Restricted permutations, Conjecture of Lin and Kim, and work of Andrews and Chern

Toufik Mansour
University of Haifa, Israel

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Avoiding a set of patterns in $\mathcal{S}_4$ (with Callan, Schork, Shattuck)

• Let $\mathcal{S}_{\mathbf{n}}$ be the symmetric group of all permutations of $[n] \equiv \{1, \ldots, n\}$.
• Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $\tau = \tau_1\tau_2\cdots\tau_k \in \mathcal{S}_k$ be two permutations.
• We say that π **contains** τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is *order-isomorphic* to τ, that is, $\pi_{i_a} < \pi_{i_b}$ if and only if $\tau_a < \tau_b$; in such a context τ is usually called a **pattern**.
• We say that π **avoids** τ, or is τ-**avoiding**, if such a subsequence does not exist.
• The set of all τ-avoiding permutations in $\mathcal{S}_n$ is denoted by $\mathcal{S}_n(\tau)$.
• For an arbitrary finite collection of patterns T, we define $\mathcal{S}_n(T) = \bigcap_{\tau \in T} \mathcal{S}_n(\tau)$.

• Two sets of patterns T and T' belong to the same **symmetry class** if and only if T' can be obtained by the action of the dihedral group of order eight - generated by the operations reverse, complement, and inverse - on T.

• The sets of patterns T and T' belong to the same **Wilf class** if and only if $|\mathcal{S}_n(T)| = |\mathcal{S}_n(T')|$ for all $n \geq 0$.

• We denote the number of Wilf classes of subsets of k patterns in $\mathcal{S}_4$ by w_k, respectively.

• It is well known that $w_1 = 3$ and $w_2 = 38$.

Theorem

We have $w_3 = 242$, $w_4 = 1100$, $w_5 = 3441$, $w_6 = 8438$, $w_7 = 15392$, $w_8 = 19002$, $w_9 = 16293$, $w_{10} = 10624$, $w_{11} = 5857$, $w_{12} = 3044$, $w_{13} = 1546$, $w_{14} = 786$, $w_{15} = 393$, $w_{16} = 198$, $w_{17} = 105$, $w_{18} = 55$, $w_{19} = 28$, $w_{20} = 14$, $w_{21} = 8$, $w_{22} = 4$, $w_{23} = 2$, and $w_{24} = 1$.

• To determine w_k for $k \in \{3, 4, \ldots, 24\}$. Since the number of subsets of $\mathcal{S}_4$ containing at least 4 patterns is given by $\sum_{k=3}^{24} \binom{24}{k} = 16776915$, it seems to be impossible to reach by using the action of the dihedral group on permutations and by constructing explicit bijections between sets of permutations.

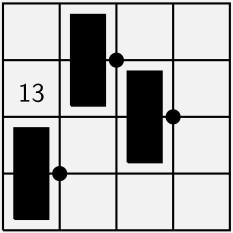
• The way out is to combine several software programs to do the work for us. Briefly, the line of argument will be as follows:

1. Fix $k \in \{3, 4, \ldots, 24\}$. In the following steps, only $T \subseteq \mathcal{S}_4$ with $|T| = k$ are considered.
2. Use **Kuszmaul's algorithm** to find the symmetry classes of the subsets T, denoted by $\mathcal{SC}_k$. Determine also for each symmetry class $T \in \mathcal{SC}_k$ the sequence $(|\mathcal{S}_n(T)|)_{n=1,\ldots,16}$.
3. Use the Maple package **INSENC** (regular insertion encoding) and try to determine the generating functions of the sequences $|\mathcal{S}_n(T)|_{n\geq 0}$ for all $T \in \mathcal{SC}_k$ on the basis of the starting sequences determined in (2).
4. Determine by **Cell Decompositions** the generating functions for those $T \in \mathcal{SC}_k$ where INSENC failed.
5. Count the number of different generating functions obtained in last steps. This number equals w_k.

• Here let us give one example for cell decomposition. Fix

$$T = \{1324, 1432, 2143, 2413, 2431, 3142, 4132\}, \tau = 132.$$

• We define the **cell diagram** of $\tau = 132$ to be the graph of 132 on a $(3+1)\times(3+1)$ board. A cell is said to be **shadow** of a dot in the cell diagram of τ if it is either directly southwest or northwest of the dot. We call the cell diagram of τ with its shadow cells a **shadow cell diagram**:



• Note that we can represent a permutation $\pi \in \mathcal{S}_n$ which contains $\tau \in \mathcal{S}_3$ either as dots on an $(n+1) \times (n+1)$ board, or, alternatively, on the shadow cell diagram – a $(3+1) \times (3+1)$ board – where the dots represent the leftmost occurrence of τ and the cells contain (possibly empty) subsequences π' of π.

