Superalgebras described by root data

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Example

$\mathfrak{sl}_3$: Lie algebra of 3×3 matrices of trace 0 ($[a,b]:=ab-ba$).
$\mathfrak{sl}_3=\tilde{\mathfrak{g}}/\mathfrak{r}$, where $\tilde{\mathfrak{g}}=\tilde{\mathfrak{n}}_-\oplus\mathfrak{h}\oplus\tilde{\mathfrak{n}}_+$:
$\mathfrak{h}$ is commutative, spanned by h_1,h_2;
$\tilde{\mathfrak{n}}_+$ is a free Lie algebra generated by e_1,e_2
$\tilde{\mathfrak{n}}_-$ is a free Lie algebra generated by f_1,f_2
we fix $\alpha_1,\alpha_2\in\mathfrak{h}^*$ by $\langle h_i,\alpha_j\rangle=a_{ij}$ for $(a_{ij}):=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$

$$(**)\quad [h,e_i]=\alpha_i(h)e_i,\quad [h,f_i]=-\alpha_i(h)f_i,\quad [e_i,f_j]=\delta_{ij}h_i.$$

- $\mathfrak{r}$ is a maximal ideal satisfying $\mathfrak{r}\cap\mathfrak{h}=0$ ($\mathfrak{r}$ is unique).
- Serre relations: $\mathfrak{r}$ is generated by $[e_1,[e_1,e_2]]$, $[f_1,[f_1,f_2]]$ $[e_2,[e_2,e_1]]$, $[f_2,[f_2,f_1]]$.

Observation: for $e_1':=f_1$, $e_2':=[e_1,e_2]$, $f_1':=e_1$, $f_2':=-[f_2,f_1]$
we have $\quad [e_1,[e_1,e_2]]=[f_1',e_2'],\qquad [f_1,[f_1,f_2]]=[e_1',f_2']$.

More examples

Using the same procedure for another matrix $A = (a_{ij})$ we get:

- semisimple Lie algebra $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ for $(a_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$;
- fin.-dim. simple Lie algebras $\mathfrak{sp}_2 = \mathfrak{o}_3$ and G_2 for $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ respectively;
- affine Lie algebras for $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$;
- a Kac-Moody algebra with infinite GK dimension for $\begin{pmatrix} 2 & a_{12} \\ a_{21} & 2 \end{pmatrix}$ with $a_{12}, a_{21} \in \mathbb{Z}_{<0}$, $a_{12}a_{21} > 4$.

Remarks:

1. All semisimple Lie algebras can be obtained by this procedure (for some $A \in Mat_{n\times n}(\mathbb{Z})$)
2. The affine Lie algebras are the infinite-dimensional Kac-Moody algebras with finite Gelfand-Kirillov dimension.

Groupoid of root data

We will now define a groupoid of root data R.

Once and forever we fix a finite set X. The objects of R (root data) are the triples $(\mathfrak{h}, a : X \to \mathfrak{h}, b : X \to \mathfrak{h}^*)$ where $\mathfrak{h}$ is a fin.-dim. vector space over $\mathbb{C}$ and a, b are “injective maps”.

We have generating arrows of three types:

1. reflections $r_x : (\mathfrak{h}, a, b) \to (\mathfrak{h}, a', b')$ defined by the source $(\mathfrak{h}, a, b)$ and reflectable elements $x \in X$;
2. tautological $t_\theta : (\mathfrak{h}, a, b) \to (\mathfrak{h}', a', b')$ determined by $\theta : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}'$. Here $a' := \theta \circ a$, $b' = ((\theta^*)^{-1}) \circ b$.
3. homothety $h_\lambda : (\mathfrak{h}, a, b) \to (\mathfrak{h}, a', b)$ determined by $\lambda : X \to \mathbb{C}^*$, with $a'(x) = \lambda(x)a(x)$.

