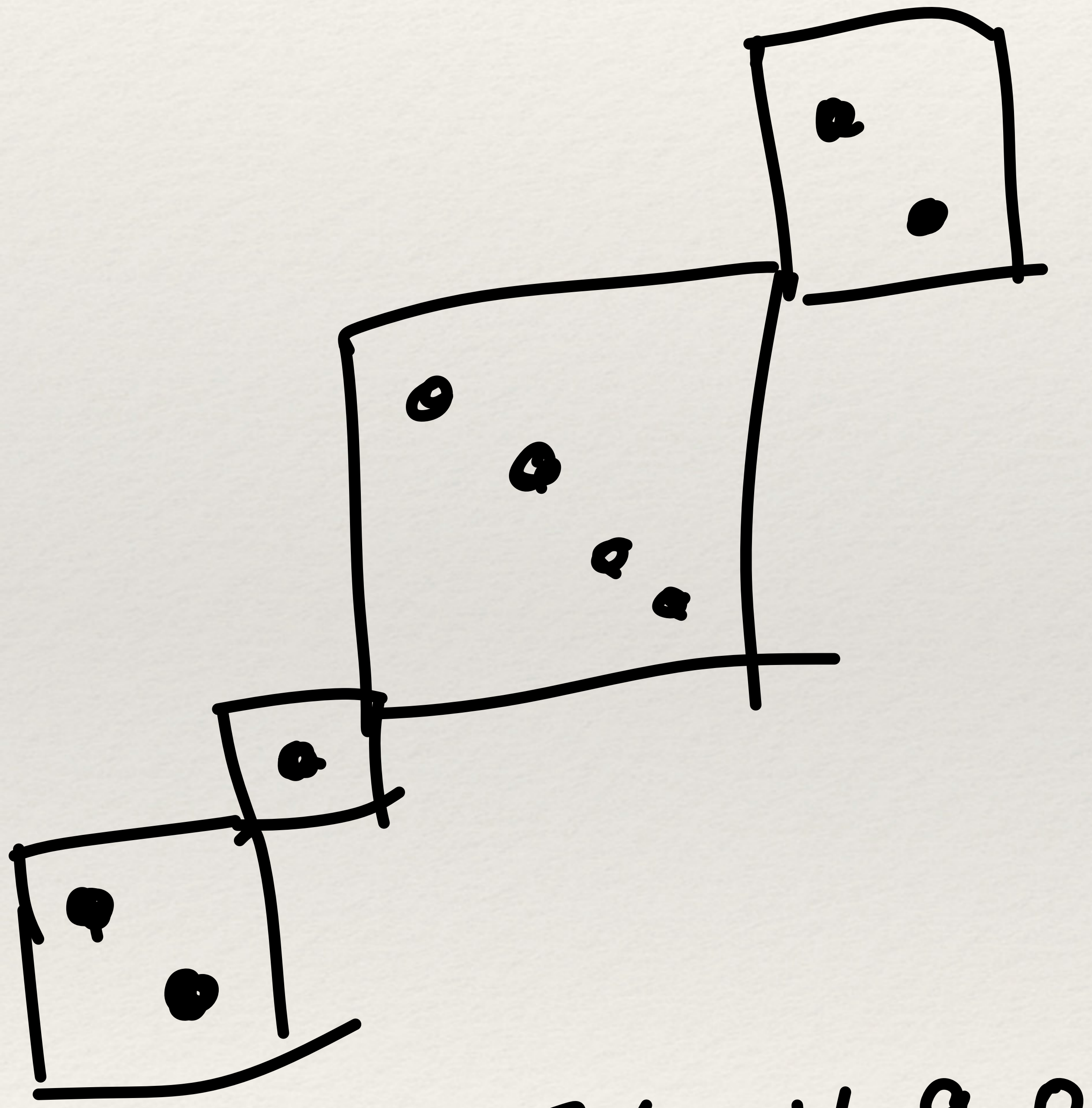
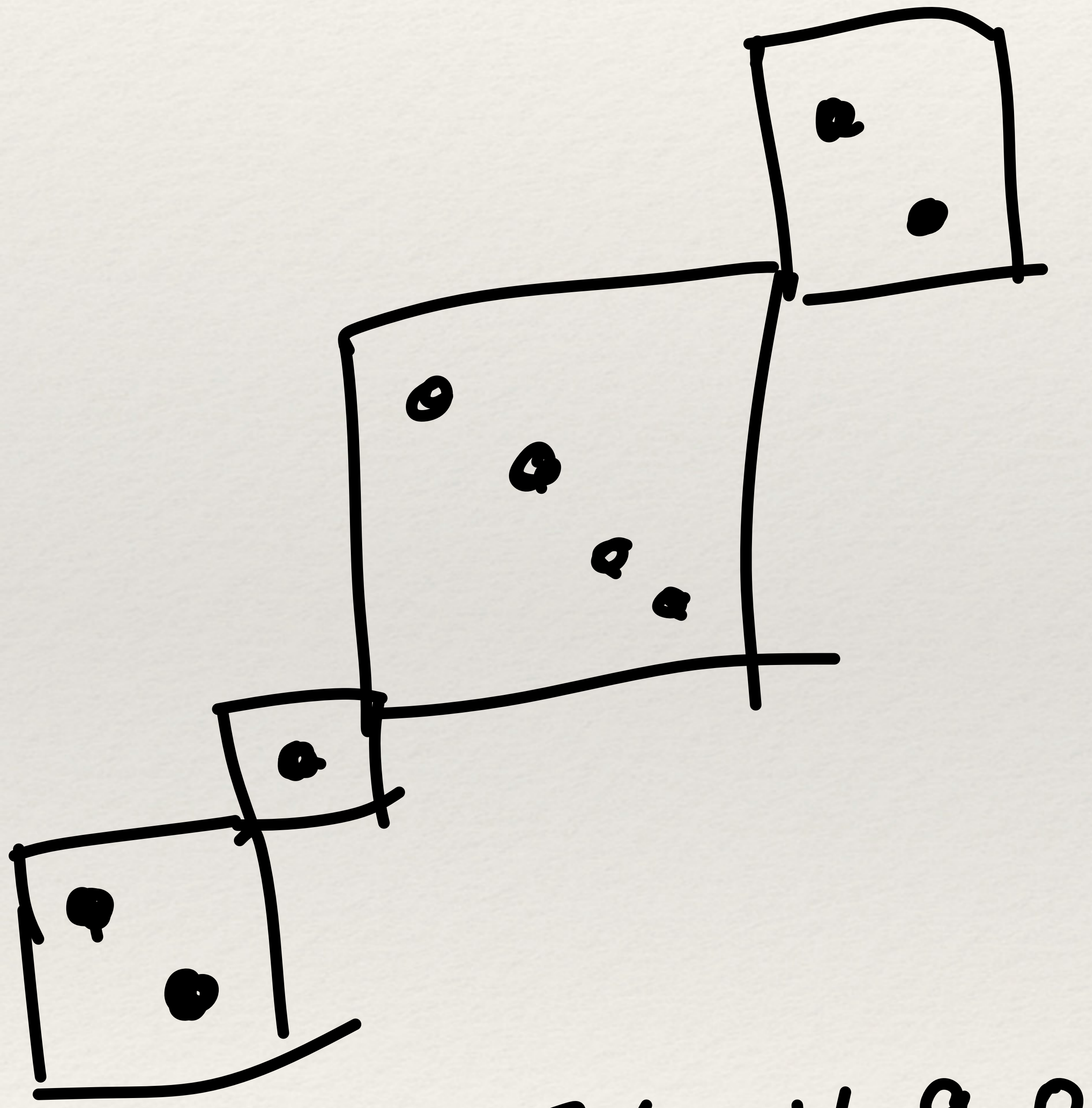


