PI theory and G-graded division algebras

In honor of Amitai Regev and Doron Zeilberger

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Table of contents

Joint work with Y. Karasik

- We learned that if A is a (f.d.) k-central simple algebra and $F = \bar{k}$, then $A \otimes_k F \cong M_n(F)$, some n.
- In fact this characterizes central simple algebras, We say central simple algebras are forms of matrix algebras.
- In particular finite dimensional k-central division algebras D are forms of matrix algebras.

- Less obvious, but still well known, every f.d. simple algebra over $F = \bar{F}$ (i.e. $\cong M_n(F)$, some n), has a division algebra form D.
- In particular the "generic object corresponding to $M_n(F)$" is a division algebra. This is the well known "generic division algebra" of degree n.

- Main problem: What happens in the G - graded case? (G is a group)
- Replace "f.d. simple" with "f.d. G-graded simple", that is, $A \cdot A \neq 0$ and A has no nontrivial G-graded 2-sided ideals.
- Replace "division algebra form" with "G-graded division algebra form", that is, nonzero homogeneous elements in D are invertible.

- $\Rightarrow$ If D is a f.d. G-graded division algebra over its e-center k and $F = \bar{k}$, then $D \otimes_k F$ is G-graded simple.

- $\Leftarrow$ One may ask (following the ungraded case): Does every f.d. G-graded simple algebra have a graded division algebra form? The answer is No!

- Main question: Which G-graded simple algebras A do admit a G-graded division algebra form?

- Address the problem in case $\mathrm{char}(F) = 0$ and G a finite group for good reasons.

- Approach: *Construct the generic G-graded algebra attached to A.*

  We use PI theory, Kemer's theory and more specifically G-graded Kemer's theory.
  ($\mathrm{char}(F) = 0$ and G-finite, joint work with Belov).

- Few words about motivation: Group grading on central simple algebras is key for studying them, e.g.

- Crossed product grading realizes $Br(k) \cong H^2(G_k, k_s^*)$.

  Every k-central simple algebra is Brauer equivalent to a crossed product algebra
  $(K/k, G = Gal(K/k), \alpha \in H^2(G, K^*))$

- Symbol algebras $\cong k^\alpha \mathbb{Z}_n \times \mathbb{Z}_n$: If $\mu_n \subset k^*$, $Br_n(k)$ (n-torsion) is generated by symbol algebras (Merkurjev-Suslin)

- Some of the main open problems in Brauer groups theory are related to these gradings. e.g. Is every Brauer class represented by an abelian crossed product? (i.e. G is abelian)

- **Recall: construction of generic Azumaya algebras and generic division algebras** (the ungraded case)

- **Polynomial identities:** A polynomial $p(x_1, \dots, x_n) \in F\langle X \rangle$ (the free associative algebra over F on a countable set X) is an identity of an algebra A if $p/_{x_i = a_i \in A} \equiv 0$.

- $[x, y] = xy - yx$ is an identity of any commutative algebra.

- $[[x, y]^2, z] \in Id(M_2(F))$.

- $s_{2n} = \sum_{\sigma \in S_{2n}} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(2n)} \in Id(M_n(F))$ (Amitsur - Levitzki).

- The set of identities $Id(A)$ is a T-ideal of $F\langle X \rangle$.

- Let $A = M_n(F)$ where F is algebraically closed.
- The field $\mathbb{Q}$ is a field of definition for $M_n(F)$:

$$M_n(\mathbb{Q}) \otimes_{\mathbb{Q}} F \cong M_n(F)$$

($\mathbb{Q}$ is unique minimal).

- Let $\mathcal{U}_{\mathbb{Q}}$ be the algebra of generic $n \times n$-matrices over $\mathbb{Q}$:

$$\langle (X^m(i,j))_{i,j} : m = 1, 2, \ldots \rangle.$$

Equivalently take $\mathcal{U}_{\mathbb{Q}} \cong \mathbb{Q}\langle X \rangle / Id(M_n(\mathbb{Q}))$. Get a domain.

- Let $f = f(x_1, \ldots, x_s)$ be a central polynomial of $M_n(F)$, coefficients in $\mathbb{Q}$. (scalar values and not all are zero)
  Example: Regev Polynomial!

- f vanishes on $M_{n-1}(F)$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n \times n}$$

  Invert f central, $\mathcal{A} = f^{-1}\mathcal{U}_{\mathbb{Q}}$. Note that if

$$\mathcal{A} \twoheadrightarrow B \neq 0$$

  then $Id(B) \supseteq Id(M_n(F)))$ but $Id(B) \not\supseteq Id(M_{n-1}(F))) \ni f$.

