Specialized Symmetric Polynomials

Joint work with S. Catiou

Please no comments or chats until slide number 13

Classical Symmetric Functions

- A symmetric polynomial in n variables $f(x_1, \dots, x_n)$ is one that satisfies $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for all permutations.
- I think of a symmetric function as the analogous object in infinitely many variables, allowing infinite sums of variables, but not infinite products.
- More technically, a symmetric function can be defined as an infinite sequence $\left(f_n(x_1, \dots, x_n)\right)_n$ of symmetric polynomials such that for every n, $f_{n+1}(x_1, \dots, x_n, 0) = f_n(x_1, \dots, x_n)$

Classical Symmetric Functions

- The elementary symmetric functions are generated by
- $e_n = \sum\{x_{i_1} \cdots x_{i_n} | i_1 < \cdots < i_n\}$
- And for partition λ= $(\lambda_1, \dots, \lambda_k)$
- $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$

- The complete symmetric functions are generated by
- $h_n = \sum\{x_{i_1} \cdots x_{i_n} | i_1 \leq \cdots \leq i_n\}$
- And, again, for a partition λ
- $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$

- The power symmetric functions are generated by
- $p_n = \sum x_i^n$
- And for each partition λ
- $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$

- The Schur functions S_λ can be defined as a determinant using the elementary or complete symmetric functions, or using semistandard Young tableaux. I will not give the details here, but I will mention that they are important because they are characters of irreducible representations of the general linear group.

- If the number of variables in infinite then each of the sets $\{e_\lambda\}, \{h_\lambda\}, \{p_\lambda\}\ and\ \{S_\lambda\}$ gives a basis for the ring of symmetric functions. And each of the sets $\{e_n\}, \{h_n\}\ and\ \{p_n\}$ is an algebraically independent set of generators.
- There are many interesting algebraic relations between these functions, see MacDonald's book "Symmetric Functions and Hall Polynomials," and many of them have gorgeous combinatorial proofs.

Classical Symmetric Polynomials

- What if there are only finitely many variables?
- Consider for example the elementary symmetric functions $e_n(y_1, \ldots, y_k)$:
- This would equal $\sum\{y_{i_1} \cdots y_{i_n} | i_1 < \cdots < i_n\}$. Clearly this is zero if k < n.
- Not obviously, $\{e_1, \ldots, e_k\}$ are still algebraically independent and generate the ring of symmetric polynomials. So the relations $e_n = 0 \text{ } when \text{ } n > k$ generate all relations among the elementary symmetric functions. Also, $\{e_\lambda\}$ where λ is restricted to have all parts less than or equal to k form a basis.
- Relations among the other symmetric functions can be derived from these and are known. In particular, for Schur functions $S_\lambda = 0$ precisely when the partition has more than k parts. And the Schur functions S_λ for partitions with k or fewer parts form a basis for symmetric polynomials in k variables.

Specialized Symmetric Polynomials

- Starting with the symmetric polynomials in $y_1, \ldots, y_n$ and given $d_1 + \cdots + d_k = n$ we specialize $y_1, \ldots, y_{d_1} \mapsto x_1, y_{d_1+1}, \ldots, y_{d_{1+d_2}} \mapsto x_2, \ldots$
- Denoting these specializations with a superscript (d):
- Example: (d) = (3,1), then $p_n = y_1^n + y_2^n + y_3^n + y_4^n$ specializes to $\mathrm{p}_\mathrm{n}^{(\mathrm{d})} = 3x_1^n + x_2^n$. e_n is zero if $n > 4$, so we will just look at $e_3 = y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4$ specializes to $e_3^{(d)} = x_1^3 + 3x_1^2x_2$. The h_n have many terms, so lets just do n=2. $h_2 = y_1^2 + y_1y_2 + y_1y_3 + y_1y_4 + y_2^2 + y_2y_3 + y_2y_4 + y_3^2 + y_3y_4 + y_4^2$. This specializes to $h_2^{(d)} = 6x_1^2 + 3x_1x_2 + x_2^2$