• In fact, the shadow cell diagram of τ represents the graph of all permutations π that contain τ in positions $i_1 i_2 i_3$ such that (1) i_1 is minimal, (2) $i_1 + i_2$ is minimal, and (3) $i_1 + i_2 + i_3$ is minimal. This *minimality* produces additional conditions on the empty cells.

For instance, the 14-cell on the left of the 13-cell.

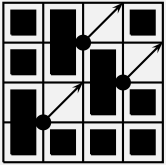
• For the class of permutations π that contain τ and avoid T, we consider each cell in the shadow cell diagram of τ and determine the *necessary restrictions* which follow for the structure of π.
(1) For each empty cell of the shadow cell diagram, we check whether adding a dot would lead to a contradiction concerning the avoidance of the patterns T. If it is possible to add a dot without contradiction, the cell is left empty, otherwise it is blackened.

(2) For each of the empty cells resulting from (1), we check whether the cell form a decreasing or increasing subsequence.

(3) For each of the empty cells resulting from (2), we check whether minimality gives a restriction (decreasing/increasing) for the possible subsequences in the cell.

Example

Let $T = \{1324, 1432, 2143, 2413, 2431, 3142, 4132\}$ and let π that contains 132 and avoids T. Then the cell decomposition diagram of permutations π can be described as



Hence the generating function for the number permutations in $S_n(T)$ is given by

$$C(x) + \frac{x^3}{(1-x)^3}.$$

Conjecture of Lin and Kim (with Shattuck)

An **inversion** within a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$ is an ordered pair (a, b) such that $1 \leq a < b \leq n$ and $\sigma_a > \sigma_b$.

The **inversion sequence** of σ is given by $a_1 \cdots a_n$, where a_i records the number of entries of σ to the right of i and less than i for each $i \in [n]$ (see Corteel, Martinez, Savage, and Weselcouch; and Mansour and Shattuck).

The (large) Schröder number S_n is defined recursively by

$$nS_n = 3(2n-3)S_{n-1} - (n-3)S_{n-2}, \qquad n \geq 3,$$

with $S_1 = 1$ and $S_2 = 2$, and arises as the enumerator of several avoidance classes of permutations corresponding to a pair of patterns of length four (see works of Gire; Kremer and West). In particular, one has the S_n enumerates $\mathcal{S}_n(\sigma,\tau)$ for the following ten inequivalent pairs (σ,τ):

$$\begin{array}{ccccc} (1234,2134) & (1324,2314) & (1342,2341) & (3124,3214) & (3142,3214) \\ (3412,3421) & (1324,2134) & (3124,2314) & (2134,3124) & (2413,3142). \end{array}$$

This answered in the affirmative a conjecture originally posed by Stanley. Moreover, outside of symmetry, there are no other such pairs (σ,τ) for which $|\mathcal{S}_n(\sigma,\tau)| = S_n$.

Lin and Kim introduced a new triangle $S_{n,k}$ for Schröder numbers in their study of inversion sequences, where

$$S_{n,k} = S_{n,k-1} + 2S_{n-1,k} - S_{n-1,k-1}, \quad 1 \leq k \leq n-2,$$

with $S_{n,n} = S_{n,n-1} = S_{n,n-2}$ for $n \geq 3$ and $S_{1,1} = S_{2,1} = S_{2,2} = 1$.

Lin and Kim showed that $S_{n,k}$ enumerates the inversion sequences $\pi_1 \cdots \pi_n \in \{I_n(021) \mid \pi_n = k \mod n\}$. Then they state the following conjecture which provides a connection between inversion sequences and pattern avoidance in permutations.

Conjecture

Let (ν,μ) be a pair of patterns of length four. Then

$$S_{n,k}=|\{\sigma_1\cdots\sigma_n\in\mathcal{S}_n(\nu,\mu)\mid \sigma_1=k\}|$$

for all $1\le k\le n$ if and only if (ν,μ) is one of the following nine pairs:

$(4321,3421)$, $(3241,2341)$, $(2431,2341)$, $(4231,3241)$, $(4231,2431)$, $(4231,3421)$, $(2431,3241)$, $(3421,2431)$, $(3421,3241)$.

