A Residue-based Approach to Vanishing Sums

Shaoshi Chen

KLMM, AMSS
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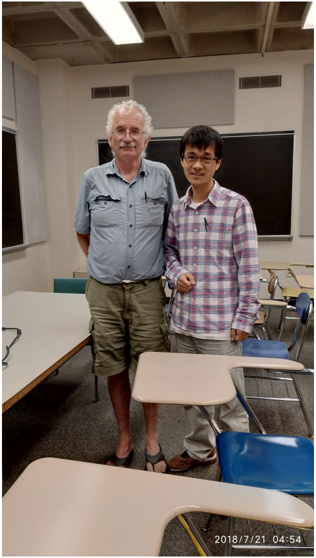
Combinatorics and Algebra from A to Z
July 26–29, 2021

A joint work with Rong-Hua Wang

Thanks to Doron!

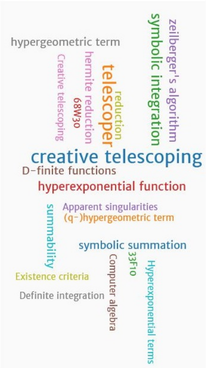


Thanks to Doron!

Author Citations for Doron Zeilberger
Doron Zeilberger is cited 2794 times by 1802 authors
in the MR Citation Database

Most Cited Publications	
Citations	**Publication**
489	**MR1379802 (97j:05001)** Petkovšek, Marko; Wilf, Herbert S.; Zeilberger, Doron $A = B$. With a foreword by Donald E. Knuth. With a separately available computer disk. *A K Peters, Ltd., Wellesley, MA,* 1996. xii+212 pp. ISBN: 1-56881-063-6 (Reviewer: Peter Paule) 05-01 (05A10 05A19 33C20 68R05) Book
146	**MR1090884 (92b:33014)** Zeilberger, Doron A holonomic systems approach to special functions identities. *J. Comput. Appl. Math.* 32 (1990), no. 3, 321–368. (Reviewer: R. A. Askey) 33C20 (05A10 33D20 68Q40)
140	**MR1163239 (93k:33010)** Wilf, Herbert S.; Zeilberger, Doron An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Invent. Math.* 108 (1992), no. 3, 575–633. (Reviewer: David R. Masson) 33C99 (03B35 05A19 33D99 68Q40)
123	**MR1103727 (92c:33005)** Zeilberger, Doron The method of creative telescoping. *J. Symbolic Comput.* 11 (1991), no. 3, 195–204. (Reviewer: R. A. Askey) 33C20
115	**MR1392498 (97d:05012)** Zeilberger, Doron Proof of the alternating sign matrix conjecture. The Foata Festschrift. *Electron. J. Combin.* 3 (1996), no. 2, Research Paper 13, approx. 84 pp. (Reviewer: David M. Bressoud) 05A10 (05A15 39A10)
68	**MR1048463 (91d:33006)** Zeilberger, Doron A fast algorithm for proving terminating hypergeometric identities. *Discrete Math.* 80 (1990), no. 2, 207–211. (Reviewer: R. A. Askey) 33C20 (05A19 11B65 11Y16 33C05)
66	**MR1487614 (99i:05137)** Foata, Dominique; Zeilberger, Doron A combinatorial proof of Bass's evaluations of the Ihara-Selberg zeta function for graphs. *Trans. Amer. Math. Soc.* 351 (1999), no. 6, 2257–2274. (Reviewer: H. M. Stark) 05C50 (11M41)
56	**MR1007910 (91a:05006)** Wilf, Herbert S.; Zeilberger, Doron Rational functions certify combinatorial identities. *J. Amer. Math. Soc.* 3 (1990), no. 1, 147–158. (Reviewer: Ira Gessel) 05A19 (33C05 68Q40)
49	**MR0808671 (87b:05015)** Wimp, Jet; Zeilberger, Doron Resurrecting the asymptotics of linear recurrences. *J. Math. Anal. Appl.* 111 (1985), no. 1, 162–176. (Reviewer: Edward A. Bender) 05A15 (39A10)
49	**MR1383799 (97c:05010)** Zeilberger, Doron Proof of the refined alternating sign matrix conjecture. *New York J. Math.* 2 (1996), 59–68, electronic. (Reviewer: Jiang Zeng) 05A15 (33D45 82B23)
	See All

My ACM Author Profile



Among 2794 citations, I contributed 106 in my 24 papers!