2 1 3 7 6 5 4 9 8

Compositions of n

2^{n-1} of length n

Example: layered permutations



2 1 3 7 6 5 4 9 8

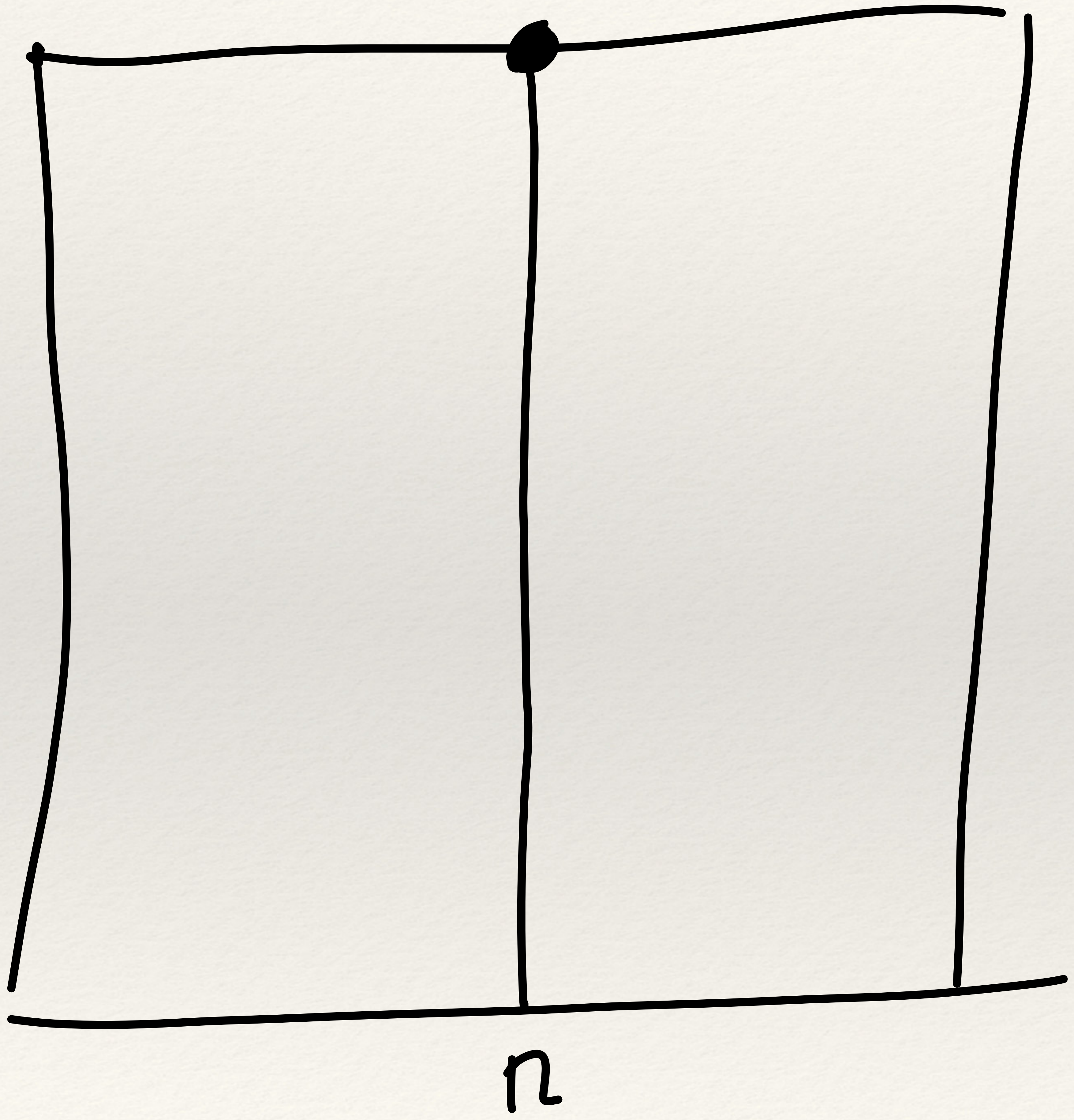
$\longleftrightarrow$

Compositions of n

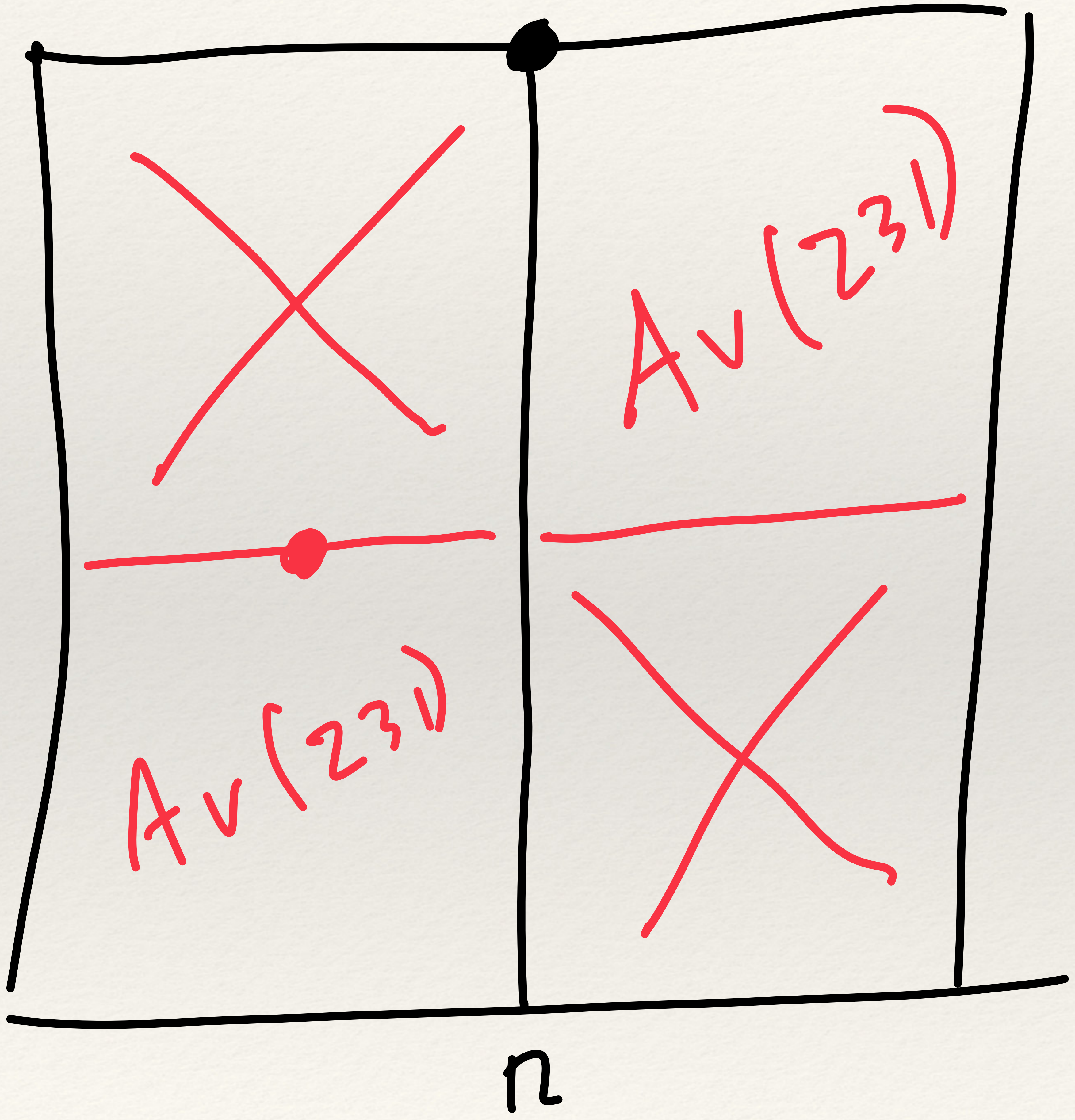
2^{n-1} of length n

$\mathrm{Av}(231, 312)$

Example: Av(231)