It is easy to see that

$$\sum_{n\geq1}\left(\sum_{k=1}^{n}S_{n,k}y^k\right)x^n$$
$$=\frac{xy(2-3x-3y+3xy)+xy(x+y-xy)(xy+\sqrt{1-6xy+x^2y^2})}{2(1-2x-y+xy)}.$$

With help of recurrence relations, generating functions, and computer programming we **proved this conjecture**. One of the harder cases is the case $(1243, 1423)$.

Here, we enumerate the members of $\mathcal{S}_n(1243, 1423)$ according to the joint distribution of the first and second letter statistics. So let

$$a_n(i,j) = \{\pi \in \mathcal{S}_n(1243, 1423) | \pi_1 = i, \pi_2 = j\}.$$

Lemma: We have

$$a_n(i,i+1) = a_{n-1}(i,i+1) + \sum_{a=1}^{i-1}\sum_{c=0}^{i-a-1}\sum_{b=a+1}^{i-c}\binom{i-a-1}{c}a_{n-c-2}(a,b), \quad 1\leq i\leq n-2,$$

$$a_n(i,i+2) = a_{n-1}(i,i+1) + a_{n-1}(i,i+2) + \sum_{a=1}^{i-1}\sum_{c=0}^{i-a-1}\sum_{b=a+1}^{i-c+1}\binom{i-a-1}{c}a_{n-c-2}(a,b), \quad 1\leq i\leq n-3,$$

with $a_n(n-1,n)=a_{n-2}$, $a_n(n-2,n)=a_{n-2}$, and

$$a_n(i,j) = a_{n-1}(i,j-1) + \sum_{a=1}^{i-1}\sum_{c=0}^{i-a-1}\binom{i-a-1}{c}a_{n-c-2}(a,j-c-2) + (1-\delta_{j,n})\cdot\left(a_{n-1}(i,j) + \sum_{a=1}^{i-1}\sum_{c=0}^{i-a-1}\binom{i-a-1}{c}a_{n-c-2}(a,j-c-1)\right)$$

We will make use of the following generating functions:

$$A(x,v,w)=\sum_{n\geq2}\sum_{a=1}^{n}\sum_{b=1}^{n}a_n(a,b)v^aw^bx^n,$$
$$A^+(x,v,w)=\sum_{n\geq2}\sum_{a=1}^{n-1}\sum_{b=a+1}^{n}a_n(a,b)v^aw^bx^n,$$
$$A^-(x,v,w)=\sum_{n\geq2}\sum_{a=2}^{n}\sum_{b=1}^{a-1}a_n(a,b)v^aw^bx^n,$$
$$C(x,v)=\sum_{n\geq2}\sum_{a=1}^{n-1}a_n(a,a+1)v^ax^n,$$
$$D(x,v)=\sum_{n\geq2}\sum_{a=1}^{n-2}a_n(a,a+2)v^ax^n,$$
$$B(x,v,w)=\sum_{n\geq4}\sum_{a=1}^{n-3}\sum_{b=a+3}^{n}a_n(a,b)v^aw^bx^n.$$

Translating the recurrence relations in terms of generating functions yields

proposition We have

$$A^+(x,v,w) = wC(x,vw) + w^2D(x,vw) + B(x,v,w),$$

$$A^-(x,v,w) = v^2wx^2 + \frac{vx}{1-v}A(x,vw,1) - \frac{v^2x}{1-v}A(vx,w,1),$$

$$C(x,vw) = vwx^2A(vwx,1,1) + vwx^2 + v^2w^2x^3 + xC(x,vw)$$

$$+\frac{vwx^2}{vwx+vw-1}A^+\left(\frac{vwx}{1-vwx},1-vwx,1\right) - \frac{(1-vwx)x^2}{vwx+vw-1}A^+\left(x,1-vwx,\frac{vw}{1-vwx}\right),$$

$$D(x,vw) = x^2A(vwx,1,1) - v^2w^2x^4 + x(C(x,vw) - vwx^2A(vwx,1,1))$$

$$+xD(x,vw) + \frac{x^2(1-vwx)}{vwx+vw-1}A^+\left(\frac{vwx}{1-vwx},1-vwx,1\right)$$

$$-\frac{x^2(1-vwx)^2}{vw(vwx+vw-1)}A^+\left(x,1-vwx,\frac{vw}{1-vwx}\right) - x^2C(x,vw),$$

$$B(x,v,w) = wxB(x,v,w) + w^3xD(x,vw)$$

$$+\frac{w^2x^2(1-vwx)}{vwx+v-1}A^+\left(x,1-vwx,\frac{vw}{1-vwx}\right) - \frac{vw^2x^2}{vwx+v-1}A^+(x,v,w) + xB(x,v,w)$$

$$+\frac{wx^2(1-vwx)^2}{v(vwx+v-1)}B\left(x,1-vwx,\frac{vw}{1-vwx}\right) - \frac{vwx^2}{vwx+v-1}B(x,v,w).$$

