The Card Guessing Game: A generating function approach

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Introduction

Blackjack: Riffle shuffle. As a player, we do card counting.

Guess the coming card.

In this project (similar to Blackjack):
Original deck is $[1, 2, 3, \ldots, n]$.
Rule: Riffle shuffle k times, guess card $\to$ reveal card, then repeat.
Goal: Try to make the correct guesses as many as possible.

We discuss the optimal guessing strategy and statistics of the number of correct guesses.

Gilbert-Shannon-Reeds (GSR) Model for Riffle Shuffles

(Bayer & Diaconis 1992)

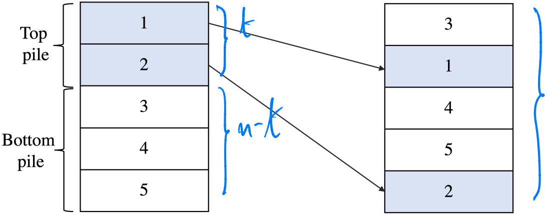
1. Split the deck into two piles.

The probability of cutting the top t cards is $\dfrac{\binom{n}{t}}{2^n}$. $0 \le t \le n$

2. Then, interleave the piles back into a single one.

Each interleaving has probability $\dfrac{1}{2^n}$ to come up.

Riffle shuffle 7 times to make deck random



(a) Before interleaving   (b) After interleaving

$\frac{1}{2^n}$.

Example of 1-time riffle shuffle of a deck of 5 cards

1-time riffle shuffle: examples

Observation:

1. # of ways = 2^n.
2. # of deck = $2^{n-1}+1$ starts with 1
3. Multiplicity:

i.d → $n+1$

others → 1

2 increasing sub. seq.

n = # of cards

$n=1$: 1 1

$n=2$:

1	1	1	2
2	2	2	1

$n=3$:

1	1	1	1	1	2	2	3
2	2	2	2	3	1	3	1
3	3	3	3	2	3	1	2

$n=4$:

1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3
3	3	3	3	3	4	2	4
4	4	4	4	4	3	4	2

1	2	2	2	3	3	3	4
4	1	3	3	1	1	4	1
2	3	1	4	2	4	1	2
3	4	4	1	4	2	2	3

$\boxed{2\ 3\ 4}$ $\boxed{1}$

1 3 2 4 → 1 3 2 4

Optimal guessing strategy

Algorithm:

1:Start by guessing number 1. → 2 → 3 ...

2:If true then continue to guess the next number in line.

3:If false then the deck is now split into two increasing subsequences. Guess the first element in the longer subsequence.

4:Continue to guess this way until until no cards remain.

This algorithm is proved to provide the maximum expected number of correct guesses.

Example

1 ✓ 3 ✗ 4 ✗ 2 ✓   {2}   2 ✗ 1 ✗ 3 ✓ 4 ✓   {3,4}

$n=4$

Permutation π												
	1	1	1	1	1	2	2	2	3	3	3	4
	2	2	3	3	4	1	3	3	1	4	1	1
	3	4	2	4	2	3	1	4	2	1	4	2
	4	3	4	2	3	4	4	1	4	2	2	3
$P(\pi)$	5/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16
#Correct guesses	4	3	3	2 ✓	3	2 ✓	3	2	3	2	2	3

All possible permutations after shuffling a 4-card deck once. The color indicates a correct guess under the optimal strategy.

Goal

The goal is to calculate the moments (i.e. mean, variance, etc) of the number of correct guesses (denote X_n) amongst all of the resulting permutations.

Generating functions and recurrences

$\boxed{Z1}$

Generating functions:

$$D_n(q) = \sum_{i=0}^{\infty} a_i q^i,$$

where a_i denotes the number of permutations with i correct guesses.