Thanks to Doron!

My 3 talks at Z's seminar:

1 / 20

October 20, 2011

Proof of the Wilf–Zeilberger Conjecture

Shaoshi Chen

Department of Mathematics
North Carolina State University, Raleigh

October 20, 2011
Rutgers Experimental Mathematics Seminar

Joint work with C. Koutschan and G. Payne

S. Chen Wilf–Zeilberger Conjecture

1 / 18

October 18, 2012

Telescopers for 3D Walks via Residues

Shaoshi Chen

Department of Mathematics
North Carolina State University, Raleigh

Rutgers Experimental Mathematics Seminar
October 18, 2012

Joint with Manuel Kauers and Michael F. Singer

Shaoshi Chen Telescopers and Residues

How to Generate All Possible Rational WZ-pairs?

Shaoshi Chen

KLMM, AMSS
Chinese Academy of Sciences

Rutgers Experimental Mathematics Seminar
Rutgers University, July 20, 2018

July 20, 2018

How to generate all possible WZ-pairs algorithmic...
vimeo.com

Happy Birthday to Amitai and Doron!

I wish both of you long life and happiness!

福如东海，寿比南山！

Wilf–Zeilberger theory

In the early 1990s, Wilf and Zeilberger developed an algorithmic theory for proving identities in combinatorics and special functions.

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$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

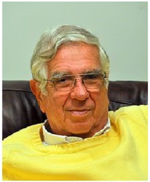
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2$$

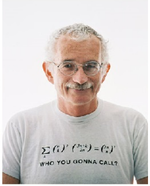
$$\int_0^{\infty} x^{\alpha-1} Li_n(-xy)dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

$$\int_{-1}^{+1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}dx = (-1)^n \pi I_n(p)$$

$$\dots$$



Herbert Wilf



Doron Zeilberger

Telescoping

Problem. For a sequence $f(k)$ in some class $\mathfrak{S}(k)$, decide whether there exists $g(k) \in \mathfrak{S}(k)$ s.t.

$$f(k) = g(k+1) - g(k)$$

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Examples.

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Examples.

- Rational sums

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Delta_k\left(-\frac{1}{k}\right) = 1 - \frac{1}{n+1}$$

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- Hypergeometric sums

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \sum_{k=0}^{n} \Delta_k \left(\frac{4k\binom{2k}{k}^2}{4^{2k}}\right) = \frac{4(n+1)\binom{2n+2}{n+1}^2}{4^{2n+2}}$$

Creative telescoping

Problem. For a sequence $f(n,k)$ in some class $\mathfrak{S}(n,k)$, find a linear recurrence operator $L \in \mathbb{F}[n, S_n]$ and $g \in \mathfrak{S}(n,k)$ s.t.

$$\underbrace{L(n,S_n)}_{\text{Telescoper}}(f) = \Delta_k(g)$$

Call g the certificate for L.

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Example. Let $f(n,k) = \binom{n}{k}^2$. Then a telescoper for f and its certificate are

$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}$$

Proving identities

$$F(n) := \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

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Creative telescoping for $f = \binom{n}{k}^2$: $L(f) = \Delta_k(g)$, where

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Since $f(n,k) = 0$ when $k < 0$ or $k > n$, we have

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2$$

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Taking sums on both sides of $L(f) = \Delta_k(g)$:

$$\sum_{k=-\infty}^{+\infty} L(f) = L\left(\sum_{k=-\infty}^{+\infty} f\right) = g(n,+\infty) - g(n,-\infty) = 0$$

Proving identities

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Creative telescoping for $f = \binom{n}{k}^2$: $L(f) = \Delta_k(g)$, where

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$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}$$

The sequence $F(n)$ satisfies

$$(n+1)F(n+1) - (4n+2)F(n) = 0$$

Proving identities

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Creative telescoping for $f = \binom{n}{k}^2$: $L(f) = \Delta_k(g)$, where

$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}$$

Verify the initial condition:

$$F(1) = 2 = \binom{2}{1}$$

Then the identity is proved!