This collection of objects and arrows (=quiver) generates a free category $\tilde{R}$. The groupoid R is defined as the one with the same objects as $\tilde{R}$, and whose arrows are equivalence classes of the arrows above, where roughly speaking, two compositions of arrows $(\mathfrak{h}, a, b) \to (\mathfrak{h}', a', b')$ are equivalent if they induce the same isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}'$.

Reflections

Cartan matrix: $A=(a_{xy})$, where $a_{xy}:=\langle a(x),b(y)\rangle$.
An element $x\in X$ is called reflectable at $v=(\mathfrak{h},a,b)$ if $a_{xx}\neq 0$ and $\frac{2a_{xy}}{a_{xx}}\in\mathbb{Z}_{\leq 0}$.
The reflection $r_x: v=(\mathfrak{h},a,b)\to v'=(\mathfrak{h},a',b')$ is given by

$$a'(y):=a(y)-2\frac{a_{yx}}{a_{xx}}a(x),\quad b'(y):=b(y)-2\frac{a_{xy}}{a_{xx}}b(x).$$

One has $r_x^2=Id$.
Cartan matrices are preserved by r_x, t_θ; for $h_\lambda:(\mathfrak{h},a,b)\to(\mathfrak{h},a',b)$ one has $A'=DA$, where D is an invertible diagonal matrix.
In this talk we consider only connected components R_0 such that for each $v\in R_0$ each $x\in X$ is reflectable.

Let $v = (\mathfrak{h}, a, b) \in R$. We assign to v a Lie superalgebra $\tilde{\mathfrak{g}}(v)$ generated by $\mathfrak{h} = \mathfrak{h}(v)$, $\tilde{e}_x, \tilde{f}_x,\ x \in X$, subject to the relations

1. $[\mathfrak{h}, \mathfrak{h}] = 0$,
2. $[h, \tilde{e}_x] = \langle b(x), h\rangle \tilde{e}_x, \quad [h, \tilde{f}_x] = -\langle b(x), h\rangle \tilde{f}_x$
3. $[\tilde{e}_x, \tilde{f}_y] = \delta_{xy} a(x)$.

Definition Let $R_0 \subset R$ be a connected component.
A root Lie algebra $\mathfrak{g}$ supported on R_0 is
a collection of Lie algebras $\mathfrak{g}(v),\ v \in R_0$ with epimorphisms $\psi_v : \tilde{\mathfrak{g}}(v) \twoheadrightarrow \mathfrak{g}(v)$ such that

1. $\operatorname{Ker} \psi_v \cap \mathfrak{h}(v) = 0$
2. for any arrow $\gamma : v \to v'$ in R_0 there exists $\mathfrak{g}(v) \xrightarrow{\sim} \mathfrak{g}(v')$ extending the corresponding isomorphism $\mathfrak{h}(v) \xrightarrow{\sim} \mathfrak{h}(v')$.

Root algebras

We call a component R_0 admissible if it admits a root algebra.
If R_0 is admissible, one has the universal root algebra $\tilde{\mathfrak{g}}(v)/\mathfrak{s}$ and the smallest root algebra $\mathfrak{g}_{KM} := \tilde{\mathfrak{g}}(v)/\mathfrak{r}$:
$\tilde{\mathfrak{g}}(v)/\mathfrak{s} \twoheadrightarrow \mathfrak{g}(v) \twoheadrightarrow \mathfrak{g}_{KM}$ for any root algebra $\mathfrak{g}(v)$.
One has $\tilde{\mathfrak{g}}(v) = \oplus_{\alpha\in\mathfrak{h}^*}\tilde{\mathfrak{g}}(v)_\alpha$ and $\tilde{e}_x \in \tilde{\mathfrak{g}}(v)_{b(x)}$.

- $\mathfrak{s}$ is generated by $[\tilde{\mathfrak{g}}(v)_{b'(x)}, \tilde{\mathfrak{g}}(v)_{b'(y)}]$ for $x \neq y$, where $v' = (\mathfrak{h}, a', b') \in R_0$;
- $\mathfrak{r}$ is the maximal ideal of $\tilde{\mathfrak{g}}(v)$ satisfying $\mathfrak{r} \cap \mathfrak{h} = 0$.