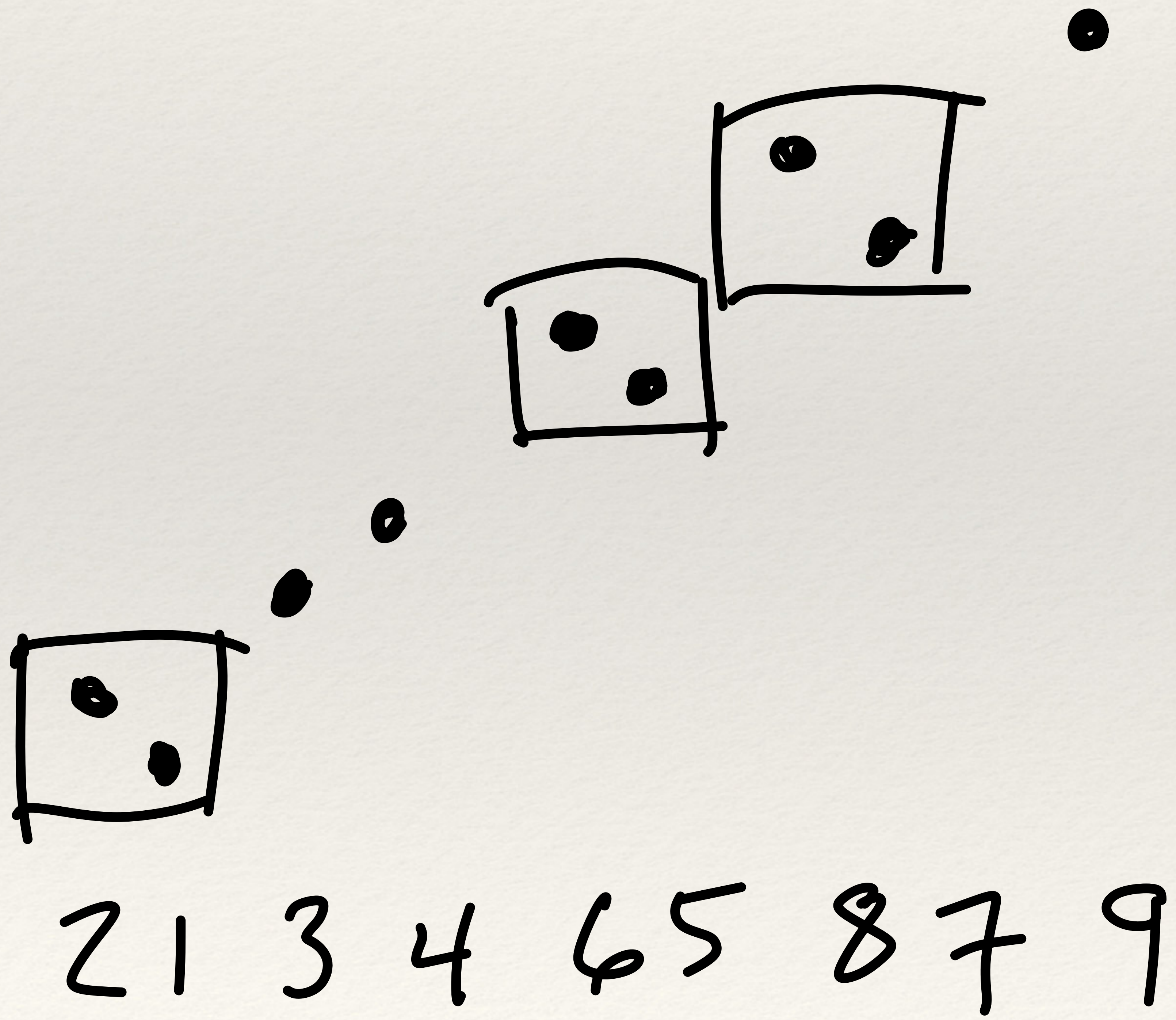
Example: Av(231)



$\longleftrightarrow$ Catalan #s

$\approx 4^n$ of length n

Example: Av(231, 312, 321)



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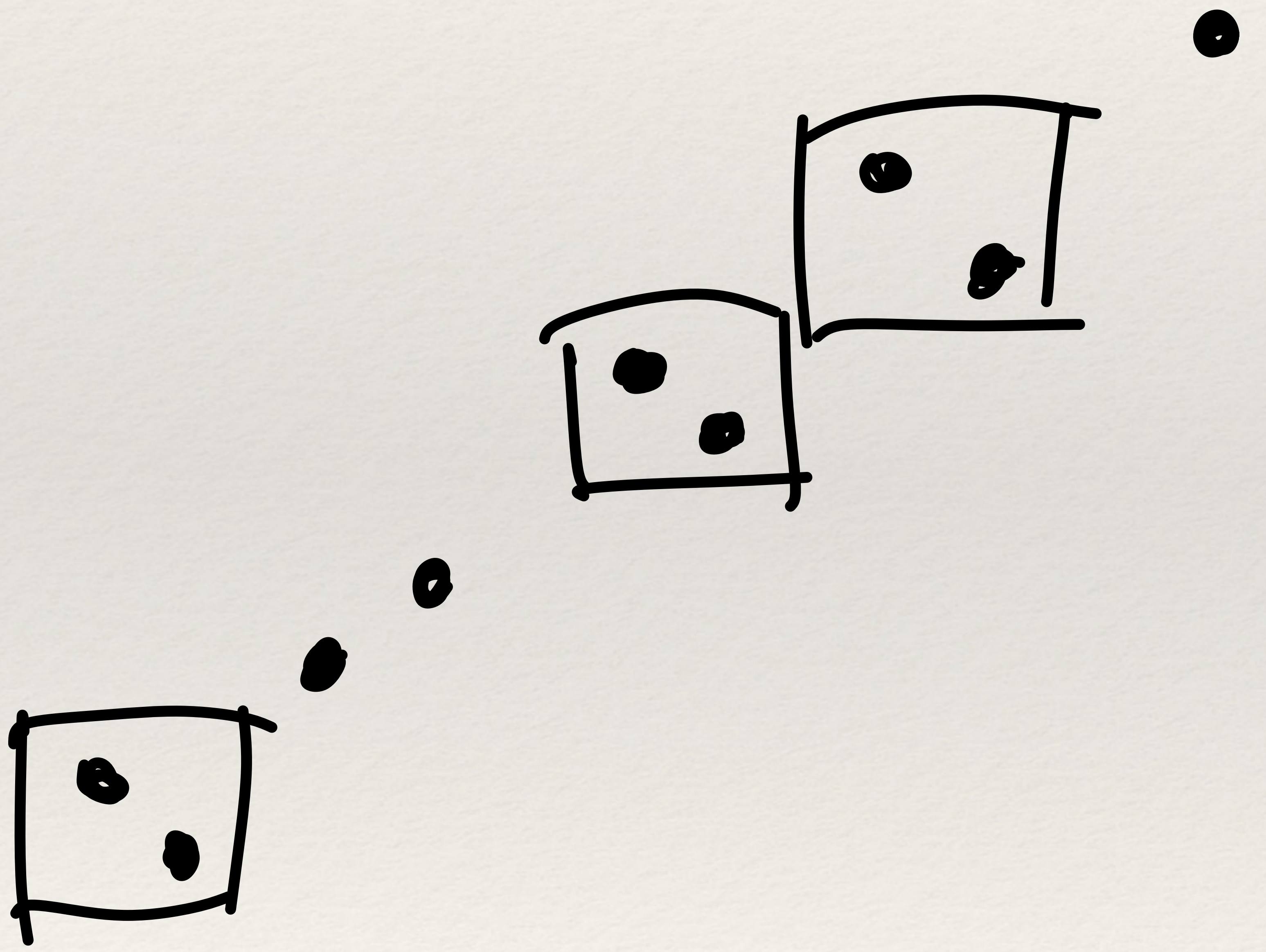

