27.7.2012

The birthday conference in honor of Amitai Regev and Doron Zeilberger

Conjugacy classes and representations of Hopf algebras and their Quantum Doubles

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Semisimple Hopf algebras over algebraically closed fields of characteristic 0 have an elusive nature. On one hand they resemble finite groups (or group algebras) in many aspects. As a result, many of the techniques in dealing with Hopf algebras try to use methods of group theory and their representation theory, some times with a great deal of success. On the other hand many elementary facts about finite groups are either hard to translate or even false.

A common key feature of groups and Hopf algebras is that the tensor product $V \otimes W$ of two H-module becomes an H-module in a non-trivial way, by using the coalgebra structure of H.

This resemblance is even stronger when (H, R) is **quasitriangular**. In this case,

$$V \otimes W \cong W \otimes V$$

as H-modules via the R-matrix, $R \in H \otimes H$. This is, in a certain sense, the essence of being quasitriangular.

Throughout, $(H, \mu, \Delta, S, \varepsilon)$ is a semisimple (hence finite dimensional) Hopf algebra over an algebraically closed field of characteristic 0 with multiplication μ, comultiplication Δ,

$$\Delta : H \to H \otimes H, \ \Delta(h) = \sum h_1 \otimes h_2,$$

an antipode $S : H \to H$ and an augmentation map $\varepsilon : H \to k$. Its dual

$$(Hom(H,k) = H^*, \Delta^*, \mu^*, S^* = s, \varepsilon^* = 1)$$

is a Hopf algebra (which is also semisimple) as well.
The Drinfeld (or quantum) double of H, $D(H)$ is the tensor product algebra

$$D(H) = H^{*cop} \otimes H$$

with induced canonical Hopf algebra structure. $D(H)$ is always quasitriangular via the image of the identity under the isomorphism

$$Hom_k(H,H) \cong H \otimes H^*.$$

When H is semisimple, so is $D(H)$.
We denote $p \bowtie h \in D(H)$ where $p \in H^*$, $h \in H$.

When applied the double construction to kG with $\{p_g\}_{g\in G}$ (the g-th projection) as a basis of kG^* dual to $\{g\}_{g\in G}$, we have

$$(p_h \bowtie g)(p_{h'} \bowtie g') = \delta_{h,gh'g^{-1}}(p_h \bowtie gg')$$

for all $g, g', h, h' \in G$.

Two module structures play a central role in this context:

1. H is a left H^*-module via the *hit* action $\leftharpoonup$ defined for all $a \in H$, $p \in H^*$,

$$\left\langle a \leftharpoonup p, p' \right\rangle = \left\langle a, pp' \right\rangle$$

2. H is a left H-module via the adjoint action

$$h_{\dot{ad}} a = \sum h_1 a S h_2$$

The combination of this two actions makes H into a $D(H)$-module via

$$(p \bowtie h) \bullet a = (h_{\dot{ad}} a) \leftharpoonup s(p)$$

If $\rho_V : H \to End_k(V)$ is a representation of H, then its character χ_V is given by:

$$\chi_V(h) = trace(\rho_V(h)).$$

The product of two characters is given by:

$$\chi_V \chi_W = \chi_{V \otimes W}$$

The product of characters (that coincides with the product Δ^* of elements in H^*) is an integral sum of irreducible characters of H. As for groups, the character ring is a ring with involution $\chi_V^* = \chi_{V^*}$.
We define the character algebra $R(H)$ to be the subalgebra of H^* generated by the characters of H. It turns out that $R(H)$ is the algebra of all cocommutative elements of H^*, that is $\Delta(x) = \Delta^{cop}(x)$.
The involution coincides with the antipode s.
The set of irreducible characters form a basis for $R(H)$.

When H is quasitriangular, then $R(H)$ **is commutative**.

Theorem(Kac-Zhu): $R(H)$ is semisimple.