By this proposition, one may express $B(x,v,w)$ in terms the generating functions A, C, D and C, D in terms of A. Thus

$$\begin{aligned}
&\frac{vx-wx-v-x+1}{vwx+v-1}A^+(x,v,w)\\
&=\frac{w^2x^2(vwx-v-1)A^+(\frac{vwx}{1-vwx},1-vwx,1)}{vwx+vw-1}\\
&+\frac{vwx^2(w^2-1)(vwx-1)A^+(x,1-vwx,\frac{vw}{1-vwx})}{(vwx+v-1)(vwx+vw-1)}\\
&+w^2x^2(vwx-v-1)A(vwx,1,1)+x^2vw^2(vw^2x^2-vwx-1).
\end{aligned}$$

By lot of luck (Computer programming), we obtain

$$\begin{aligned}
&A^{+}(x,v,w)\\
&=\frac{(w^2-1)(1-x)vwx^2\sqrt{1-6vwx+v^2w^2x^2}\sqrt{(1-x)(1-x-4vwx)}}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}\\
&+\frac{(1-x-2vwx)vw(1-x)x^2\sqrt{1-6vwx+v^2w^2x^2}}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}\\
&+\frac{(1-2v^2x^2+4v^2x-2v^2-x^2-2x-2vwx(1-x))vw^3x^2\sqrt{1-6vwx+v^2w^2x^2}}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}\\
&+\frac{(1-w^2)(vwx^2-vwx-3x+1)vwx^2\sqrt{(1-x)(1-x-4vwx)}}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}\\
&+\frac{(3(w^2-1)x^2+4(1-2w)x-w^2+4w-1)vwx^2}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}\\
&-\frac{((w^2-1)x^3+8(w^2+1)x^2-(7w^2+4w+11)x+4w+4)v^2w^2x^2}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}\\
&-\frac{2((1-x)(w^2x^2+x^2+3x-3)+vwx(1-x)^2)v^3w^3x^2}{4(vwx-vw-2x+1)(vx-wx-v-x+1)}.
\end{aligned}$$

and

$$\begin{aligned}
&A^{-}(x,v,w)\\
&= \frac{((1+v-2vx)x + (v^2x - v^2 + vx - 1)wx - (1-x)(1-vx)vw^2)v^2wx^2\sqrt{1-6vwx+v^2w^2x}}{2(vwx-2vx-w+1)(vwx-vw-2x+1)}\\
&+ \frac{(6vx^2 - 3(v+1)x + 2 - (2v^2x^3 + 2v(v+1)x^2 - (3v^2+2v+3)x + 2v + 2)w)v^2wx^2}{2(vwx-2vx-w+1)(vwx-vw-2x+1)}\\
&+ \frac{(v^2x^3 - v^2x^2 + vx^3 + 3vx^2 - 3vx - x^2 - 3x + 3 - (1-x)(1-vx)vwx)v^3w^3x^2}{2(vwx-2vx-w+1)(vwx-vw-2x+1)}.
\end{aligned}$$

So we can state the following result: The generating function for the joint distribution of the first and second letter statistics on $\mathcal{S}_n(1243, 1423)$ for $n \geq 1$ is given by

$$A^+(x, v, w) + A^-(x, v, w).$$

Lin and Kim conjectured

Conjecture

Let (ν,μ) be a pair of patterns of length four. Then

$$\sum_{\pi\in\mathcal{S}_n(\sigma,\tau)} q^{\mathrm{desc}(\pi)} v^{\mathrm{first}(\pi)} = \sum_{e\in I_n(\geq,-,>)} q^{\mathrm{dist}(e)-1} v^{\mathrm{last}(e)}$$

for the following six pairs (ν,μ):

$$(1243,1324),(1243,1342),(1243,1423),(1324,1342),(1324,1423),(1342,1423).$$

As before with lot of tricks we **proved this conjecture**. In particular, we showed that the generating function over $n \geq 1$ of both sides of

$$\sum_{\pi\in\mathcal{S}_n(\sigma,\tau)} q^{\mathrm{desc}(\pi)} v^{\mathrm{first}(\pi)} = \sum_{e\in I_n(\geq,-,>)} q^{\mathrm{dist}(e)-1} v^{\mathrm{last}(e)}$$