- Here would be the general formulas
- $p_i^{(d)} = d_1 x_1^i + \cdots + d_k x_k^i$
- $e_i^{(d)} = \sum\left\{\binom{d_1}{i_1} \cdots \binom{d_k}{i_k} x_1{}^{i_1} \cdots x_k^{i_k} \,|\, i_1 + \cdots + i_k = i\right\}$
- $h_i^{(d)} = \sum\left\{\binom{d_1+i_1-1}{i_1} \cdots \binom{d_k+i_k-1}{i_k} x_1{}^{i_1} \cdots x_k^{i_k} \,|\, i_1 + \cdots + i_k = i\right\}$

- **These formulas make sense even if the d_i are not positive, are not integers, are not real! Can define specialized symmetric polynomials for any d_i**
- I always take the d_i to be non-zero, but not every one does
- Can also define $S_\lambda^{(d)}$ for any (d), using either tableaux or determinants
- All of the relations from MacDonald's book that hold between the classical symmetric functions in infinitely many variable continue to hold among their specializations.
- The main question I wish to study is:
- **Question:** What are the algebraic relations among the specialized symmetric polynomials?

- I mean what are the identities other than the ones known for symmetric functions in infinitely many variables?
- For example, in infinitely many variables the p_λ are linearly independent. So I want to find linear relations among the $p_\lambda^{(d)}$
- The problem is not solved, but I do have some results. Maybe you want to work on it.

For personal reasons, a sad slide

- Using the language of plethysms or Λ-calculus, if $d = (d_1^{m_1}, \dots, d_k^{m_k})$ then $e_\lambda^{(d)} = e_\lambda[d_1 X^{(1)} + \dots + d_k X^{(k)}]$, $h_\lambda^{(d)} = h_\lambda[d_1 X^{(1)} + \dots + d_k X^{(k)}]$,$p_\lambda^{(d)} = p_\lambda[d_1 X^{(1)} + \dots + d_k X^{(k)}]$, and $S_\lambda^{(d)} = S_\lambda[d_1 X^{(1)} + \dots + d_k X^{(k)}]$, where each X^i is a set of m_i variables

Example

- If $d_1 + \cdots + d_k = n$, an integer, then $e_m^{(d)} = 0$ for $m > n$. But if n is not an integer then no $e_m^{(d)}$ is zero.
- If $d = (d_1, d_2)$ then we have the identities
- $\left(d_1 d_2^3 + 2d_1^2 d_2^2 + d_1^3 d_2\right) p_{3,3} - 6\left(d_1 d_2^2 + d_1^2 d_2\right) p_{3,2,1} + 4 d_1 d_2 p_{3,1^3} + 4\left(-d_1^3 + d_1^2 d_2 +\right.$

Example

- I did most of my computations using the p_λ because they are the easiest to program. In the classical case it is much easier to describe the relations between the e_λ or S_λ. I don't know if it would be easier in this case, but it is not obvious. Here are the smallest relations in $\Lambda^{(d_1,d_2)}$ in terms of the specialized elementary and Schur polynomials ((d) superscipts surpressed):
- $4d_1d_2(d_1+d_2)e_4 - 4d_1d_2(d_1+d_2-3)e_{3,1} - 2\big(d_1d_2^2 + d_1^2d_2 - 2d_1^2 - 2d_2^2 +$

Field Dependence

- We usually assume that the d_i lie in the base field, but we can be more general. Given fields $F \subset K$ we can assume that $d_i \in K$ and ask for identities with coefficients in *F*. For example, if $(d_1, d_2) = (\sqrt{2}, 1)$, then we have this identity with coefficients in **Q**:
- $-6p_7 - 16p_{6,1} - 22p_{5,2} + 20p_{4,1^2} + 30p_{4,3} + 40p_{4,2,1} - 18p_{4,1^3} - 50p_{3^2,1} + 9p_{3,2^2} + 40p_{3,2,1^2} + p_{3,3} - 18p_{2^3,1} - 11p_{2^2,1^3} + p_{1^7}$
- This is not an absolute identity
- One way to think of absolute identities is that the d_i are algebraically independent over *F*, and that we are looking for relations with coefficients from *F*