2 1 3 4 6 5 8 7 9

$\longleftrightarrow$ Fibonacci #s

$\approx 1.62^n$ of length n

Growth rates of permutation classes

Let $\mathcal{C}$ be a class,
and $\mathcal{C}_n$ be the permutations
of length n in $\mathcal{C}$.

$$\boxed{\mathrm{gr}(\mathcal{C}) = \limsup_{n \to \infty} \sqrt[n]{|\mathcal{C}_n|}}$$

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$$\boxed{\text{if } |\mathcal{C}_{n+m}| \geq |\mathcal{C}_n||\mathcal{C}_m|}$$

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Examples:

- $\mathrm{gr}(\text{layered}) = 2$
- $\mathrm{gr}(\mathrm{Av}(231)) = 4$
- $\mathrm{gr}(\mathrm{Av}(231, 312, 321)) \approx 1.62$

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Regev (1981):

$$\mathrm{gr}(\mathrm{Av}(k \cdots 21)) = (k-1)^2$$

(One of the many implications of "Asymptotic values for degrees associated with strips of Young diagrams".)

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Easy:

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Is there an elementary proof of the reverse?

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The set of all growth rates

- What does the set of all growth rates of all classes look like?

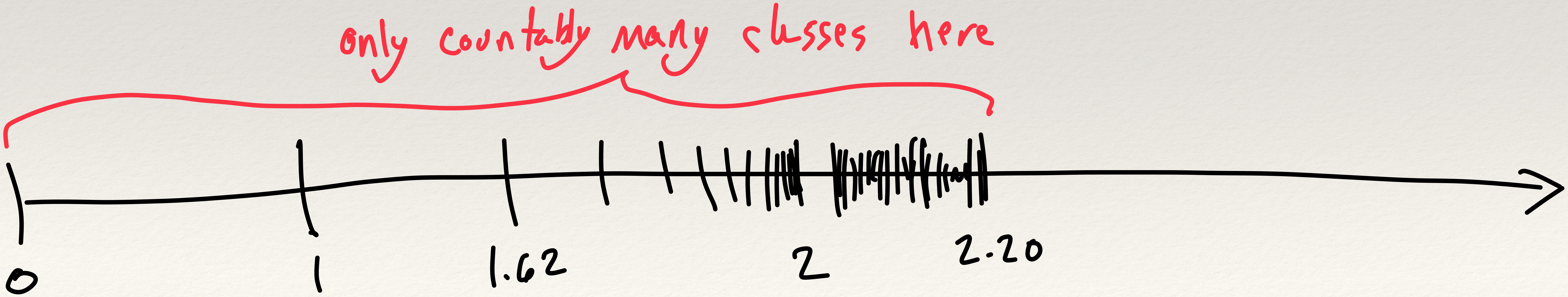
Kaiser and Klazar (2003): it begins

$$0,\ 1,\ \underbrace{\approx 1.62,\ \dots\ ,}_{\text{all come from g.f. denominators of } 1-x-x^2-\dots-x^k,\ \text{accumulate at } 2}\ 2$$

The set of all growth rates

- What does the set of all growth rates of all classes look like?

V(2011): extend list to ≈ 2.20, where there are uncountably many classes.



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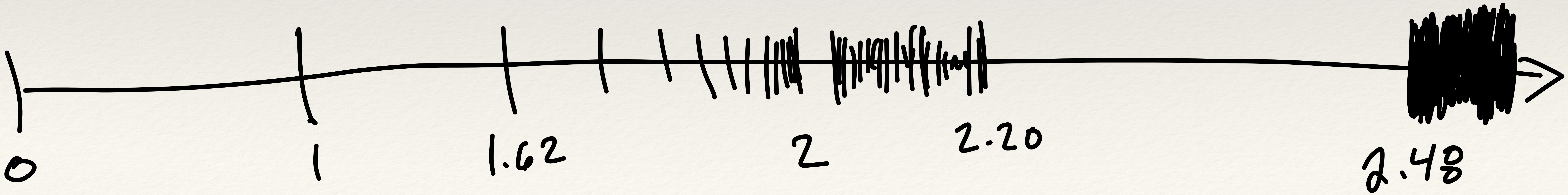
Albert and Linton (2009), V (2010): above ≈2.48, every real # is the growth rate of some class.



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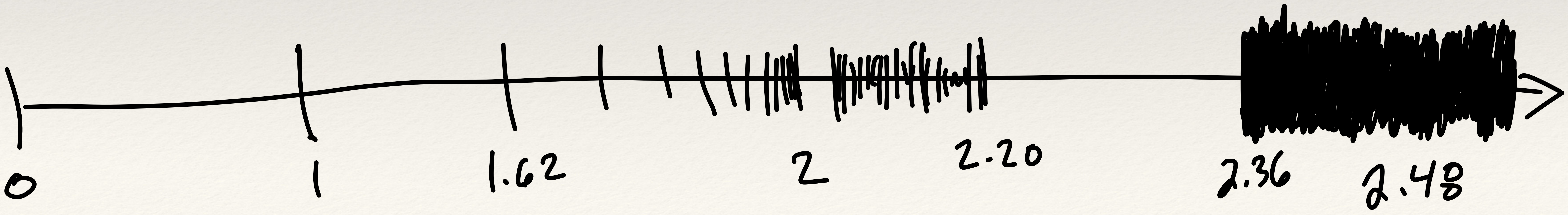
Bevan (2018): Make that ≈ 2.36.