Let $\{V_0, \dots V_{n-1}\}$ be a complete set of non-isomorphic irreducible H-modules of dimension d_i. Let $\{E_0, \dots E_{n-1}\}$ and $\mathrm{Irr}(H) = \{\chi_0, \dots, \chi_{n-1}\}$ be the associated central primitive idempotents and irreducible characters of H respectively, where $E_0 = \Lambda$, the idempotent integral of H and $\chi_0 = \varepsilon$. We have:

$$\lambda = \chi_H = \sum_{i=0}^{n-1} d_i \chi_i$$

is an integral for H^* satisfying $\langle \lambda, \Lambda \rangle = 1$.
When $R(H)$ is commutative, it has also a basis of (central) primitive idempotents $\{F^i\}_{i=0}^{n-1}$ where $F^0 = \frac{1}{d}\lambda$. Define the **conjugacy class** $\mathfrak{C}^i$ as:

$$\mathfrak{C}^i = \Lambda \leftharpoondown F^i H^*.$$

We have shown that:
Theorem:[CW] $\mathfrak{C}^i$ is an irreducible $D(H)$-module via the action $\bullet$. Moreover,

$$H = \bigoplus_i \mathfrak{C}^i$$

as a $D(H)$-module.

We generalize also the notions of **Class sum** and of a representative of a conjugacy class as follows:

$$C^i = \Lambda \leftharpoonup dF_i \qquad \eta^i = \frac{C^i}{\dim(F_i H^*)}.$$

As for groups $C^i \in Z(H)$.
We refer to η^i as a **normalized class sum**.
We have:
$\{F^i\}$ and $\{\eta^i\}$ are dual bases for $R(H)$ and $Z(H)$ respectively.

These definitions boil down to the usual ones when applied to finite groups G.

Let $H = kG$, a quasitriangular Hopf algebra (with $R = 1 \otimes 1$) and the projections $\{p_g\}$ be a basis for $kG)^*$. We have,

$$\Lambda = \frac{1}{|G|} \sum_{g \in G} g \qquad g \leftharpoonup p_{g'} = \delta_{g,g'} g$$

For $\sigma_i \in G$ we have

$$\mathfrak{C}^i = \mathfrak{C}_{\sigma_i} = \{g \sigma_i g^{-1}\}_{g \in G}$$

The orthogonal idempotents of $R(kG)$ are precisely $\{F^i\}$ where

$$F^i = \sum_{g \in \mathfrak{C}^i} p_g$$

Then

$$\frac{1}{|G|} \sum_{g \in G} g \leftharpoonup |G| \sum_{g \in \mathfrak{C}^i} p_g = \sum_{g \in \mathfrak{C}^i} g = C^i$$

and

$$\{ \sum_{g \in G} g \leftharpoonup (\sum_{g \in \mathfrak{C}^i} p_g) p_h \}_{h \in G} = \{\delta_{g,h} g,\, g \in \mathfrak{C}^i\} = \mathfrak{C}^i.$$

When (H, R) quasitriangular, we have the following Hopf algebras maps:

$$f_R : H^{*cop} \to H,\ f_R(p) = \sum \left\langle p, R^1 \right\rangle R^2,$$

$$f_R^* = f_{R^\tau} : H^{*op} \mapsto H,\ f_{R^\tau}(p) = \sum \left\langle p, R^2 \right\rangle R^1,$$

and the Drindeld map, with $Q = R^\tau R$

$$f_Q(p) = (f_{R^\tau} * f_R)(p) = \sum \left\langle p, R^2 r^1 \right\rangle R^1 r^2.$$

H is **factorizable** if the map f_Q is a monomorphism. In particular $D(H)$ is always factorizable. We have a Hopf projection

$$\Phi : D(H) \to H,\ \Phi(p \bowtie h) = f_R(p)h$$

and its dual Hopf injection

$$\Phi^* : H^* \longrightarrow D(H)^*,\ \Phi^*(p) = \sum f_{R^\tau}(p_1) \otimes p_2$$

In particular, if V_i is an irreducible H-module with associated character χ_i, then it is also an irreducible $D(H)$-module with associated character $\Phi^*(\chi_i) \in R(D(H))$.

In order to distinguish between H and $D(H)$, elements of the $D(H)$-theory will usually appear with $\widehat{}$.

