Log-convexity of P-recursive sequences

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Joint work with Zuoru Zhang and Guojie Li

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Outline

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What we focus on?

Inequalities

Let $\{a_n\}_{n\geq 0}$ be a sequence.

- If
$$a_n^2 \leq a_{n-1}a_{n+1} \quad \forall\, n \geq 1,$$
  we say that $\{a_n\}_{n\geq 0}$ is log-convex.

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Example

$\{n!\}_{n\geq 0} = 1, 1, 2, 6, 24, \ldots$ *is log-convex since*

$$n!^2 = \frac{n}{n+1}(n-1)!(n+1)! < (n-1)!(n+1)!.$$

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Example

$\{\binom{N}{n}\}_{n\geq 0} = 1, 4, 6, 4, 1, 0, 0, \ldots$ *is log-concave since*

$$\binom{N}{n}^2 = \left(1 + \frac{N+1}{n(N-n)}\right)\binom{N}{n-1}\binom{N}{n+1} \geq \binom{N}{n-1}\binom{N}{n+1}.$$

What we focus on?

- Let φ, ψ be the operator given by

$$\varphi\{a_n\}_{n\geq 0} = \{a_n a_{n+2} - a_{n+1}^2\}_{n\geq 0}, \quad \psi\{a_n\}_{n\geq 0} = \{a_{n+1}^2 - a_n a_{n+2}\}_{n\geq 0}$$

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- *r-log-convex*

$$\varphi\{a_n\}_{n\geq 0}, \quad \varphi^2\{a_n\}_{n\geq 0}, \quad \ldots, \quad \varphi^r\{a_n\}_{n\geq 0}$$

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  are all non-negative sequences.

- Higher order Turán inequalities

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 \geq 0.$$

Why we study these properties?

They are interesting.

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$$\begin{gathered}\text{Riemann hypothesis}\\ \Updownarrow\\ \text{real-rootness of certain Jensen polynomials}\\ \Downarrow\\ \text{higher order Turán inequality}\end{gathered}$$

- W. Chen, D.X.Q. Jia, L.X.W. Wang. Higher order Turán inequalities for the partition function, *Trans. Amer. Math. Soc.*, 2019.
- M. Griffin, K. Ono, L. Rolen, D. Zagier. Jensen polynomials for the Riemann zeta function and other sequences, *PNAS*, 2019.

How to prove

- Algebraic method
  - P. Brändén. Polynomials with the half-plane property and matroid theory. *Adv. Math.*, 2007.
  - L. Liu and Y. Wang. On the log-convexity of combinatorial sequences. *Adv. Appl. Math.*, 2007.

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  W.Y.C. Chen, L.X.W. Wang, and G.Y.B. Xie, Finite differences of the logarithm of the partition function. *Math. Comp.*, 2016.
- Computer proof

  M. Kauers and P. Paule, A computer proof of Molls log-concavity conjecture, *Proc. Amer. Math. Soc.* 2007.

Our approach

Prove the r-log-convexity by aid of computer.

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- Guess an upper bound and a lower bound for $u_n = a_{n+1}a_{n-1}/a_n^2$.
- Show that they are really the bounds.
- Find the relation of u_n and $\hat{u}_n = b_{n+1}b_{n-1}/b_n^2$ with $\{b_n\} = \varphi\{a_n\}$.

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We focus on two kind of sequences.

- P-recursive sequences
- the partition function $p(n)$

Outline

1. Introduction
2. P-recursive sequences
3. The partition function

P-recursive sequences

- Recurrence relation with polynomials as coefficients:

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0.$$

P-recursive sequences

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$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_d(n)a_{n+d} = 0.$$

- It must be a linear combination of $e^{Q(\rho,n)}s(\rho,n)$, where

$$Q(\rho,n) = \mu_0 n \log n + \sum_{j=1}^{\rho} \mu_j n^{j/\rho},$$

and

$$s(\rho,n) = n^r \sum_{j=0}^{t-1} (\log n)^j \sum_{s=0}^{M-1} b_{sj} n^{-s/\rho},$$

with $\rho, t, M \in \mathbb{Z}^+$ and $\mu_j, r, b_{sj} \in \mathbb{C}$.

G. D. Birkhoff and W. J. Trjitzinsky, Analytic theory of singular difference equations, *Acta Math.*, 1933.

J. Wimp and D. Zeilberger, Resurrecting the asymptotics of linear recurrences, *J. Math. Anal. Appl.*, 1985.

P-recursive sequences

Theorem

Let $\{a_n\}_{n\geq 0}$ be a P-recursive sequence with asymptotic

$$e^{Q(\rho,n)}s(\rho,n).$$

Then

$$u_n = \frac{a_{n-1}a_{n+1}}{a_n^2} = 1 + \sum_{i=1}^{m} \frac{r_i(\log n)}{n^{\alpha_i}} + o\left(\frac{1}{n^{\beta}}\right),$$

where m is a nonnegative integer, α_i are real numbers, $r_i(x)$ are rational functions of x and

$$0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \beta.$$

Examples

Let I_n be the number of involutions on $\{1, \ldots, n\}$. We have

$$I_n = I_{n-1} + (n-1)I_{n-2}.$$

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We find that

$$a_n = c \cdot e^{-n/2+\sqrt{n}} n^{n/2} \left(1 + \frac{7}{24\sqrt{n}} - \frac{119}{1152n} + o\left(\frac{1}{n}\right)\right).$$

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$$u_n = 1 + \frac{1}{2n} - \frac{1}{4n^{3/2}} + \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right),$$

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Similarly, $a_n = I_n/n!$ satisfies

$$u_n = 1 - \frac{1}{2n} - \frac{1}{4n^{3/2}} + \frac{5}{8n^2} + o\left(\frac{1}{n^2}\right),$$

Guess the bounds for r_n

With the asymptotic expression of a_n, we may guess the bounds for $r_n = a_{n+1}/a_n$.