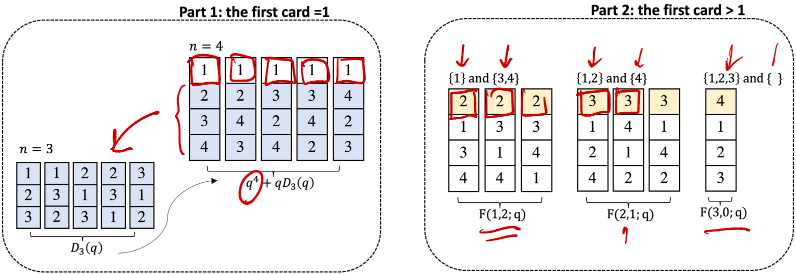
Recurrence:

$D_n(q) = q\,D_{n-1}(q) + G_n(q)$

$$D_n(q) = \underbrace{qD_{n-1}(q) + q^n}_{\text{the first card} = 1} + \underbrace{\sum_{i=0}^{n-2} F(n-1-i, i; q)}_{\text{the first card} > 1}, \qquad \text{(Main recurrence)}$$

where $D_0(q) = 1$.

Example:



Recurrence structure $D_4(q) = \left(q^4 + qD_3(q)\right) + F(1,2;q) + F(2,1;q) + F(3,0;q)$

The catch!

Z2

$$F(m,n;q) = \underbrace{qF(m-1,n;q)}_{\text{next card from longer subsq.}} + \underbrace{F(m,n-1;q)}_{\text{next card from shorter subsq.}}, \quad (1)$$

for $m \geq n$, where $F(m,0;q) = q^m$. Also, $F(m,n;q) := F(n,m;q)$ whenever $m < n$.

It is easy to show the formula of $F(m,n;q)$ (once you know what it looks like).

Proposition

For $m \geq n$,

$$F(m,n;q) = \sum_{i=0}^{n} \left[\binom{m+n}{i} - \binom{m+n}{i-1} \right] q^{m+n-i}. \quad (2)$$

More catch!!

Assume $G_n(q) = q^n + \sum_{i=0}^{n-2} F(n-1-i, i; q)$.
It can be shown that

Proposition

For $r \geq 1$, the closed-form formula for $G_n^{(r)}(q)|_{q=1}$ can be obtained by evaluating the binomial sums:

$$G_{2k}^{(r)}(q)|_{q=1} = (2k)_r - (2k-1)_r + 2\sum_{i=0}^{k-1}(k-i)\left[\binom{2k-1}{i} - \binom{2k-1}{i-1}\right](2k-1-i)_r,$$

$$G_{2k+1}^{(r)}(q)|_{q=1} = (2k+1)_r - (2k)_r + 2\sum_{i=0}^{k}\left(k+\frac{1}{2}-i\right)\left[\binom{2k}{i} - \binom{2k}{i-1}\right](2k-i)_r,$$

where $(a)_r$ is the falling factorial, i.e. $(a)_r = a(a-1)(a-2)\dots(a-r+1)$.

Now the moments!

Z3

Procedure: `Factorial Moment (fixed` r`, formula in` n`)`

Step 1: Compute $G_n^{(r)}(q)|_{q=1}$ by the binomial sum, n symbolic

Step 2: Use the method of undetermined coefficient to calculate $D_n^{(r)}(q)|_{q=1}$.

Step 3: Apply (4) to obtain $E[X(X-1)\dots(X-r+1)]$.

- Step 2 is acquired through the relation:

$$D_n(q) = qD_{n-1}(q) + G_n(q). \qquad \text{(Main recurrence)}$$

$$D_n^{(r)}(q)|_{q=1} = D_{n-1}^{(r)}(q)|_{q=1} + rD_{n-1}^{(r-1)}(q)|_{q=1} + G_n^{(r)}(q)|_{q=1}. \qquad (3)$$

- Step 3 is acquired from the relation:

$$E[X(X-1)\dots(X-r+1)] = \frac{D_n^{(r)}(q)|_{q=1}}{2^n}. \qquad (4)$$

Example: The first moment $E[X]$

$r=1$

- Step 1:

$$G'_{2k}(q)|_{q=1} = \frac{k-1}{2}4^k + k\binom{2k}{k} + 1, \quad \checkmark$$

$$G'_{2k+1}(q)|_{q=1} = \frac{2k-1}{2}4^k + \frac{4k+1}{2}\binom{2k}{k} + 1. \quad \checkmark$$