is given by

$$\frac{vx}{1-vqx} + \frac{xv(vqx-v-x)t(xv)}{2(vq^2x^2-vqx^2-qx+vx-v-x+1)(vqx-vx-1)}$$
$$+\frac{(x+(qx^2+qx+3x^2-2x-1)v)vx}{2(vqx-vx-1)(vqx-1)(vq^2x^2-vqx^2-qx+vx-v-x+1)}$$
$$-\frac{(2q^2x^2+3q^2x-qx^2-qx-3q+2x-1+(1-2q)(qx-1)vqx)v^3x^2}{2(vqx-vx-1)(vqx-1)(vq^2x^2-vqx^2-qx+vx-v-x+1)},$$

where $t(x) = \sqrt{(1-2q)^2x^2-2x(1+2q)+1}$.

Work of Andrews and Chern: Further enumeration results concerning a recent equivalence of restricted inversion sequences (with Shattuck)

Lin conjectured the following equivalence involving the ascents statistic on the avoidance classes $I_n(\geq,\neq,>)$ and $I_n(>,\neq,\geq)$:

$$\sum_{e\in I_n(\geq,\neq,>)} q^{\mathrm{asc}(e)} = \sum_{e\in I_n(>,\neq,\geq)} q^{n-1-\mathrm{asc}(e)}, \qquad n\geq 1. \tag{1}$$

This equivalence was shown by Andrews and Chern using a functional equation approach.

Here, we consider some further combinatorial aspects of (**??**). In particular, we consider a refinement of both sides of (**??**) by introducing a variable p marking the number of descents in members of each class. We compute an explicit formula for the generating function of the joint distribution of desc and asc on $I_n(\geq, \neq, >)$, and also of desc and $n-1-$ asc on $I_n(>, \neq, \geq)$, using the kernel method.

Here we solve the case $\mathcal{A}_n = I_n(\geq, \neq, >)$.

We first define two new concepts related to the relative sizes of the non-ascent entries within an inversion sequence.

Let the **height** of $e = e_1 e_2 \cdots e_n \in I_n$ be given by

$$\mathrm{hgt}(e) = \max\{e_i : 1 \leq i \leq n-1 \text{ and } e_i \geq e_{i+1}\}.$$

If $a = \mathrm{hgt}(e)$ with $j \in [n-1]$ minimal such that $e_j = a$, then let the **depth** of e be defined as $\mathrm{dep}(e) = e_{j+1}$.

Let $e \in \mathcal{A}_n$ has height and depth values of a and b, respectively. If $a > b$, then there exists a single descent ab and at most two runs of the letter a, the first of which has length one. On the other hand, if $a = b$ within e, there can exist only a single run of a. Within a (maximal) subsequence of the form $ab \cdots b$, any letter beyond the second will be referred to as a **redundant bottom**, regardless of whether or not a and b are distinct.

We now decompose $\mathcal{A}_n$ into disjoint subsets as follows. Given $1 \leq i \leq n-1$ and $1 \leq j \leq n$, let $\mathcal{B}_n(i,j)$ denote the subset of $\mathcal{A}_n$ whose members have height i and last letter j, where the last letter is not a redundant bottom. Let $\mathcal{C}_n(i,j)$ be defined the same as $\mathcal{B}_n(i,j)$, but where the last letter is a redundant bottom. Note that $\mathcal{C}_n(i,j)$ can be nonempty only when $n \geq 3$ and $1 \leq j \leq i \leq n-2$. Define the distribution polynomial $b_n(i,j) = b_n(i,j;p,q)$ by

$$b_n(i,j) = \sum_{\pi \in \mathcal{B}_n(i,j)} p^{\mathrm{desc}(\pi)} q^{\mathrm{asc}(\pi)},$$

and likewise for $c_n(i,j) = c_n(i,j;p,q)$.

Let

$$b_n = \sum_{i=1}^{n-1} \sum_{j=1}^{n} b_n(i,j), \qquad n \geq 2,$$

and

$$c_n = \sum_{i=1}^{n-2} \sum_{j=1}^{i} c_n(i,j), \qquad n \geq 3,$$

and put $b_1 = 0$ and $c_1 = c_2 = 0$. Note that b_n and c_n are polynomials in p and q. Then we seek a formula for $a_n = a_n(p,q)$ defined as

$$a_n = b_n + c_n + q^{n-1}, \qquad n \geq 1.$$

Note that a_n gives the joint distribution of desc and asc on $\mathcal{A}_n$, where the q^{n-1} term accounts for the sequence $12\cdots n$ which belongs to no subset $\mathcal{B}_n(i,j)$ or $\mathcal{C}_n(i,j)$.