Example: Identity about Harmonic Numbers

$$\sum_{k=1}^{n} \underbrace{(-1)^{k+1}\frac{1}{k}\binom{n}{k}}_{F(n,k)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \triangleq H_n.$$

1 Creative telescoping for $F(n,k)$ yields $L(n,S_n)(F) = \Delta_k(G)$ with

$$L = S_n - 1 \quad \text{and} \quad G = \frac{(-1)^k}{n+1}\binom{n}{k-1}.$$

2 Summing both sides of $L(F) = \Delta_k(G)$ for k from 1 to n gets

$$\sum_{k=1}^{n} L(F) = L\left(\sum_{k=1}^{n} F\right) - F(n+1,n+1) = \sum_{k=1}^{n} \Delta_k(G)$$

$$= G(n,n+1) - G(n,1) \quad \Rightarrow \quad L\left(\sum_{k=1}^{n} F\right) = \frac{1}{n+1}$$

Example: Identity on T-shirt

C:\Users\schen\Desktop\Example-WZmethod.mw* - [Server 1] - Maple 17

> $with(SumTools) : with(Hypergeometric) :$

Identity: $\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 \binom{3\cdot n+k}{2\cdot n} = \binom{3\cdot n}{n}^2}$

> $f := \text{binomial}(n, k)^2 \cdot \text{binomial}(3\cdot n+k, 2\cdot n);$

$$f := \text{binomial}(n, k)^2 \,\text{binomial}(3\, n+k, 2\, n) \tag{1}$$

> $RHS := \boxed{ZeilbergerRecurrence(f, n, k, y, 0..n);}$

$$RHS := (-729\, n^4 - 1458\, n^3 - 1053\, n^2 - 324\, n - 36)\, y(n) + (16\, n^4 + 48\, n^3 + 52\, n^2 + 24\, n + 4)\, y(n+1) = 0 \tag{2}$$

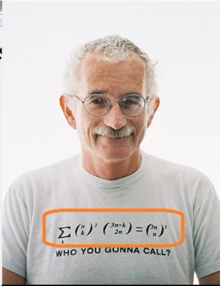
> $LHS := \text{binomial}(3\cdot n, n)^2;$

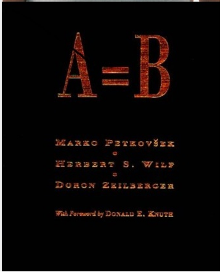
$$LHS := \text{binomial}(3\, n, n)^2 \tag{3}$$

> $normal(expand((-729\, n^4 - 1458\, n^3 - 1053\, n^2 - 324\, n - 36)\cdot LHS + (16\, n^4 + 48\, n^3 + 52\, n^2 + 24\, n + 4) \cdot eval(LHS, n = n+1)));$

$$0 \tag{4}$$

>





Integrability criterion via residues

Problem: Given $f \in \mathbb{C}(x)$, decide whether

$$f = D_x(g) \quad \text{for some } g \in \mathbb{C}(x).$$

Partial fraction decomposition:

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(x-\beta_i)^j}, \quad \text{where } p \in \mathbb{C}[x], \alpha_{i,j}, \beta_i \in \mathbb{C}.$$

Def. The residue of f at β_i is $\alpha_{i,1}$, denoted by $\operatorname{res}_x(f, \beta_i)$.

Theorem.

$$f = D_x(g) \ \text{ for some } g \in \mathbb{C}(x) \quad \Leftrightarrow \quad \text{All residues of } f \text{ are zero.}$$

Discrete residues

Def. For $\beta \in \mathbb{C}$, the $\mathbb{Z}$-orbit of β in $\mathbb{C}$ is

$$[\beta] := \{\beta + i \mid i \in \mathbb{Z}\}.$$

Partial fraction decomposition:

$$f = p + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(x - \beta_i + \ell)^j}, \tag{1}$$

where $p \in \mathbb{C}[x]$, $m, n_i, d_{i,j} \in \mathbb{N}$, $\alpha_{i,j,\ell}, \beta_i \in \mathbb{C}$ and β_i's are in distinct $\mathbb{Z}$-orbits.

Def. Let $f \in \mathbb{C}(x)$ be of the form (1). The sum $\sum_{\ell=0}^{d_{i,j}} \alpha_{i,j,\ell} \in \mathbb{C}$ is called the discrete residue of f at the $\mathbb{Z}$-orbit $[\beta_i]$ of multiplicity j, denoted by $\mathrm{dres}_x(f, [\beta_i], j)$.

Summability criterion via residues

Problem: Given $f \in \mathbb{C}(x)$, decide whether

$$f = \Delta_x(g) \quad \text{for some } g \in \mathbb{C}(x).$$

If g exists, we say f is summable in $\mathbb{C}(x)$.