The set of all growth rates

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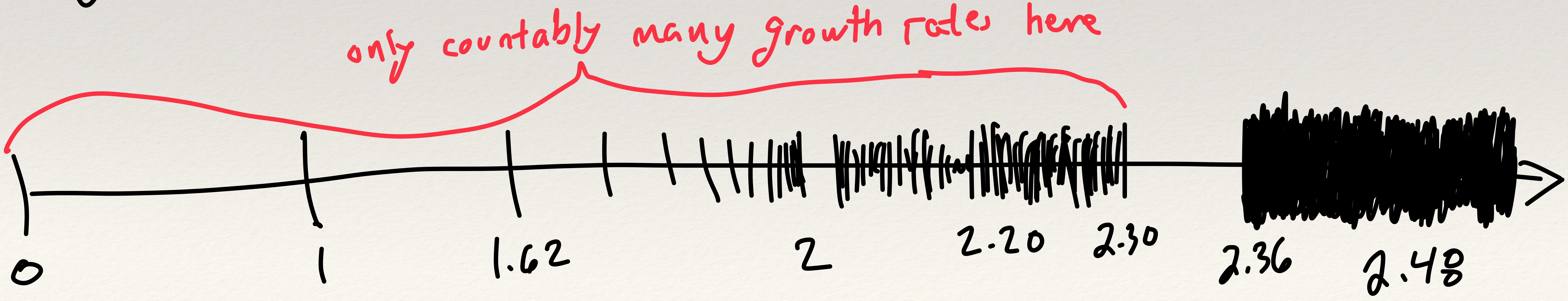
V(2019) and Pantone and V(2020): extend list to ≈ 2.30, where there are uncountably many growth rates.



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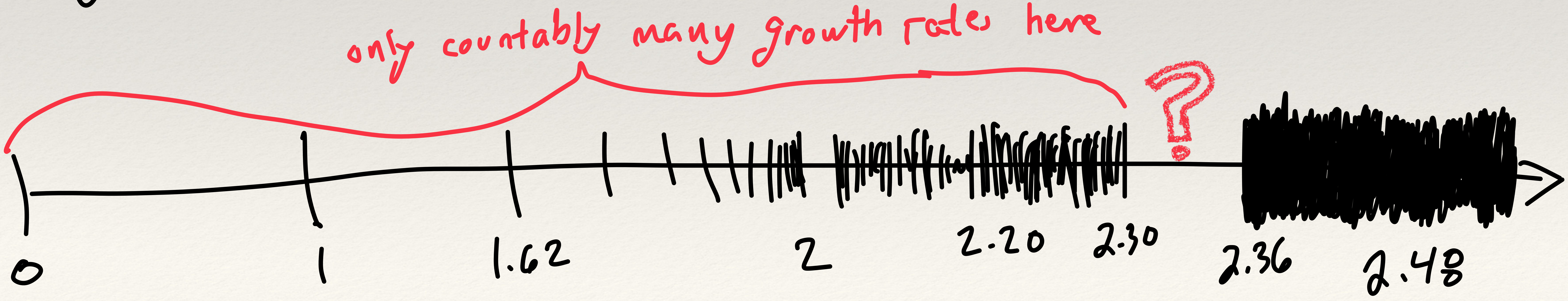


Growth rates of certain classes

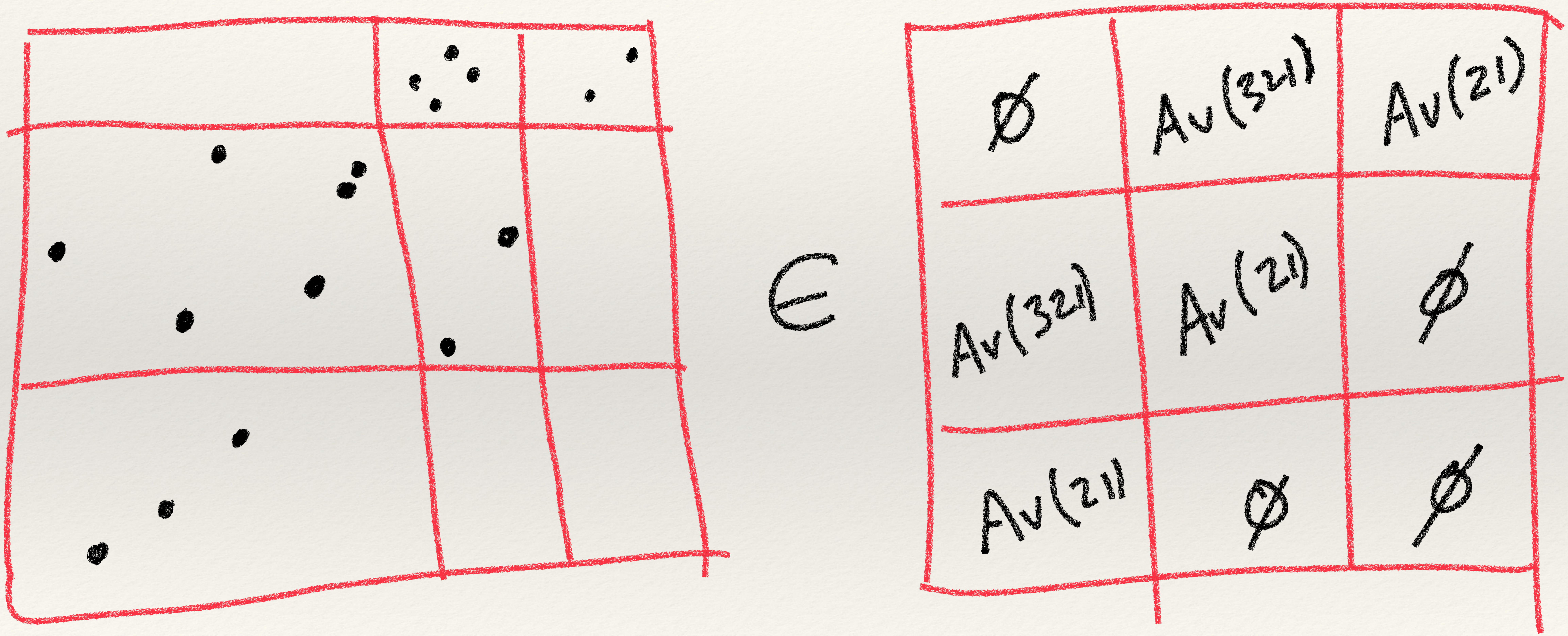
- What about avoiding a single pattern?
- $\mathrm{gr}(\mathrm{Av}(231)) = \mathrm{gr}(\mathrm{Av}(321)) = 4$
- $\mathrm{gr}(\mathrm{Av}(4321)) = 9$ (Regev 1981)
- $\mathrm{gr}(\mathrm{Av}(1342)) = 8$ (Bóna 1997)
- $\mathrm{gr}(\mathrm{Av}(1324)) = ?$ (probably ≈ 11.6)