For $D(H)$ (in fact for any factorizable Hopf algebra) the following sets have the same cardinality and the same set of indices.

$$\{\widehat{\chi}_i\}, \{\widehat{F}_i\}, \{\widehat{E}_i = f_{\widehat{Q}}(\widehat{F_i})\}, \{\widehat{\eta}_i = f_{\widehat{Q}}(\frac{\widehat{\chi}_i}{\widehat{d}_i})\}$$

of respectively irreducible characters, primitive idempotents of $R(D(H))$, central primitive idempotents of $D(H)$ and normalized class sums of $D(H)$.
Let $\{F^t\}$ denote the set of primitive orthogonal idempotents of $R(H)$. Since Φ^* is an algebra injection from $R(H)$ to $R(D(H))$ we have that $\Phi^*(F^t) = \widehat{F}^t$ is an idempotent in $R(D(H))$. Hence for each primitive idempotent $\widehat{F}_i \in R(D(H))$ there exists a unique t so that

$$\widehat{F}_i\widehat{F}^t = \widehat{F}_i, \text{ and } \widehat{F}_i\widehat{F}^k = 0,\ k \neq t$$

We may say that $\widehat{F}_i$ "belongs" to $\widehat{F}^t$

We define an equivalence relation on $\{\widehat{F}_i\}$ by: Two primitive idempotents are equivalent if they belong to the same idempotent $\widehat{F}^t$. That is:

$$\widehat{F}_i \cong \widehat{F}_j \Longleftrightarrow \widehat{F}_i \widehat{F}^t = \widehat{F}_i \text{ and } \widehat{F}_j \widehat{F}^t = \widehat{F}_j.$$

We denote the equivalence class of indices of idempotents belonging to $\widehat{F}^t$ by I_t. We also denote by I_{t^*} the equivalence class related to $s(\widehat{F}^t)$.

We show
Lemma: Let $\{\widehat{\eta}_j\}$, $\{\eta^t\}$ be the sets of normalized class sums in $D(H)$ and H respectively. Then

$$\Phi(\widehat{\eta}_j) = \eta^t \Longleftrightarrow j \in I_t.$$

Important elements of I_t

Let $\widehat{\chi}_{j_{\mathfrak{C}^t}}$ denote the character of the irreducible $D(H)$-module $\mathfrak{C}^t$. That is,

$$\left\langle \widehat{\chi}_{j_{\mathfrak{C}^t}}, p \bowtie h \right\rangle = Trace_{(\mathfrak{C}^t, \bullet)}(p \bowtie h).$$

Motivated by Zhu Shenglin et al we prove a key observation.

Theorem: Let $\mathfrak{C}^t$ be a conjugacy class of H and η^t its normalized class sum. Let $\widehat{\chi}_{j_{\mathfrak{C}^t}} \in R(D(H))$ be the character of $\mathfrak{C}^t$ as an irreducible $D(H)$ module. Then

$$\Phi f_{\widehat{Q}} \left(\frac{S\widehat{\chi}_{j_{\mathfrak{C}^t}}}{\dim \mathfrak{C}^t} \right) = \Phi(S\widehat{\eta}_{j_{\mathfrak{C}^t}}) = \eta^t.$$

Hence by the previous lemma

$$j_{\mathfrak{C}^t} \in I_{t^*}$$

Another equivalence relation arises from Nichols-Richmond equivalence relation on characters.

Motivation: Hopf subalgebras of $(kG)^*$ are in $1-1$ correspondence with normal subgroups of G via the representations of their quotients. Explicitly, each Hopf subalgebra $B \subset (kG)^*$ is of the form $k(G/N)^*$ for some normal group $N \triangleright G$.

Given a normal subgroup N of G, we have a natural equivalence relation on the irreducible characters of G, where each equivalence class consists of all irreducible characters with the same restriction to N (up to scalar multiplication by dimension).

This equivalence relation is expressed in the following Nichols-Richmond equivalence relation on simple subcoalgebras of H^*, with respect to a Hopf subalgebra B.

Let B be a Hopf subalgebra of H^*. Define an equivalence relation on simple subcoalgebra of H^* as follows:

$$B_j \cong B_k \Longleftrightarrow B_j \subset BB_k.$$

Denote the equivalence class of B_j by $[B_j]$. Note that $[B_j] = BB_j$.
Note that the Nichols-Richmond equivalence relation yields a decomposition of H^*

$$H^* = \bigoplus_{[B_t]} BB_t$$

where $\{B_t\}$ is a set of representatives of each equivalence class. Since each simple subcoalgebra B_j is generated as a coalgebra by the irreducible character χ_j, we have equivalently,

$$\chi_j \cong_B \chi_k \Longleftrightarrow \lambda_B \frac{\chi_j}{d_j} = \lambda_B \frac{\chi_k}{d_k},$$

where λ_B is the integral of the Hopf algebra B.