Theorem (ChenSinger2012). Let $f = a/b \in \mathbb{C}(x)$ be s.t. $a, b \in \mathbb{C}[x]$ and $\gcd(a,b) = 1$. Then

$$f \text{ is summable in } \mathbb{C}(x) \quad \Leftrightarrow \quad \mathrm{dres}_x(f, [\beta], j) = 0$$

for any $\mathbb{Z}$-orbit $[\beta]$ with $b(\beta) = 0$ of any multiplicity j.

Example. Consider

$$f = \frac{1}{x(x+1)(x+2)} = \frac{1/2}{x+2} + \frac{-1}{x+1} + \frac{1/2}{x}.$$

Then $\mathrm{dres}_x(f, [0], 1) = 1/2 - 1 + 1/2 = 0$. So f is summable.

Integrability and summability of monomials

Continuous case: For any $\beta \in \mathbb{C}$ and $m \in \mathbb{Z}$, we have

$$(x-\beta)^m = D_x(f) \text{ for some } f \in \mathbb{C}(x) \quad \Leftrightarrow \quad m \neq -1.$$

If $m \neq -1$, we have $f = (m+1)^{-1}(x-\beta)^{m+1}$.

Definition: For $p \in \mathbb{F}[x]$ and $m \in \mathbb{Z}$, we define

$$p(x)^{\underline{m}} = \begin{cases} p(x)p(x-1)\cdots p(x-m+1), & m > 0; \\ 1, & m = 0; \\ \frac{1}{p(x+1)\cdots p(x-m)}, & m < 0. \end{cases}$$

Discrete case: For any $\beta \in \mathbb{C}$ and $m \in \mathbb{Z}$, we have

$$(x-\beta)^{\underline{m}} = \Delta_x(f) \text{ for some } f \in \mathbb{C}(x) \quad \Leftrightarrow \quad m \neq -1.$$

If $m \neq -1$, we have $f = (m+1)^{-1}(x-\beta)^{\underline{m+1}}$.

Nicole's theorem

Theorem (Nicole, 1717, *Traité du calcul des différences finies*).
Let $P \in \mathbb{C}[x]$ be such that $\deg(P) \le n-2$. Then

$$f = \frac{P(x)}{(x+\beta_1)\cdots(x+\beta_n)}, \quad \text{where } \beta_i - \beta_j \in \mathbb{Z}\setminus\{0\} \text{ for } i \neq j,$$

is summable in $\mathbb{C}(x)$.

Proof. By the assumption on the β_i, we can write

$$f = \frac{\tilde{P}(x)}{(x-\beta)^{\underline{m}}}, \quad \text{where } m \ge n,\ \beta \in \mathbb{C}, \text{ and } \deg_x \tilde{P}) \le m-2.$$

Since $\deg_x(\tilde{P}) \le m-2$, we have

$$\tilde{P} = \sum_{i=0}^{m-2} c_i (x-\beta)^{\underline{i}} \quad \Rightarrow \quad \frac{\tilde{P}}{(x-\beta)^{\underline{m}}} = \sum_{i=0}^{m-2} c_i (x-\beta-m)^{\underline{i-m}}$$

Note that $i-m \neq -1$ if $0 \le i \le m-2$.

Residue theorem and Lagrange's formula

Theorem. If f is holomorphic in an open set containing a circle C and its interior, except for poles at $\alpha_1, \ldots, \alpha_m$ inside C . Then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{i=1}^{m} \mathrm{res}_z(f, \alpha_i).$$

If $f(z)$ is also holomorphic outside C, then

$$\sum_{i=1}^{m} \mathrm{res}_z(f, \alpha_i) = -\mathrm{res}_z(f, \infty) \quad \text{where } \mathrm{res}_z(f, \infty) \triangleq \int_C \frac{f(1/z)}{z^2} dz.$$

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If f has a pole of order n at α_0, then

$$\mathrm{res}_z(f, \alpha_0) = \lim_{z\to\alpha_0} \frac{1}{(n-1)!} D_z^{n-1} (z-\alpha_0)^n f(z).$$

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If $f = P/Q$ with $P, Q \in \mathbb{C}[z]$ and Q squarefree, then for any $z_0 \in \mathbb{C}$ with $Q(z_0) = 0$, we have

$$\mathrm{res}_z(f,\alpha_0) = \frac{P(\alpha_0)}{D_z(Q)(\alpha_0)} \qquad \text{Lagrange's formula.}$$

Residue theorem and Lagrange's formula

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If $f = P/Q$ with $P, Q \in \mathbb{C}[z]$ and $\deg_z(P) \le \deg_z(Q) - 2$, then

$$\mathrm{res}_z(f, \infty) = \int_C \frac{f(1/z)}{z^2} dz = 0,$$

where all zeros of Q are inside the circle C.