The arrays $b_n(i,j)$ and $c_n(i,j)$ satisfy the following system of recurrences.

lemma We have

$$\begin{aligned} b_n(i,j) &= \delta_{i,n-2} \cdot q^{n-2} + qc_{n-1}(i,i) + q\sum_{\ell=i+1}^{j-1} b_{n-1}(i,\ell) \\ &\quad + q\sum_{k=1}^{i-1} b_{n-2}(k,i) + q\sum_{\ell=1}^{i-1} c_n(i,\ell) \\ &\quad + q^2\sum_{\ell=1}^{i-1}\sum_{s=1}^{n-i-2} c_{n-s}(i,\ell), \quad 1 \le i \le n-2 \text{ and } i < j \le n, \end{aligned}$$

with $b_n(n-1,n) = 0$ for $n \geq 2$,

$$\begin{aligned}b_n(i,i) &= \delta_{i,n-1}\cdot q^{n-2}\\ &\quad+\sum_{k=1}^{i-1} b_{n-1}(k,i)+q\sum_{\ell=1}^{i-1}\sum_{s=1}^{n-i-1} c_{n-s+1}(i,\ell),\ 1\le i\le n-1,\end{aligned}$$

$$\begin{aligned}b_n(i,j) &= \delta_{i,n-1}\cdot pq^{n-2}+p\sum_{k=1}^{j} b_{n-1}(k,i)\\ &\quad+p\sum_{k=j+1}^{i-1}\sum_{s=0}^{i-k-1}\binom{i-k-1}{s}q^{s+1}c_{n-s-1}(k,j),\end{aligned}$$

for $\quad 1\le j<i\le n-1,$

$$c_n(i,j) = b_{n-1}(i,j) + c_{n-1}(i,j), \quad 1 \le j < i \le n-2,$$

$$c_n(i,i) = \delta_{i,n-2} \cdot q^{n-3} + c_{n-1}(i,i) + \sum_{k=1}^{i-1} b_{n-2}(k,i), \quad 1 \le i \le n-2.$$

Furthermore, we have the following recurrences

$$b_n(i,j) = \sum_{\ell=1}^{i} \sum_{t=0}^{j-i-1} \binom{j-i-1}{t} q^{t+1} c_{n-t}(i,\ell) + \sum_{\ell=1}^{i-1} \sum_{t=0}^{j-i-1} \sum_{s=1}^{n-i-t-2} \binom{j-i-1}{t} q^{t+2} c_{n-s-t}(i,\ell)$$

and

$$\begin{aligned}
&\frac{1}{pq}\Big(b_n(i,j) - q\sum_{k=j+1}^{i-1} b_{n-1}(k,j)\Big)\\
&= \delta_{j,n-2}\cdot q^{n-3} + \sum_{k=j+1}^{i-1} c_{n-1}(k,j) + \sum_{k=1}^{j}\sum_{\ell=1}^{k} c_{n-1}(k,\ell)\\
&\quad + \sum_{\ell=2}^{j}\sum_{k=1}^{\ell-1} b_{n-2}(k,\ell) + q\sum_{k=2}^{j}\sum_{\ell=1}^{k-1}\sum_{r=1}^{n-k-2} c_{n-r-1}(k,\ell).
\end{aligned}$$

Define now the following generating functions: $A(x) = \sum_{n\geq 1} a_n x^n$, $B(x) = \sum_{n\geq 2} b_n x^n$ and $C(x) = \sum_{n\geq 3} c_n x^n$. Then clearly,

$$A(x) = B(x) + C(x) + \frac{x}{1-xq}.$$

Note that $A(x)$ is the (ordinary) generating function for the joint distribution of desc and asc on $I_n(\geq, \neq, >)$ for $n \geq 1$. In order to study $B(x)$ and $C(x)$, we refine them as follows.

Define

$$B_0(x,v)=\sum_{n\geq2}\sum_{i=1}^{n-1}b_n(i,i)v^ix^n,$$

$$B^+(x,v,w)=\sum_{n\geq2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}b_n(i,j)v^iw^jx^n,$$

$$B^-(x,v,w)=\sum_{n\geq3}\sum_{i=2}^{n-1}\sum_{j=1}^{i-1}b_n(i,j)v^iw^jx^n,$$

with

$$C_0(x,v)=\sum_{n\geq3}\sum_{i=1}^{n-2}c_n(i,i)v^ix^n$$

and

$$C^-(x,v,w)=\sum_{n\geq4}\sum_{i=2}^{n-2}\sum_{j=1}^{i-1}c_n(i,j)v^iw^jx^n.$$

Translating our recurrences in terms of these generating functions yields the following system of functional equations.