Back to $D(H)$. Set $\widehat{N} = D(H)^{co\Phi}$, that is

$$\widehat{N} = \{x \in D(H) | \sum x_1 \otimes \Phi(x_2) = x \otimes 1\}$$

In fact, this is the Hopf analogue of the group-theory kernels. Set

$$\widehat{B} = f_{\widehat{Q}}^{-1}(\widehat{N}).$$

by [CW], $\widehat{B}$ is a Hopf subalgebra of $D(H)^*$.

Remark: When $T = (R^\tau)^{-1}$, then (H, T) is quasitriangualr as well. To avoid ambiguity, we define f_T, Φ_T etc. as for f_R, Φ, etc. We proved:

$$\widehat{B} = \Phi_T^*(H^*)$$

We show,

Proposition For any $\widehat{\chi}_j, \widehat{\chi}_k \in R(D(H))$,

$$j, k \in I_t \iff \widehat{\chi}_j \cong_{\widehat{B}} \widehat{\chi}_k.$$

equivqlently,

$$\widehat{B}_j \cong_{\widehat{B}} \widehat{B}_k \iff j, k \in I_t$$

In particular, we can choose as a representative of each equivalence class the simple subcoalgebra $B_{j_{\mathfrak{C}t}}$.

We can show now our main result:

Main theorem: Let (H, R) be a semisimple quasitriangular Hopf algebra, $\mathfrak{C}^t$ a conjugacy class and $\widehat{\chi}_{j_{\mathfrak{C}^t}} \in R(D(H))$ its corresponding irreducible character. Let $\widehat{B}_{j_{\mathfrak{C}^t}}$ be the simple subcoalgebra of $D(H)^*$ generated by $\widehat{\chi}_{j_{\mathfrak{C}^t}}$. Then:

$$D(H)^* = \bigoplus_t \Phi_T^*(H^*)\widehat{B}_{j_{\mathfrak{C}^t}}.$$

Hence, each irreducible character of $D(H)$ is a constituent of

$$\Phi_T^*(\chi_i)\widehat{\chi}_{j_{\mathfrak{C}^t}}.$$

for some irreducible H-character χ_i and some conjugacy class $\mathfrak{C}^t$.
Equivalently, each irreducible $D(H)$-representation is a direct summand of $V_i \otimes \mathfrak{C}^t$ where V_i is an irreducible H-representation and $\mathfrak{C}^t$ is some conjugacy class.

The special case of 1-dimensional representations of H.

For any Hopf algebra H, if $\eta \in G(H^*)$ (that is, η is an algebra homomorphism in $Hom(H,k)$) and $(M,\cdot)$ is an H-module, then we have a modified action of H on M given by

$$h \underset{\eta}{\cdot} m = (\eta \rightharpoonup h)\cdot m = \sum \langle \eta, h_2\rangle\, h_1 \cdot m$$

If $(M,\cdot)$ is an irreducible H-module, so is $(M, \underset{\eta}{\cdot})$. Moreover, any 1-dimension representation of H is isomorphic to $k\Lambda^{\eta^{-1}} = k(\Lambda \leftharpoonup \eta^{-1})$ where $\eta \in G(H^*)$ and so, for all $h \in H$,

$$h\Lambda^{\eta^{-1}} = \langle \eta, h\rangle\, \Lambda^{\eta^{-1}}.$$

Consider $(f_R(\eta^{-1}) \otimes \eta) \in G(D(H)^*)$[Radford], then the action $\bullet$ of $D(H)$ on $\mathfrak{C}^s$, gives rise to the modified $(f_R(\eta^{-1}) \otimes \eta)$-action of $D(H)$ on $\mathfrak{C}^s$, which we denote by $\underset{\eta}{\bullet}$. Explicitly,

$$(p \bowtie h)\underset{\eta}{\bullet} c = ((\eta \rightharpoonup h)\underset{ad}{\cdot} c) \leftharpoonup s(p \leftharpoonup f_R(\eta^{-1})).$$

Note that when $H = kG$ then $R = 1 \otimes 1$, hence the action $\underset{\eta}{\bullet}$ of H^* on $\mathfrak{C}^s$ boils down to $\bullet$.