Proof of Nicole's theorem via residues

Theorem (Nicole, 1717). Let $P \in \mathbb{C}[x]$ be such that $\deg(P) \leq n-2$. Then

$$f = \frac{P(x)}{(x+\beta_1)\cdots(x+\beta_n)}, \quad \text{where } \beta_i - \beta_j \in \mathbb{Z}\setminus\{0\} \text{ for } i \neq j,$$

is summable in $\mathbb{C}(x)$.

Proof. By Partial fraction decomposition,

$$f = \sum_{i=1}^{n} \frac{\alpha_i}{x+\beta_i}, \quad \text{where } \alpha_i \in \mathbb{C}.$$

Then f is summable iff $\sum_{i=1}^{n} \alpha_i = 0$. By Cauchy's residue theorem,

$$\sum_{i=1}^{n} \alpha_i = -\mathrm{res}_x(f,\infty) = -\frac{1}{2\pi i}\oint_{\Gamma_0} f\left(\frac{1}{x}\right) d\frac{1}{x} = \frac{1}{2\pi i}\oint_{\Gamma_0} \underbrace{\frac{P(1/x)x^{n-2}}{(1+\beta_1 x)\cdots(1+\beta_n x)}}_{\tilde{f}} dx.$$

Since $\deg(P) \leq n-2$, $\tilde{f}$ is analytic at 0 and then $\mathrm{res}_x(f,\infty) = 0$.

From Nicole's theorem to vanishing sums

Corollary. Let $P \in \mathbb{C}[x]$ be such that $\deg(P) \leq n-1$. Then

$$\sum_{k=0}^{n} r_k = 0, \quad \text{where } r_k = \frac{P(-k)}{(-1)^k k!(n-k)!}.$$

Proof. Consider the rational function

$$f = \frac{P(x)}{x(x+1)\cdots(x+n)}.$$

By Nicole's theorem, f is summable in $\mathbb{C}(x)$. Then Lagrange's formula implies that

$$\mathsf{dres}_x(f,[0],1) = \sum_{k=0}^{n} \mathsf{res}_x(f,-k) = \sum_{k=0}^{n} \frac{P(-k)}{(-1)^k k!(n-k)!} = 0.$$

Vanishing sums via residues

Example.

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

Consider the rational function

$$f = \frac{P}{Q} = \frac{n!}{x(x+1)\cdots(x+n)} = \sum_{i=0}^{n} \frac{\alpha_i}{x+i}.$$

By Lagrange's formula, $\operatorname{res}_x(f, -i) = P(-i)/D_x(Q)(-i)$,

$$\alpha_i = \frac{n!}{(-i)(-i+1)\cdots(-1)\cdot 1\cdot 2\cdots(-i+n)} = (-1)^i \binom{n}{i}.$$

By Nicole's theorem, f is summable if $n \geq 1$. Then

$$\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

Vanishing sums via residues

Example.

$$\sum_{i=0}^{n} \binom{2i}{i} \binom{2n-2i}{n-i} \frac{1}{2i-1} = 0.$$

Consider the rational function

$$f = \frac{P}{Q} = \frac{-2^n \prod_{i=1}^{n-1} (2(x+i)+1)}{x(x+1)\cdots(x+n)} = \sum_{i=0}^{n} \frac{\alpha_i}{x+i}.$$

By Lagrange's formula, $\mathrm{res}_x(f,i) = P(i)/D_x(Q)(i)$,

$$\alpha_i = \binom{2i}{i} \binom{2n-2i}{n-i} \frac{1}{2i-1}.$$

By Nicole's theorem, f is summable if $n \geq 1$. Then

$$\sum_{i=0}^{n} \alpha_i = \sum_{i=0}^{n} \binom{2i}{i} \binom{2n-2i}{n-i} \frac{1}{2i-1} = 0.$$

Vanishing sums via residues

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^j=0,\quad \text{where } 0\le j<n.$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}p(k)=0,\quad \text{where } p\in\mathbb{C}[x] \text{ with } 0\le \deg(p)<n.$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+x}{n-k}\binom{k+x+1}{k}=0,\quad \text{where } n\ge 2.$$

$$\sum_{k=0}^{2n-1}(-1)^{k-1}\frac{n-k}{\binom{2n}{k}}=0.$$

$$\sum_{k=0}^{2n+1}(-1)^{k-1}\binom{2k}{k}\binom{4n-2k+2}{2n-k+1}=0$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{m+k}{k}}+\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{1}{\binom{n+k}{k}}=0.$$

$$\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\binom{2k}{k}\frac{1}{\binom{n+k+1}{k}}=0$$

Remark. Related work by G. P. Egorychev, I.-Ch. Huang, Z.G. Liu etc.