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Example

Let M_n be the n-th Motzkin number. Then

$$M_n \approx C \cdot \frac{3^n}{n^{3/2}} \left(1 - \frac{39}{16n} + \frac{2665}{512n^2}\right)$$

$$\Longrightarrow r_n = \frac{M_{n+1}}{M_n} \approx 3 - \frac{9}{2n} + \frac{207}{16n^2}$$

$$\Longrightarrow f_n = 3 - \frac{9}{2n} + \frac{191}{16n^2} < r_n < 3 - \frac{9}{2n} + \frac{223}{16n^2} = g_n.$$

Prove the bounds for r_n

By the recurrence relation of a_n, we can show

$$f_n < r_n < g_n \implies f_{n+1} < r_{n+1} < g_{n+1}, \quad n \geq N.$$

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Example

$$(n+4)M_{n+2} - (2n+5)M_{n+1} - 3(n+1)M_n = 0$$

$$\implies (n+4)r_{n+1}r_n - (2n+5)r_n - 3(n+1) = 0$$

$$\implies r_{n+1} = \frac{2n+5}{n+4} + \frac{3(n+1)}{(n+4)r_n} > \frac{2n+5}{n+4} + \frac{3(n+1)}{(n+4)g_n}.$$

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For $n \geq 75$, we have

$$\frac{2n+5}{n+4} + \frac{3(n+1)}{(n+4)g_n} > f_{n+1}.$$

r-log-convexity

Lemma

$$\hat{u}_n = u_{n+1}^2 \frac{(u_n - 1)(u_{n+2} - 1)}{(u_{n+1} - 1)^2}.$$

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Finally, we follow the process

$$r_n \implies u_n \implies \hat{u}_n \implies \cdots \implies u_n^{(r)} > 1.$$

An example

```
In[38]:= L = (n + 4) N^2 - (2 n + 5) N - 3 (n + 1);

In[65]:= rLogBound[L, n, N, {1, 1, 2, 4}, 2, 5]
```

$1+\frac{1069}{32\,n^4}-\frac{63}{8\,n^3}+\frac{3}{2\,n^2}$<=s[1]<=$1+\frac{1133}{32\,n^4}-\frac{63}{8\,n^3}+\frac{3}{2\,n^2}$ for n>=1

$3+\frac{116039}{1024\,n^4}-\frac{153}{4\,n^3}+\frac{207}{16\,n^2}-\frac{9}{2\,n}$<=a_ {n+1}/a_n<=

$3+\frac{118087}{1024\,n^4}-\frac{153}{4\,n^3}+\frac{207}{16\,n^2}-\frac{9}{2\,n}$ for n>=428

```
a_n preserves the bounds for n>=685

the bounds hold for n>=340

Out[65]=
{True, 685}
```

Outline

Bounds for $p(n)$

We utilize the Hardy-Ramanujan-Rademacher formula and the error estimation given by Lehmer to derive an estimation for $p(n)$.

$$p(n) = T(n) + R(n), \quad T(n) = \frac{d}{\mu(n)^2}\left(1 - \frac{1}{\mu(n)}\right) e^{\mu(n)}.$$

where

$$d = \frac{\pi^2}{6\sqrt{3}}, \quad \mu(n) = \frac{\pi}{6}\sqrt{24n-1}.$$

We have

$$\left|\frac{R(n)}{T(n)}\right| < 2e^{-\mu(n)/2}, \quad n \geq 30.$$

```
In[79]:= pnsBound[5, z]
```

Out[79]=

$$\left\{1+\frac{-\frac{\pi^4}{3}+\frac{\pi^6}{9}}{z^5}+\frac{4\,\pi^4}{9\,z^4}-\frac{\pi^4}{9\,z^3},\ \frac{4422.31}{z^6},\ \frac{2005.85}{z^6},\ 176\right\}$$

For $n \geq 176$, we have

$$1-\frac{\pi^4}{9z^3}+\frac{4\pi^4}{9z^4}+\frac{-\frac{\pi^4}{3}+\frac{\pi^6}{9}}{z^5}-\frac{4423}{z^6}<u_n=\frac{p(n-1)p(n+1)}{p(n)^2}<$$
$$1-\frac{\pi^4}{9z^3}+\frac{4\pi^4}{9z^4}+\frac{-\frac{\pi^4}{3}+\frac{\pi^6}{9}}{z^5}+\frac{2006}{z^6},$$

where

$$z=\frac{\pi}{6}\sqrt{24n-1}.$$

Higher order Turán inequalities

```
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Out[79]=
```

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$$z=\frac{\pi}{6}\sqrt{24n-1}.$$

Theorem (Chen-Jia-Wang)

For $n \geq 95$, $p(n)$ satisfies the higher order Turán inequality.

Thanks for your attention !

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Happy Birthday !