Based on the above, we obtain a family of irreducible $D(H)$-modules that are modifications of $\mathfrak{C}^s$ on one hand and have the form $V_i \otimes \mathfrak{C}^s$ on the other.

Proposition: Let $\eta \in G(H^*)$, then

$$H\Lambda^{\eta^{-1}} \otimes \mathfrak{C}^s \cong (\mathfrak{C}^s, \underset{\eta}{\bullet})$$

as $D(H)$-modules. In particular,

$$k\Lambda \otimes \mathfrak{C}^s \cong \mathfrak{C}_s.$$

Remark If $\eta \neq \eta'$ then $\Lambda^\eta \neq \Lambda^{\eta'}$, yet, $H\Lambda^\eta \otimes \mathfrak{C}^s$ and $H\Lambda^{\eta'} \otimes \mathfrak{C}^s$ may be isomorphic as $D(H)$-modules.

A common way to construct $D(kG)$ modules is known (e.g. [Di,Ma,Gou]). We present below our translated form.

Given a conjugacy classe $\mathfrak{C}^\sigma$ and an irreducible representations M^σ of its centralizer $Z_G(\mathfrak{C}^\sigma)$, an irreducible $D(kG)$-representation V^σ is constructed as:

$$V^\sigma = kG \underset{kZ_G(\sigma)}{\otimes} M^\sigma.$$

The action of $D(kG)$ on V^σ is given by:

$$h \underset{D(kG)}{\bullet} (a \underset{(kZ_G(\sigma),\cdot)}{\otimes} m) = ha \underset{(kZ_G(\sigma),\cdot)}{\otimes} m$$

$$p \underset{D(kG)}{\bullet} (a \underset{(kZ_G(\sigma),\cdot)}{\otimes} m) = \langle s(p), a_1 \sigma S a_3 \rangle \, a_2 \underset{(kZ_G(\sigma),\cdot)}{\otimes} m.$$

Example - Representations of $D(kS_3)$: In the following example we compare the Hopf approach and the common $D(kG)$ approach for the irreducible representations of $D(kS_3)$.

Set $x = (1,2)$ and $y = (1,2,3)$.
So $x^2 = 1$, $y^3 = 1$ and $xy = y^2x$.
Conjugacy classes of S_3 are given by:

$$\mathfrak{C}_1 = \{1\} \quad \mathfrak{C}_x = \{x, yx, y^2x\} \quad \mathfrak{C}_y = \{y, y^2\}$$

Their centralizers are given by:

$$Z_G(1) = S_3 \quad Z_G(x) = \{1, x\} \quad Z_G(y) = \{1, y, y^2\}$$

The irreducible representations of kS_3 are:

$$V_1 = (k, \dot{\varepsilon}) \cong k\Lambda \qquad V_2 = (k, \dot{sgn}) \cong k\Lambda^{sgn}$$

$$V_3 = He_\omega = sp_k\{e_\omega, xe_\omega\}$$

where ω a third root of unity and

$$e_\omega = \frac{1}{3}(1 + \omega y + \omega^2 y^2)$$

The irreducible representations of $D(kS_3)$ are constructed as follows. From $\mathfrak{C}_1$ with centralizer kS_3, we obtain 3 irreducible representations of $D(kS_3)$, which are actually the 3 irreducible representations of S_3.

1. $M_1 = kS_3 \underset{kS_3}{\otimes} V_1 = V_1$
2. $M_2 = kS_3 \underset{kS_3}{\otimes} V_2 = V_2$
3. $M_3 = kS_3 \underset{kS_3}{\otimes} V_3 = V_3$

From $\mathfrak{C}_x$ with centralizer $Z_G(x) = \{1, x\}$, we obtain 2 irreducible representation of $D(kS_3)$:

$$4.\ M_4 = kS_3 \underset{(kZ_G(x),\dot{\varepsilon})}{\otimes} k \cong (\mathfrak{C}_x, \bullet).$$

That is, M_4 is the module obtained from the trivial representation $(k,\dot{\varepsilon})$, of $kZ_G(x)$.

$$5.\ M_5 = kS_3 \underset{(kZ_G(x),\dot{sgn})}{\otimes} k \cong (\mathfrak{C}_x, \underset{\text{sgn}}{\bullet}).$$

That is, M_5 is the module obtained from the sign representation $(k,\dot{sgn})$ of $kZ_G(x)$. Note that $sgn \in G(kS_3)^*$ and so $(\mathfrak{C}_x, \underset{\text{sgn}}{\bullet})$ is a modified action of $D(kS_3)$ on $\mathfrak{C}_x$.