How to find rational functions for vanishing sums

To show the identity of the form

$$\sum_{k=0}^{n} a(n,k) = 0,$$

where $a(n,k)$ are hypergeometric in n and k. Consider

$$\sum_{k=0}^{n} \frac{a(n,k)}{x+k} = \frac{P_n(x)}{x(x+1)\cdots(x+n)}.$$

It suffices to show that $\deg_x(P_n(x)) \leq n-1$.

Idea. Find a linear recurrence in n for $P_n(x)$ and estimate the degree recursively.

How to find rational functions for vanishing sums

Example (Gould's *Combinatorial Identities*, page 63).

$$\sum_{k=0}^{2n+1} a(n,k) = 0, \quad \text{where } a(n,k) = (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n+k+1}{k}}.$$

Consider

$$\sum_{k=0}^{2n+1} \frac{a(n,k)}{x+k} = \frac{P_n(x)}{x(x+1)\cdots(x+2n+1)}.$$

By creative telescoping, we get

$$b_3 P_{n+3}(x) + b_2 P_{n+2}(x) + b_1 P_{n+1}(x) + b_0 P_n(x) = 0,$$

where $b_3, b_2, b_1, b_0 \in \mathbb{Q}[n,x]$ with

$$(\deg_x(b_3), \deg_x(b_2), \deg_x(b_1), \deg_x(b_0)) = (2,4,6,5).$$

Since $(\deg_x(P_1), \deg_x(P_2), \deg_x(P_3)) = (2,4,6)$, we get $\deg_x(P_n(x)) \leq 2n$.

Vanishing sums from orthogonal polynomials

Example. Laguerre polynomials

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!}.$$

Consider the rational function

$$f = \frac{L_{n-1}^{\alpha}(x)}{x(x+1)\cdots(x+n)} = \sum_{k=0}^{n} \frac{a(n,k)}{x+k} \quad \Rightarrow \quad \sum_{k=0}^{n} a(n,k) = 0.$$

When $\alpha = 0$, we get

$$\sum_{k=0}^{n} \sum_{j=0}^{n-1} \binom{n}{k} (-1)^k \frac{k^j}{j!} = 0.$$

Using Legendre polynomial yields

$$\sum_{k=0}^{n+1} \sum_{j=0}^{\lceil n/2 \rceil} (-1)^k \binom{n}{k} \binom{n}{j} \binom{2n-2j}{n-2j} (-k)^{n-2j} = 0.$$

Creative vanishing sums

Example.

$$F_n := \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^n = (-1)^n n!.$$

Let $L = S_n + (n+1)$. Then

$$\begin{aligned} L\left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^n\right) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k k^{n+1} + (n+1) \sum_{k=0}^{n+1} \binom{n}{k} (-1)^k k^n \\ &= \sum_{k=0}^{n+1} (-1)^k k^n \left(k\binom{n+1}{k} + (n+1)\binom{n}{k}\right) \\ &= \sum_{k=0}^{n+1} (-1)^k k^n (n+1) k \binom{n+1}{k}. \end{aligned}$$

Using Nicole's theorem, we get $L(F_n) = 0$.

Remark. Note that k^n is not holonomic.

Summary

Theorem (Nicole, 1717). Let $P \in \mathbb{C}[x]$ be s.t. $\deg(P) \leq n-2$. Then

$$f = \frac{P(x)}{(x+\beta_1)\cdots(x+\beta_n)}, \quad \text{where } \beta_i - \beta_j \in \mathbb{Z}\setminus\{0\} \text{ for } i \neq j,$$

is summable in $\mathbb{C}(x)$.

Applications. Proving and discovering vanishing sums via residues:

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} \binom{2k}{k} \frac{1}{\binom{n+k+1}{k}} = 0.$$

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Thank you!

Holonomic Polynomial Sequences I.

Degree Growth

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