Lemma We have $B(x) = B_0(x,1) + B^+(x,1,1) + B^-(x,1,1)$ and $C(x) = C_0(x,1) + C^-(x,1,1)$, where

$$\begin{aligned}
B^+(x,v,w) &= \frac{qvw^2x^3(w+1)}{1-qvwx} + \frac{qwx}{1-w}(C_0(x,vw) - wC_0(wx,v)) \\
&+ \frac{qwx}{1-w}(B^+(x,v,w) - wB^+(wx,v,1)) \\
&+ \frac{qwx^2}{1-w}(B^+(x,1,vw) - w^2B^+(wx,1,v)) \\
&+ \frac{qw}{1-w}(C^-(x,vw,1) - C^-(wx,v,1)) \\
&+ \frac{q^2w}{1-w}\left(\frac{x}{1-x}C^-(x,vw,1) - \frac{wx}{1-wx}C^-(wx,v,1)\right), \\
B_0(x,v) &= \frac{vx^2}{1-qvx} + xB^+(x,1,v) + \frac{q}{1-x}C^-(x,v,1), \\
B^-(x,v,w) &= \frac{pqv^2wx^3}{(1-qvx)(1-qvwx)} + \frac{px}{1-w}(B^+(x,w,v) - B^+(x,1,vw)) \\
&+ \frac{pqvx}{1-v-qvx}\left(C^-(x,v,w) - C^-\left(\frac{vx}{1-qvx}, 1-qvx, w\right)\right),
\end{aligned}$$

$$C^-(x,v,w)=\frac{x}{1-x}B^-(x,v,w),$$
$$C_0(x,v)=\frac{vx^3}{(1-x)(1-qvx)}+\frac{x^2}{1-x}B^+(x,1,v),$$
$$B^+(x,v,w)=\frac{qw}{1-w-qwx}\left(C_0(x,vw)-C_0(\frac{wx}{1-qwx},v(1-qwx))\right)$$
$$+\frac{qw}{1-w-qwx}\left(C^-(x,vw,1)-C^-(\frac{wx}{1-qwx},v(1-qwx),1)\right)$$
$$+\frac{q^2wx}{(1-x)(1-w-qwx)}\left(C^-(x,vw,1)-\frac{w(1-x)}{1-wx-qwx}C^-(\frac{wx}{1-qwx},v(1-qwx),1)\right)$$

and

$$
\begin{aligned}
&\frac{1}{pq}B^-(x,v,w)\\
&=\frac{vx}{p(1-v)}(B^-(x,v,w)-B^-(vx,1,w))+\frac{v^2wx^3}{1-qvwx}\\
&+\frac{vx}{1-v}(C^-(x,v,w)-C^-(vx,1,w))\\
&+\frac{x}{1-w}\Bigg(\frac{v}{1-v}(C^-(x,vw,1)+C_0(x,vw)-C^-(vx,w,1)-C_0(vx,w))\\
&-\frac{vw}{1-vw}(C^-(x,vw,1)+C_0(x,vw)-C^-(vwx,1,1)-C_0(vwx,1))\Bigg)\\
&+\frac{x^2}{1-w}\Bigg(\frac{v}{1-v}(B^+(x,1,vw)-vB^+(vx,1,w))-\frac{vw}{1-vw}(B^+(x,1,vw)-vwB^+(vwx,1,1))\Bigg)\\
&+qx^2\Bigg(\frac{v}{(1-v)(1-vw)(1-x)}C^-(x,vw,1)-\frac{v^2w}{(1-v)(1-vw)(1-vwx)}C^-(vwx,1,1)\\
&-\frac{v^2}{(1-v)(1-w)(1-vx)}C^-(vx,w,1)+\frac{v^2w}{(1-v)(1-w)(1-vwx)}C^-(vwx,1,1)\Bigg).
\end{aligned}
$$