Absolute Identities

- Let $(d) = (d_1, \dots, d_k)$ have k parts.
- Let $a_1, \dots, a_{k+1}$ and $b_1, \dots, b_{k+1}$ be sequences of distinct positive integers
- Then the determinant $|p^{(d)}_{a_i+b_j}|$ is an (absolute) identity. We will call it a determinental identity.
- **Proof:** We will use the notation $\Lambda^{(d)}$ for the algebra of specialized symmetric polynomials and $D^{(d)}$ for the algebra of diagonal matrices with trace

$$\mathrm{tr}\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & x_k \end{pmatrix} = d_1x_1 + \cdots + d_kx_k$$

- With respect to this trace $tr(X^n) = p^{(d)}_n(x_1, \dots, x_k)$

- Since the algebra of diagonal matrices is k dimensional

$$\boldsymbol{\sum sgn(\sigma) tr\left(x_{\sigma(1)} y_1\right) \cdots tr\left(x_{\sigma(k+1)} y_{\{k+1\}}\right) = 0}$$

For any trace function, since it is alternating in k+1 variables.

Now let $x_i = X^{a_i}$ and $y_j = X^{b_j}$ $\blacksquare$

- This theorem is exercise 2.9 in Lascoux's book and it generalizes a theorem of G. Bellavitis, 1857.
- Call the above identity (in x and y) $C_{k+1}(x; y)$. It is an analogue of the Capelli identity
- One variable pure trace identities in $D^{(d)}$ correspond to algebraic relations among the $p_\lambda^{(d)}$

Some results

- $\Lambda^{(d)}$ has rank k, so any $k+1$ elements are algebraically dependent.
- For any $n_1 < \cdots < n_k$ each of these sets is algebraically independent: $\{e_{n_i}\}, \{p_{n_i}\}, \{h_{n_i}\}$
- The above assumes that each d_i is in the base field. On the other extreme, if the d_i are algebraically independent, then the rank is $2k$
- In this case there is a $\Delta \in \Lambda^{(d)}$ such that $\Lambda^{(d)} \subseteq F[p_0, \dots, p_{2k-1}, \Delta^{-1}]$

Connections to PI

- Let X be an n×n diagonal matrix with entries $x_1, \dots, x_n$. Then the usual trace of X^k is the power symmetric function $p_k(x_1, \dots, x_n)$.
- So, an algebraic relation among the symmetric functions in n variables can be interpreted as a trace identity for n×n diagonal matrices in one variable.
- In 1996 I proved that all trace identities for diagonal matrices (in any number of variables) are consequences of commutativity and the Cayley-Hamilton theorem.

- May generalize to the weighted trace function to get $p_\lambda^{(d)}$ instead of p_λ
- For example, the absolute identity for $d = (d_1, d_2)$ from four slides back translates to $tr(x^5)tr(x^3)tr(x) + 2tr(x^4)tr(x^3)tr(x^2) - tr(x^3)^2 - tr(x^4)^2tr(x) - tr(x^5)tr(x^2)^2$
- If the d_i are all positive integers then we are studying trace identities among block scalar matrices, i.e., diagonal matrices with diagonal of the form $(x_1, \dots, x_1, \dots, x_k, \dots, x_k)$.
- The case in which the d_i are each ± 1 was studied by Kantor and Trishin in 1999. It is related to the theory of supersymmetric functions.
- From the point of view of p.i. algebras there is no reason to stick to one variable identities. In fact, in the general case not all identities follow from the single variable ones.