Example: The first moment $E[X]$ (continue)

- Step 2:

$$D'_{2k+1}(q)|_{q=1} = D'_{2k-1}(q)|_{q=1} + \frac{k}{2}4^k + k\binom{2k}{k} + 1 + \frac{2k+1}{2}4^k + \frac{4k+1}{2}\binom{2k}{k} + 1.$$

Simplifying the equation and writing expression in terms of n,

$$D'_n(q)|_{q=1} = D'_{n-2}(q)|_{q=1} + \frac{3n-1}{8}2^n + \frac{3n-2}{2}\binom{n-1}{(n-1)/2} + 2.$$

Example: The first moment $E[X]$ (continue)

- Step 2: method of undetermined coefficients

  For the term $\dfrac{3n-1}{8}2^n$, we assume the solution to be in the form

  $$(an+b)2^n. \qquad \text{(ansatz1)}$$

  For the term $\dfrac{3n-2}{2}\binom{n-1}{(n-1)/2}$, we assume a solution of the form

  $$\left(a_0\sqrt{n} + \frac{a_1}{\sqrt{n}} + \frac{a_2}{n^{3/2}} + \dots\right)2^n. \qquad \text{(ansatz2)}$$

Example: The first moment $E[X]$ (continue)

$$D_n'(q)|_{q=1} = (n-1)2^{n-1} + 2^n\sqrt{\frac{2n}{\pi}} \left(1 - \frac{3}{4n} - \frac{53}{96n^2} - \frac{443}{384n^3} - \frac{75949}{18432n^4} - \frac{4621519}{221184n^5} - \dots\right).$$

- Step 3:

  Now as $E[X] = \frac{D_n'(q)|_{q=1}}{2^n}$, the expectation for the case when $n = 2k+1$ is given by

$$E[X] = \frac{n}{2} + \sqrt{\frac{2n}{\pi}} - \frac{1}{2} - \sqrt{\frac{2}{\pi n}} \left(\frac{3}{4} + \frac{53}{96n} + \frac{443}{384n^2} + \frac{75949}{18432n^3} + \frac{4621519}{221184n^4} + \dots\right).$$

Some other results:

For $n = 2k$:

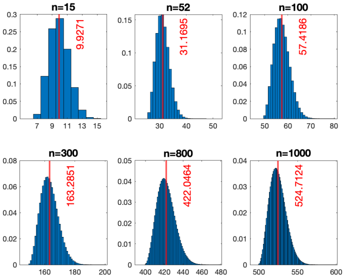
$$E[X] = \frac{n}{2} + \sqrt{\frac{2n}{\pi}} - \frac{1}{2} - \sqrt{\frac{2}{\pi n}}\left(\frac{3}{4} + \frac{49}{96n} + \frac{439}{384n^2} + \frac{76709}{18432n^3} + \dots\right).$$

$$E[(X-\mu)^2] = \left(\frac{3}{4} - \frac{2}{\pi}\right)n - \frac{3}{4} + \frac{3}{\pi} - \sqrt{\frac{2}{\pi n}} + \frac{11}{12\pi n} + \dots.$$

$$E[(X-\mu)^3] =$$
$$\sqrt{\frac{2}{\pi}}\left(\left(\frac{4}{\pi} - \frac{5}{4}\right)n^{3/2} + \left(\frac{43}{16} - \frac{9}{\pi}\right)n^{1/2} - \frac{3\sqrt{2\pi}}{4} + 3\sqrt{\frac{2}{\pi}} + \dots\right).$$

Non-normal distribution

The skewness coefficient is given by $\frac{m_3}{m_2^{3/2}}$, where $m_r := E[(X-\mu)^r]$. We see that the skewness of X_n does not tend to zero. Therefore, the number of correct guesses is not asymptotically normally distributed.



Probability histograms of the number of correct guesses when n varies. The red vertical line indicates the corresponding expected value $E[X_n]$.

Generalization to k Riffle Shuffles: Is it possible?

This problem seems very difficult at the moment.

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