Again, with computer programming, we showed the following.
Lemma The generating function $A(x)$ is given by

$$A(x)=\frac{1}{1-x}B^{+}(x,1,1)+\frac{1-x+qx}{(1-x)^2}B^{-}(x,1,1)+\frac{x}{(1-x)(1-qx)},$$

where

$$\begin{aligned}&K(x,v)B^{-}(x/v,1,v)\\&=A_1(x,v)B^{-}(x,1,1)+A_2(x,v)B^{+}(x,1,1)+A_3(x,v),\end{aligned}$$

with

$$\begin{aligned}
K(x,v) &= (x-v)(qx+v-1)(qvx+v^2-vx-v+x)\\
&\quad - qx^2(qvx-qx+v^2-vx-2v+2x)p,\\
A_1(x,v) &= (x-v)(qvx+v^2-vx-v+x)xq\\
&\quad + \frac{(v-x)x^2((qv-q-v+2)x^2+(v^2-v-2)x-v^2+2v)qp}{(1-x)^2},\\
A_2(x,v) &= \frac{(x-v)(1-v)(qx^2+vx-x^2-v+x)xp}{1-x},\\
A_3(x,v) &= \frac{(x-v)(qx^2-qx+vx-x^2-v+x+1)(1-v)pqx^3}{(1-x)(1-qx)}.
\end{aligned}$$

Note that the kernel equation $K(x, v) = 0$ (see prior lemma) has two power series solutions $v_1(x)$ and $v_2(x)$, where

$$v_1(x) = 1 + (\sqrt{pq} - q)x - \frac{q((2p+1)\sqrt{pq} - 2pq - p)}{2\sqrt{pq}}x^2 + \cdots,$$

$$v_2(x) = 1 - (\sqrt{pq} + q)x - \frac{q((2p+1)\sqrt{pq} + 2pq + p)}{2\sqrt{pq}}x^2 + \cdots.$$

Substituting $v = v_1(x)$ and $v = v_2(x)$ into

$$K(x,v)B^-(x/v,1,v)$$
$$= A_1(x,v)B^-(x,1,1) + A_2(x,v)B^+(x,1,1) + A_3(x,v),$$

and solving for $B^-(x,1,1)$ and $B^+(x,1,1)$, we obtain

$$B^-(x,1,1)$$
$$= \frac{px(v_2(x)-1)(v_1(x)-1)}{((q-1)x^2 - v_1(x)v_2(x) + xv_1(x) + xv_2(x))(p-q)},$$
$$B^+(x,1,1)$$
$$= \frac{qx(qx-1)(qx-x+1)(v_2(x)-1)(v_1(x)-1)}{(1-x)(1-qx)((q-1)x^2 - v_1(x)v_2(x) + xv_1(x) + xv_2(x))(p-q)}$$
$$+ \frac{qx^2((q-1)x(v_1(x)v_2(x)+x^2) + (x^2-x+1-qx)(v_1(x)+v_2(x)) + (1-2q)x^2 + (1}{(1-x)(1-qx)((q-1)x^2 - v_1(x)v_2(x) + xv_1(x) + xv_2(x))}$$

Hence, we have the following result.
The generating function $A(x)$ is given by

$$A(x)=\frac{(1-v_1(x)-v_2(x)-(q-1)x)x}{(q-1)x^2-v_1(x)v_2(x)+x(v_1(x)+v_2(x))}.$$

Refereneces

[1] Permutation patterns and cell decompositions (with Schork); this paper is part of the ACA 2017 Jerusalem Special Issue.
[2] Wilf classification of subsets of four letter patterns (with Schork)
[3] Wilf classification of subsets of eight and nine four-letter patterns (with Schork)
[5] Wilf classification of subsets of six and seven four-letter patterns (with Schork)
[5] Enumeration and Wilf-classification of permutations avoiding five patterns of length 4
[6] Enumeration of permutations avoiding a triple of 4-letter patterns is almost all done (with Callan and Shattuck)
[7] A Wilf class composed of 19 symmetry classes of quadruples of 4-letter patterns (with Arikan and Kilic)
[8] Enumeration of 2-Wilf Classes of Four 4-letter Patterns (with Callan)
[9] Enumeration of 3- and 4-Wilf classes of four 4-letter patterns (with Callan)
[10] A Wilf class composed of 7 symmetry classes of triples of 4-letter patterns (with Callan)

[11] On permutations avoiding 1324, 2143, and another 4-letter pattern (with Callan)
[12] On permutations avoiding 1243, 2134, and another 4-letter pattern (with Callan)
[13] Wilf classification of triples of 4-letter patterns I+II (with Callan and Shattuck)
[14] Enumeration of permutations avoiding a triple of 4-letter patterns is almost all done (with Callan and Shattuck)
[15] On a conjecture of Lin and Kim concerning a refinement of Schröder numbers (with Shattuck)
[16] Equivalence of the descents statistic on some (4,4)-avoidance classes of permutations (with Shattuck)
[17] Further enumeration results concerning a recent equivalence of restricted inversion sequences (with Shattuck)

Happy Birthday Amitai and Doron