We say that R_0 is symmetrizable if $a_{xy} = a_{yx}$ for each x, y for some $v \in R_0$.
Theorem (O. Gabber- V. Kac, 1981) If R_0 is symmetrizable, then R_0 admits a unique root algebra.

Superworld: Groupoid of root data

The objects of R (root data) are the quadruples
$(\mathfrak{h}, a : X \to \mathfrak{h}, b : X \to \mathfrak{h}^*, p : X \to \mathbb{Z}_2)$
An element $x \in X$ is called *reflectable* at $v = (\mathfrak{h}, a, b, p)$ if the following conditions hold.

1. If $a_{xx} = 0$ then $p(x) = 1$;
2. If $a_{xx} \neq 0$ and $p(x) = 0$ then $\frac{2a_{xy}}{a_{xx}} \in \mathbb{Z}_{\leq 0}$.
3. If $a_{xx} \neq 0$ and $p(x) = 1$ then $\frac{a_{xy}}{a_{xx}} \in \mathbb{Z}_{\leq 0}$.

The reflection $r_x : v \to v' = (\mathfrak{h}, a', b', p')$ is defined as follows.

If $a_{xx} \neq 0$, then $p' := p$ and

$$a'(y) := a(y) - 2\frac{a_{yx}}{a_{xx}}a(x), \quad b'(y) := b(y) - 2\frac{a_{xy}}{a_{xx}}b(x).$$

If $a_{xx} = 0$ then $p(x) = 1$ and $(a'(y), b'(y), p'(y))$ is given by

$$\begin{cases} (-a(x), -b(x), p(x)) & \text{if } x = y, \\ (a(y), b(y), p(y)) & \text{if } x \neq y, \quad a_{xy} = 0, \\ (a(y) + \frac{a_{yx}}{a_{xy}}a(x), b(y) + b(x), 1 + p(y)) & \text{if } a_{xy} \neq 0. \end{cases}$$

Root (super)algebras

We call a component R_0 quasi-symmetric if for all $v \in R_0$ $a_{xy} = 0$ implies $a_{yx} = 0$ for all $x, y \in X$.

$$\text{Symmetricity} \Longrightarrow \text{Quasi-symmetricity} \Longleftrightarrow \text{Admissibility}$$

The Kac-Moody superalgebras $\mathfrak{g}_{KM}$ were classified in V. Kac in Adv. in Math., 1977, J. W. van de Leur Comm. in Algebra, 1989, and by C. Hoyt and V. Serganova ("Kac-Moody superalgebras and integrability", 2011).

Root (super)algebras: examples and classification

Example $|X| = 1$, $p(x) = 1$, $a_{xx} = 0$. In this case $\dim \mathfrak{h} \geq 2$. For $\dim \mathfrak{h} = 2$, $\tilde{\mathfrak{g}}(v) = \mathfrak{g}(v)$ is of dimension $(4|2)$ and $\mathfrak{g}_{KM} = \mathfrak{gl}(1|1)$ has dimension $(2|2)$. Root algebra is not unique!

Example If $\mathfrak{g}$ is an indecomposable fin-dim. Kac-Moody algebra, then $\mathfrak{g}$ is a root algebra for some admissible R_0; moreover, $\mathfrak{g} \cong \mathfrak{g}_{KM}$ if $\mathfrak{g} \neq \mathfrak{gl}(1|1)$.

Theorem (Hinich, Serganova, G. 2021)

1. Let R_0 be admissible and symmetrizable.
   The root superalgebra is unique if $\mathfrak{g}_{KM} \neq \mathfrak{gl}(1|1), A(n|n)^{(i)}$.
   and is not unique if $\mathfrak{g}_{KM} = \mathfrak{gl}(1|1), A(n|n)^{(1)}$.
2. In the non-symmetrizable affine cases the root superalgebra is unique for $S(2|1, a)$ and is not unique for $\mathfrak{q}(n)^{(2)}$.

Remark: relations in [1] were studied in H. Yamane, PRIMS 1999.