- Artin - Procesi: $\mathcal{A}$ is *Azumaya* of degree n over its center $\mathcal{R}$.

- $\mathcal{A}$ specializes *precisely* to all forms of $M_n(F)$.
- Localizing the center: $S = Z(\mathcal{U}_{\mathbb{Q}}) \setminus \{0\}$

$$\mathcal{D} = S^{-1}\mathcal{U}_{\mathbb{Q}}$$

is the so called *generic division algebra* of degree n over the rationals.

Construction of generic Azumaya algebras (the graded case)

How finite dimensional G-simple algebras look like? (two examples)

1. Twisted group algebra (*fine grading*):
   $F^{\alpha}H$, $H \leq G$, $\alpha \in H^2(H, F^*)$, $u_{h_1}u_{h_2} = \alpha(h_1, h_2)u_{h_1 h_2}$.

2. *Elementary grading* on $M_r(F)$ by $\mathfrak{g} = (g_1, \ldots, g_r) \in G^{(r)}$:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & g_i^{-1}g_j & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{r\times r}$$

Bahturin, Sehgal and Zaicev: Classification of f.d. G-simple algebras A over F, $char(F) = 0$ and $\bar{F} = F$.

- Every f.d. G-simple algebra combines fine and elementary gradings, i.e.

$$A \cong F^{\alpha}H \otimes M_r(F)$$

More precisely . . .

- Grading presentation (may assume *connected*):

$$P_A = (H \leq G,\ \alpha \in H^2(H, F^*),\ (g_1, \dots, g_r) \in G^{(r)})$$

$$F^{\alpha}H \otimes \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & g_i^{-1}g_j & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{r \times r}$$

- Basis: $\{u_h \otimes e_{i,j} : h \in H,\ 1 \leq i, j < r\}$
- The homogeneous degree of $u_h \otimes e_{i,j}$ is: $g_i^{-1} h g_j$.
- e-center of A: $F = Z(A) \cap A_e$

- Given A (G-graded simple), want to construct a generic G-graded Azumaya algebra $\mathcal{A}$ that maps precisely to all G-graded forms of A.

- **Main steps in the construction**:

  Show existence and find explicitly a unique minimal field of definition k for $A \cong F^{\alpha}H \otimes M_r(F)$.
  k turns out to be a *cyclotomic* extension of $\mathbb{Q}$.

  More precisely ...

- By the Universal Coefficient Theorem

$$H^2(H, F^*) \cong Hom(M(H), F^*)$$

so α corresponds to $\eta_\alpha : M(H) \to F^*$ and we let $\mu = Im(\eta_\alpha)$, $M(H)$ is the Schur multiplier of H.

- Schur's theory: $\mathbb{Q}(\mu)$ is a minimal field of definition for $F^\alpha H$ and hence a field of definition for $A \cong F^\alpha H \otimes M_r(F)$ but possibly not minimal!

- Surprisingly, the matrices can lower the field of definition. (main tools-PI theory).

- Let k - the minimal field of def'n for A, and $k\langle X_G\rangle$ the free G-graded algebra.
- Divide by $\Gamma = Id_{k,G}(A)$ and get $k\langle X_G\rangle/\Gamma$. Note: The T-ideal Γ is defined over k.
- Construct an e-central polynomial f_e (combination of Regev's polynomial and group representation). Invert f_e.

Theorem

1. *$\mathcal{A} = f_e^{-1}k\langle X_G\rangle/\Gamma$ is G-graded Azumaya in the sense of Artin-Procesi.*
2. *$\mathcal{A}$ specializes precisely to all G-graded forms of A.*

Invert the entire e-center of $\mathcal{A}$ and obtain $\mathcal{D}_A$, the generic G-graded simple algebra over k corresponding to A.

When is $\mathcal{D}_A$ a G-graded division algebra?
There are 3 conditions (necessary and sufficient) on the presentation of A

$$P_A = (H \leq G,\ \alpha \in H^2(H, F^*),\ (g_1, \ldots, g_r) \in G^{(r)})$$

1. H is normal in G.
2. Every H-coset is represented the same number of times in $(g_1, \ldots, g_r) \in G^{(r)}$.

and the 3rd condition ...

(3) The cohomology class α is G-invariant. What does it mean?

G acts on H so it acts on the Schur multiplier $M(H)$.
If $\eta_\alpha : M(H) \to F^*$ (corresponds to $\alpha \in H^2(H, F^*)$) and $B_\alpha = ker(\eta_\alpha) \subseteq M(H)$, then α invariant means B_α is G-invariant.

We may ask "When does A admit a division algebra form which is G-graded?".

Well, this is another story.

THANK YOU FOR YOUR ATTENTION!

And once again Congratulations Amitai and Doron!