- **Theorem:** All absolute identities in $\Lambda^{(d)}$ are consequences of the determinental ones.
- Just to clarify: I mean either identities involving the $p_\lambda^{(d)}$ or one variable trace identities for $D^{(d)}$. I don't know about the general identities of $D^{(d)}$, although I **conjecture** that all absolute identities are consequences of $C_{k+1}(x;y)$ and commutativity.
- The determinental identity of minimal degree has degree k^2+k. This is not the identity of minimal degree in the k=2 case, and probably is not in general. I do not know the degree of the minimal identity.

Some Results and Questions

- Ioppolo, Koshlukov and La Matina studied the case of k=2 in JPAA, 2021. Although they studied the mixed trace identities $D^{(d_1,d_2)}$ their results can be translated to symmetrized symmetric polynomials to prove that all relations between the $p_\lambda^{(d)}$ are consequences of these:

$$d_1 d_2(d_1+d_2)p_{n+3} - 3d_1 d_2 p_{n+2,1} - \left(d_1^2 - d_1\ d_2 + d_2^2\right)p_{n+1,2}$$
$$+(d_1+d_2)p_{n+1,1,1} - d_1 d_2 p_{n,3} + (d_1+d_2)p_{n,2,1} - p_{n,1,1,1}$$

And

$$d_1 d_2(d_1+d_2)^2 p_{3,3} - 6d_1 d_2(d_1+d_2)p_{3,2,1} + 4d_1 d_2 p_{3,1^3}$$
$$-(d_1-d_2)^2(d_1+d_2)p_{2^3} + 3\left(d_1^2 + d_1 d_2 + d_2^2\right)p_{2^2,1^2}$$
$$-3(d_1+d_2)p_{2,1^4} + p_{1^6}$$

Some Results and Questions

- Let X be a generic diagonal matrix. I know that X is algebraic over $\Lambda^{(d)}$ but I don't know if it is integral. I'm thinking that this would be an analogue of the Cayley-Hamilton equation.
- For example, if $d = (d_1, d_2)$ then we have the identity $d_1 d_2 (d_1 + d_2) X^3 - 3 d_1 d_2 p_1 X^2 - \left(d_1^2 - d_1 d_2 + d_2^2\right) p_2 X + (d_1 + d_2) p_{1,1} X - d_1 d_2 p_3 + (d_1 + d_2) p_{2,1} - p_{1^3} = 0$
- I think that it is and I have a conjecture about the degree. Let $d = (d_1^{m_1}, \dots, d_t^{m_t})$. Then I conjecture that the degree should be $\prod (m_i + 1) - 1$
- I checked it in the cases $(d^m), (d_1, d_2), (d_1, d_1, d_2)$ and (d_1, d_2, d_3)

- Take another look at the identity
- $d_1 d_2(d_1+d_2)X^3 - 3d_1 d_2 p_1 X^2 - \left(d_1^2 - d_1 d_2 + d_2^2\right)p_2 X + (d_1+d_2)p_{1,1}X - d_1 d_2 p_3 + (d_1+d_2)p_{2,1} - p_{1^3} = 0$
- If we just take the trace of both sides of the equation we get 0. But if we multiply both sides by X and take trace we get a relation among the p_λ. Note that the last term will be $-p_{1^4}$, which equals p_1^4. This implies that p_1 is monic algebraic over $F[p_2, p_3, \ldots]$.
- Using the technique called multilinearization, this implies that $tr(a)tr(b)tr(c)tr(d)$ can be written as a linear combination of terms each with three or fewer traces.

- The computations that I did supporting the conjecture about the Cayley-Hamilton type equation also suggest that p_1 is monic over $F[p_2, p_3, \ldots\}$ and so $D^{(d)}$ satisfies a trace identity of the form