From $\mathfrak{C}_y$ with centralizer $Z_G(y) = \{1, y, y^2\}$, we obtain 3 irreducible representations of $D(kS_3)$,

$$6.\ M_6 = kS_3 \underset{(Z_G(y),\dot{\varepsilon})}{\otimes} k \cong (\mathfrak{C}_y, \bullet)$$

That is, M_6 is the module obtained from the trivial representation $(k, \dot{\varepsilon})$, of $kZ_G(y)$. Now,

$$7.\ M_7 = kS_3 \underset{(Z_G(y),\dot{\omega})}{\otimes} k$$

That is, M_7 is the module obtained from the $(k,_{\dot{\omega}})$ representation of $kZ_G(y)$,

$$y_{\dot{\omega}}\alpha = \omega\alpha,\ \alpha \in k.$$

Finally,

$$8.\ M_8 = kS_3 \underset{(Z_G(y),_{\dot{\omega^2}})}{\otimes} k.$$

That is, M_8 is the module obtained from the $(k,_{\dot{\omega^2}})$ representation of $kZ_G(y)$, given by:

$$y_{\dot{\omega^2}}\alpha = \omega^2\alpha,\ \alpha \in k.$$

The Hopf decomposition described in the main theorem into the irreducible $D(kS_3)$-representations, is given in the following:

1. $\mathfrak{C}_x \otimes V_1 = \mathfrak{C}_x \cong M_4$
2. $\mathfrak{C}_x \otimes V_2 = (\mathfrak{C}_x, s\dot{g}n) \cong M_5$
3. $\mathfrak{C}_x \otimes V_3 \cong M_4 \oplus M_5$
4. $\mathfrak{C}_y \otimes V_1 = \mathfrak{C}_y \otimes V_2 \cong \mathfrak{C}_y \cong M_6$
5. $\mathfrak{C}_y \otimes V_3 \cong M_7 \oplus M_8$

Note that, while $\mathfrak{C}_x \otimes V_3$ is a direct sum of already known deformations of $\mathfrak{C}_x$, this is no longer true for $\mathfrak{C}_y \otimes V_3$.

NOT NECESSARILY FOR THIS LECTURER

Let $\sigma \in kG,\ \eta \in G((kG)^*)$.

Set $\overline{\eta} = \eta_{|kZ_G(\sigma)} \in G(kZ_G(\sigma)^*)$. Let $\overline{M}$ be a $kZ_G(\sigma)$-module. Denote by $\dot{\overline{\eta}}$ the modified $kZ_G(\sigma)$-module action on $\overline{M}$

$$y_{\dot{\overline{\eta}}}\overline{m} = (\eta \rightharpoonup y) \cdot \overline{m}$$

for $\overline{m} \in \overline{M}, y \in kZ_G(\sigma)$.

Proposition: Let $\sigma \in kG,\ \eta \in G((kG)^*)$. Then

$$kG \underset{(kZ_G(\sigma),_{\dot{\overline{\eta}}})}{\otimes} k \cong (\mathfrak{C}_\sigma, \underset{\eta}{\bullet})$$

as $D(kG)$-modules.

Corollary:

$$kG \underset{(kZ_G(\sigma),_{\dot{\overline{\eta}}})}{\otimes} k \cong (\mathfrak{C}_\sigma, \underset{\eta}{\bullet}) \cong \mathfrak{C}_s \otimes k\Lambda^{\eta^{-1}}$$

Remark:

If $\eta, \eta' \in kG^*,\ \eta \neq \eta'$, but $\eta_{|kZ_G(\sigma))} = \eta'_{kZ_G(\sigma))}$, then the modules $\mathfrak{C}_s \otimes k\Lambda^{\eta^{-1}}$ and $\mathfrak{C}_s \otimes k\Lambda^{\eta'^{-1}}$ are isomorphic .