Grid classes

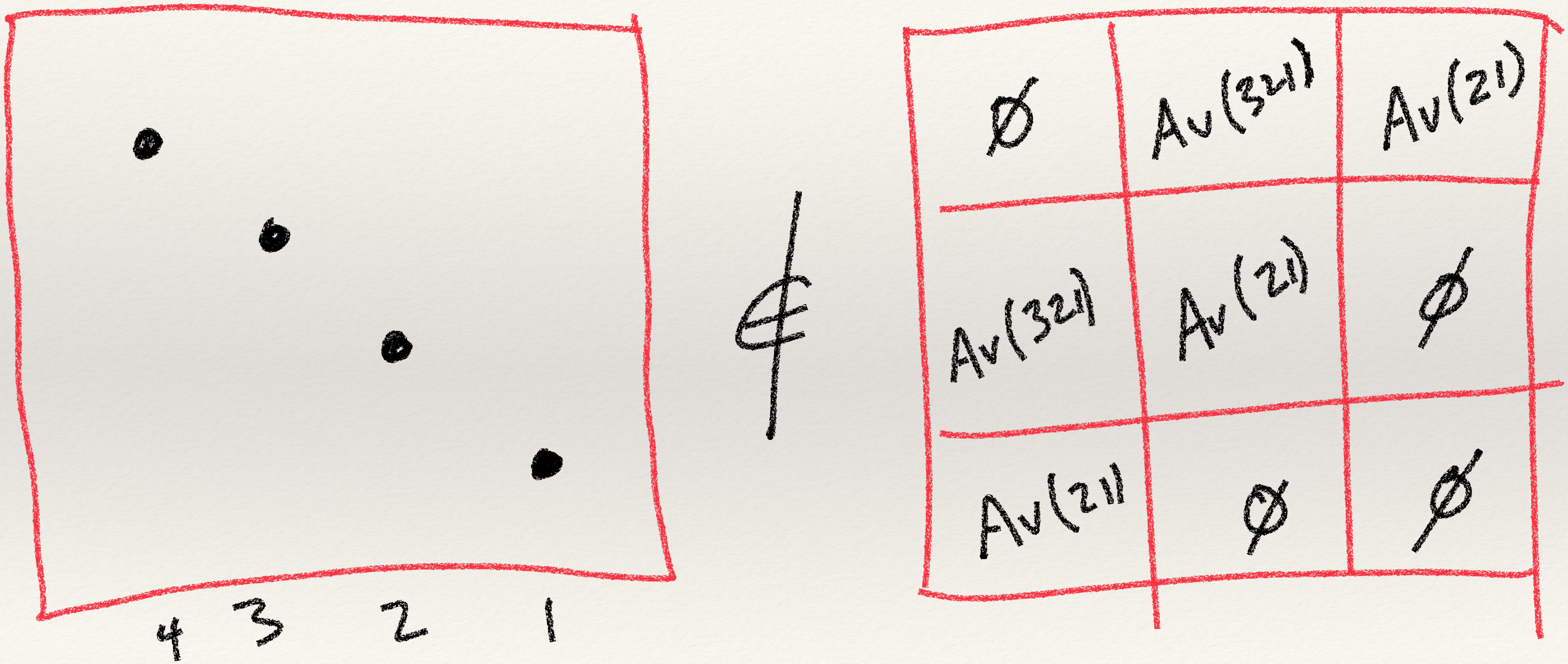
A (generalized) grid class is defined by a matrix of permutation classes. It contains those perms that can be subdivided in a compatible manner.

$\emptyset$	$\mathrm{Av}(321)$	$\mathrm{Av}(21)$
$\mathrm{Av}(321)$	$\mathrm{Av}(21)$	$\emptyset$
$\mathrm{Av}(21)$	$\emptyset$	$\emptyset$

Grid classes



Grid classes



Growth rates of grid classes

Building on Bevan (2015):

Albert and V (2019): Define

a matrix Γ by

$$\Gamma(k,\ell) = \sqrt{\text{gr(class in that cell)}}.$$

Then

$$\text{gr(grid class)} = \text{largest eigenvalue of } \Gamma^t \Gamma.$$

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(Because square of largest singular value of Γ.)

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Av(321)	Av(21)	$\emptyset$
Av(21)	$\emptyset$	$\emptyset$

Growth rates of grid classes

In this example,

$$\Gamma = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$\Gamma^t \Gamma = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$\mathrm{gr}(\mathrm{grid}) \approx 7.34.$

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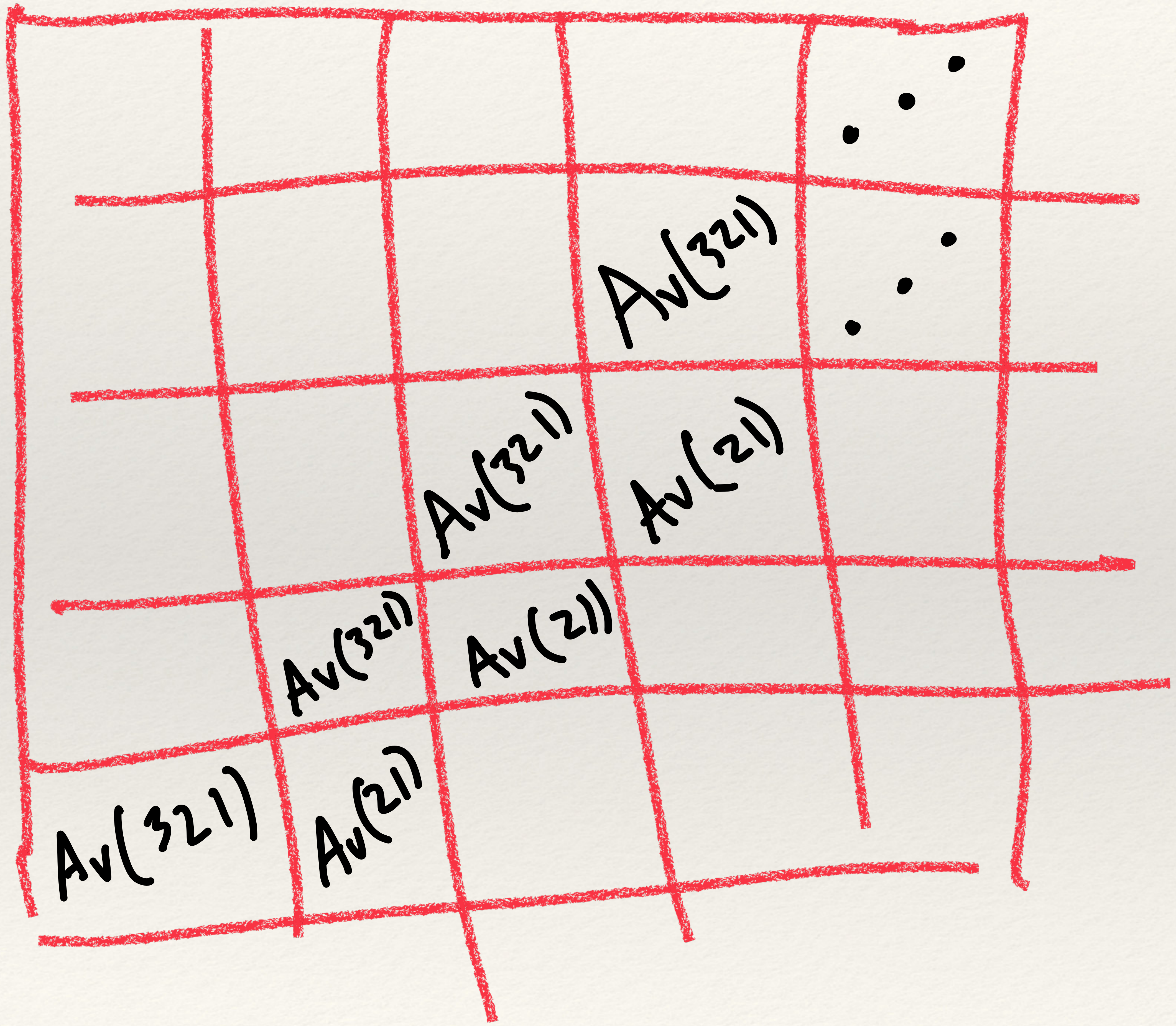
Avoiding a monotone pattern