$$tr(x_1) \cdots tr(x_m) =$$

A linear combination of terms with fewer than m traces, and

$$m = \prod(m_i + 1)$$

A Crash Course In Codimensions and Cocharacters

- Let X_i =diag$(x_{i1}, \dots, x_{ik})$ be a generic diagonal matrix
- Let $V_n \subset F[x_{11}, \dots, x_{nk}]$ be the vector space of the evaluations of all degree n, multilinear pure trace polynomials in $X_1, \dots, X_n$
- For example, V_3 is spanned by $tr(X_1X_2X_3), tr(X_1X_3X_2), tr(X_1X_2)tr(X_3), tr(X_1X_3)tr(X_2), tr(X_2X_3)tr(X_1)$ and $tr(X_1)tr(X_2)tr(X_3)$
- If the trace is the (d)-trace, then the dimension of this space is the n-th pure trace codimension of $D^{(d)}$, denoted $c_n\ (D^{(d)})$
- V_n is also a module for the symmetric group S_n, which acts by permuting the X_i. The character is called the n-th cocharacter of $D^{(d)}$, denoted $\chi_n(D^{(d)})$

Trace Codimensions

- If $d = (d_1, d_2)$, distinct, then $c_n\left(D^{(d)}\right) = 2^n - n$
- If $d = (1^k)$, the classical case, then $c_n\left(D^{(d)}\right) \simeq \frac{k^n}{k!}$. In fact, $c_n(D^{(d)})$ in this case is related to the Stirling numbers and equals the number of ways to place n distinguished objects into k indistinguished boxes, some possibly empty.
- In general, $c_n\left(D^{(d)}\right)$ is asymptotic to ak^n for some constant *a*, where *a* is between $1/k!$ And $1/\prod m_i!$ Note that the (d_1, d_2) case is consistent with the latter estimate and the (1^k) case is consistent with both. My guess is that $a = 1/\prod m_i!$

Trace Cocharacters

- The irreducible characters of the symmetric group S_n are indexed by the partitions of n, and are denoted χ^λ
- The cocharacter $\chi_n(D^{(d)})$ can be decomposed into a sum of irreducible parts $\sum m_\lambda \chi^\lambda$. We are interested in the multiplicities m_λ
- In the classical case, $(d) = (1^k)$ the pure trace cocharacter has a number of nice descriptions, computed by Regev and myself, 1995
- m_λ will be zero if λ has more than k parts.
- Let $W^\lambda(k)$ be the irreducible GL(k)-module corresponding to the partition λ. The symmetric group S_k is contained in GL(k) as the permutation matrices. Then the multiplicity m_λ equals the dimension of $\left(W^\lambda\right)^{S_k}$, the subspace of W^λ invariant under the S_k-action.

- Another description of $\chi_n(D_k)$ can be gotten from Molien's theorem.
- A third description from my new paper is an estimate: m_λ is bounded above by the number of semistandard tableaux of shape λ whose type is a partition with at most k parts, and bounded below by the number of semistandard tableaux of shape λ whose type is a strict partition with at most k parts.
- In the case of $d = (d_1, d_2)$, $d_1 \neq d_2$ and $\lambda = (\lambda_1, \lambda_2)$ the multiplicities equal $\lambda_1 - \lambda_2 + 1$, if $\lambda_2 \geq 2$, and equal $\lambda_1 - \lambda_2$, if $\lambda_2 \leq 1$.

- In general if A and B are any two algebras with trace, then $A \oplus B$ also has a trace and the trace cocharacters are related by $\chi_n(A \oplus B) \leq \sum \chi_i(A) \otimes \chi_{n-i}(B)$. We will write $\chi(A \oplus B) \leq \chi(A) \otimes \chi(B)$, for short
- If $d = \left(d_1^{m_1}, \ldots, d_k^{m_k}\right)$, then $D^{(d)} = D^{(d_1^{m_1})} \oplus \cdots \oplus D^{(d_k^{m_k})}$
- Also, the cocharacter of $D^{(d^m)}$ is independent of d. Also, the cocharacter of $D^{(1^m)}$, which I will denote D_m, is known from the previous slides.
- This gives the upper bound $\chi\left(D^{(d)}\right) \leq \chi\left(D_{m_1}\right) \otimes \cdots \otimes \chi\left(D_{m_k}\right)$.
- A lower bound is $\chi(D_m) \leq \chi(D^{(d)})$