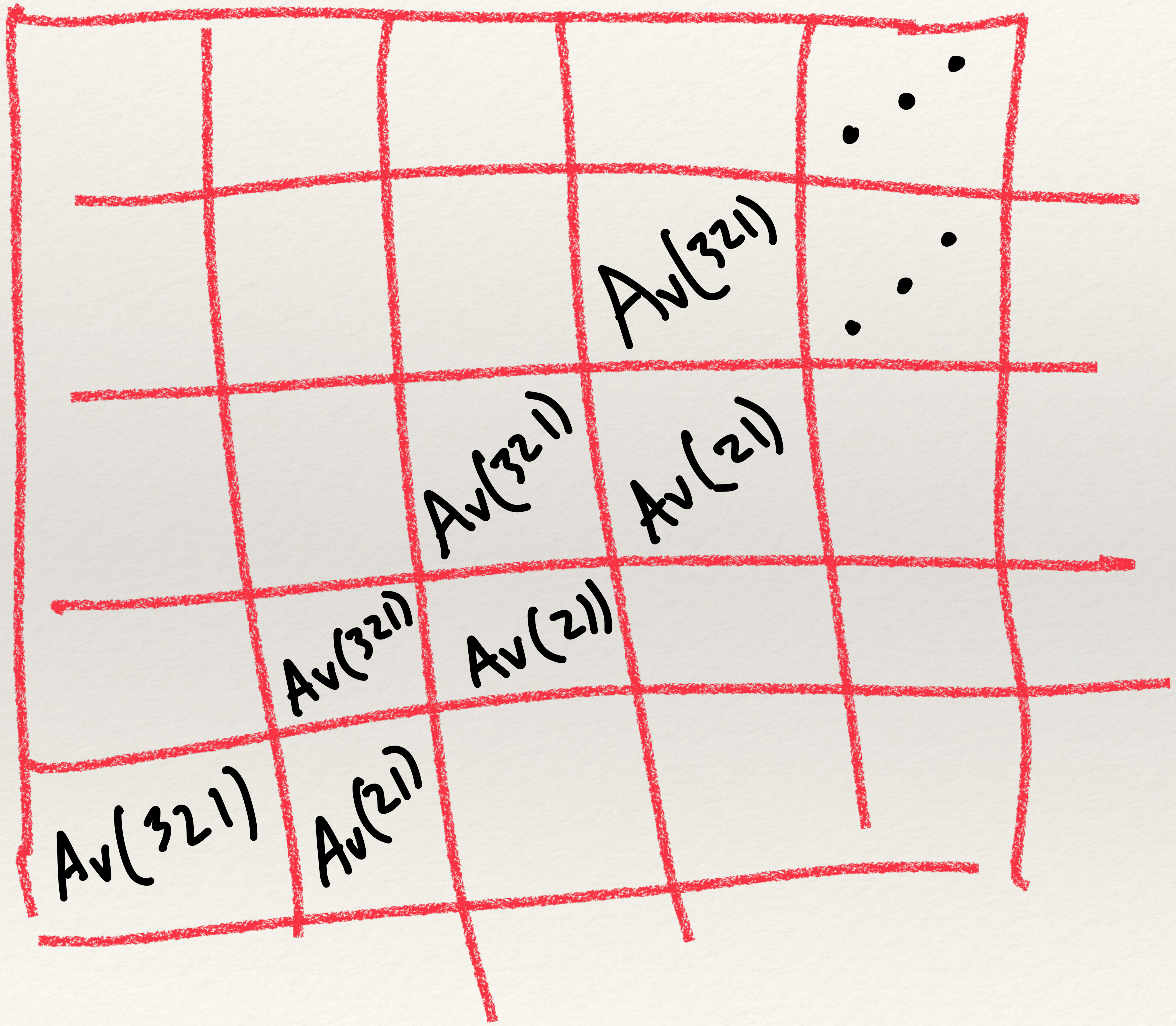
$$\Gamma^t\Gamma = \begin{bmatrix} & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \\ & & 2 & 5 & 2 \\ & 2 & 5 & 2 & \\ 2 & 5 & 2 & & \\ 5 & 2 & & & \end{bmatrix}$$

Now, there is a formula for the eigenvalues of a tri-diag Toeplitz...

If we start with a $t \times t$ grid...

$$5 + 4\cos\left(\frac{1}{t+1}\right) \longrightarrow 9.$$

Avoiding a monotone pattern



This generalizes in the obvious way to prove

$$\mathrm{gr}(\mathrm{Av}(k\cdots 21)) \geq (k-1)^2.$$

Merges of classes

Albert, Pantone, and V (2019) generalized this construction to obtain results about merges.

Given two classes $\mathcal{C}$ and $\mathcal{D}$, their merge is all perms whose entries can be partitioned into a member of $\mathcal{C}$ and a member of $\mathcal{D}$ (sometimes thought of as a red-blue coloring of these entries).

Merges of classes

Let $\mathcal{C} \odot \mathcal{D}$ = merge of $\mathcal{C}$ and $\mathcal{D}$.

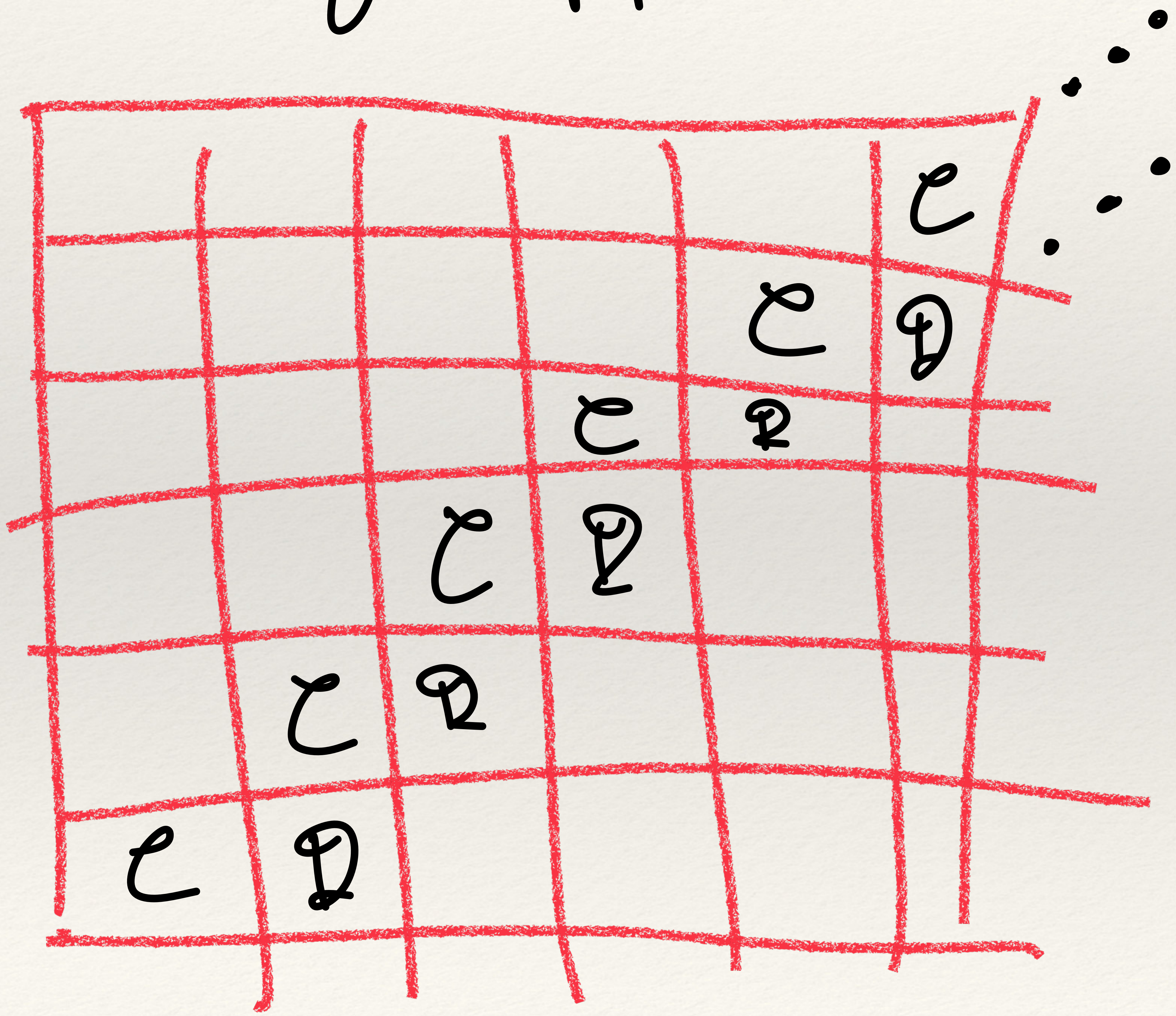
Easy: $$|(\mathcal{C} \odot \mathcal{D})_n| \leq \sum_{i=0}^{n} \binom{n}{i}^2 |\mathcal{C}_i| |\mathcal{D}_{n-i}|,$$ so

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) \leq \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})}\right)^2.$$

Note: $\mathrm{Av}(4321) = \mathrm{Av}(321) \odot \mathrm{Av}(21)$.

When is that an equality?

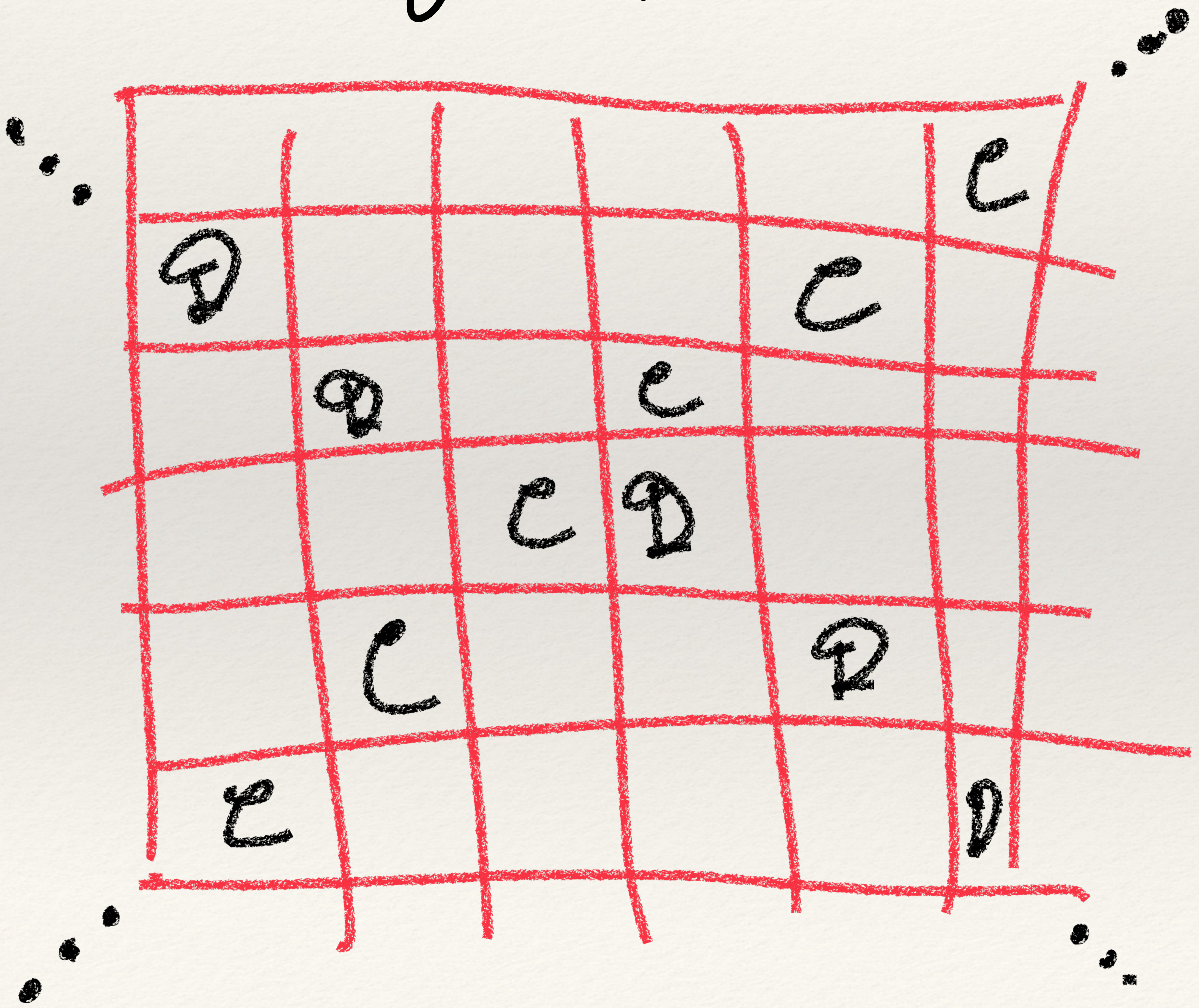
This easy upper bound is the truth if...



$$\subseteq \mathcal{C} \odot \mathcal{D}$$

When is that an equality?

This easy upper bound is also the truth if…



$$\subseteq \mathcal{C} \odot D$$

Some corollaries

Bóna (2005):

$$\mathrm{gr}(\mathrm{Av}(54213)) = \left(\sqrt{\mathrm{gr}(\mathrm{Av}(21))} + \sqrt{\mathrm{gr}(\mathrm{Av}(4213))}\right)^2$$

$$= \left(\sqrt{\mathrm{gr}(\mathrm{Av}(21))} + \sqrt{\mathrm{gr}(\mathrm{Av}(1342))}\right)^2$$

$$= (1 + \sqrt{8})^2$$

$$= 9 + 4\sqrt{2}.$$

Some corollaries

Bóna (2007):

$$\mathrm{gr}\left(\mathrm{Av}\left(\begin{smallmatrix}\boxed{\alpha} & & \\ & \bullet & \\ & & \boxed{\beta}\end{smallmatrix}\right)\right) = \left(\mathrm{gr}\left(\mathrm{Av}\left(\begin{smallmatrix}\boxed{\alpha} & \\ & \bullet\end{smallmatrix}\right)\right) + \mathrm{gr}\left(\mathrm{Av}\left(\begin{smallmatrix}\bullet & \\ & \boxed{\beta}\end{smallmatrix}\right)\right)\right)^2.$$

One more question

Is it always true that

$$gr(\mathcal{C} \odot \mathcal{D}) = \left(\sqrt{gr(\mathcal{C})} + \sqrt{gr(\mathcal{D})} \right)^2 \; ?$$

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HAPPY BIRTHDAY!