THE THREAD OF RAMSEY THEORY IN ZEILBERGER'S WORK

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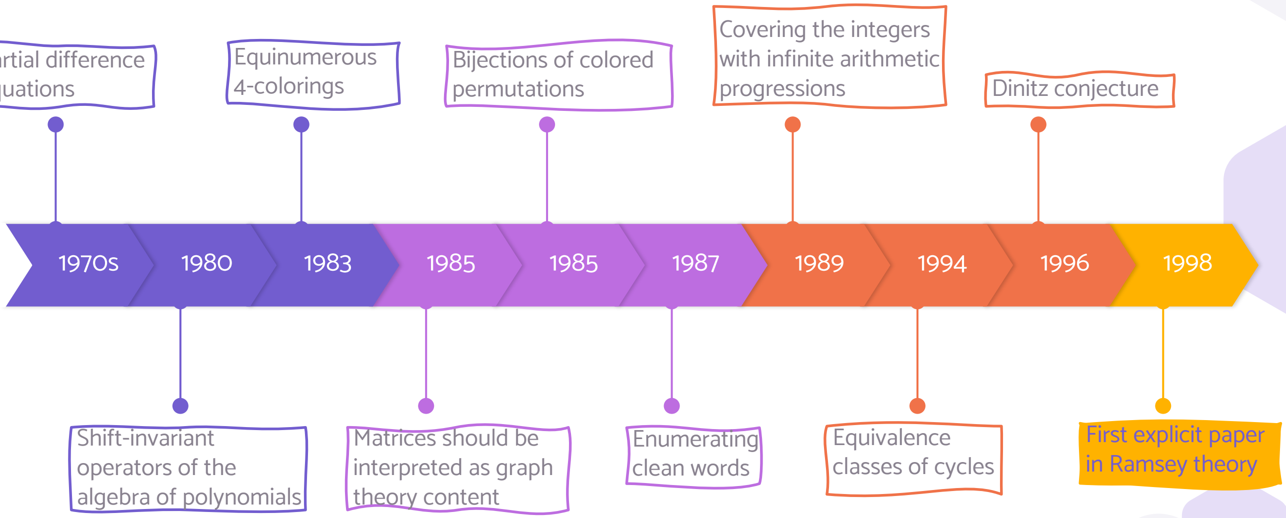
INTRODUCTION

or: Why Am I Doing What I Am Doing?

FROM PDEs TO RAMSEY



Most speakers have no clue how to give a general talk. They start out, very nicely, with ancient history, and motivation, for the first five minutes, but then they start racing into technical lingo that I doubt even the experts can fully follow.

Please! Expand these first five minutes into fifty minutes, tell us about the history, background, motivation, and you don't have to even mention your own results.

– Doron Zeilberger
Opinion 104

1980: Some Comments on Rota's Umbral Calculus

- The shift operator is at the heart of umbral calculus.
- Discusses relationship with Fourier analysis

Connection

> **Szemerédi's Theorem.** *Let A be a subset of integers with positive upper density. Then A contains arbitrarily long arithmetic progressions.*

> **Furstenberg's Multiple Recurrence Theorem.** *For any finite measure space, let A be a measurable set with positive measure and let T be the shift operator. Then there exists a positive integer d such that*

$$\bigcap_{j=0}^{k-1} T^{-jd}(A)$$

has positive measure.

> **Gowers** develops new Fourier analysis tools to give a new proof of Szemerédi's Theorem and, consequentially, a significantly improved upper bound for the minimal integer $n=w(k;r)$ such that every r-coloring of $[1,n]$ admits a monochromatic k-term arithmetic progression.

1983: A Direct Combinatorial Proof of a Positivity Result

- Co-authored with Joe Gillis

- Given a prescribed number of each of 4 types of hats (colors) from n people, it is more likely that a redistribution of hats to the n people has an even number of incorrect types than odd number of incorrect types.

- If we have an equal number of each type of hat, we have an equinumerous 4-coloring of $[1,n]$.

Connection

> **On the Existence of Rainbow 4-Term Arithmetic Progressions**

Conlon, Jungić, and Radoičić

> *For any n, there exists an equinumerous 4-coloring of [1,n] that admits no 4-term arithmetic progression with 4 distinct colors (rainbow arithmetic progression).*

> (However, earlier Jungić and Radoičić had shown that equi-numerous 3-colorings always admit rainbow 3-term arithmetic progressions.)

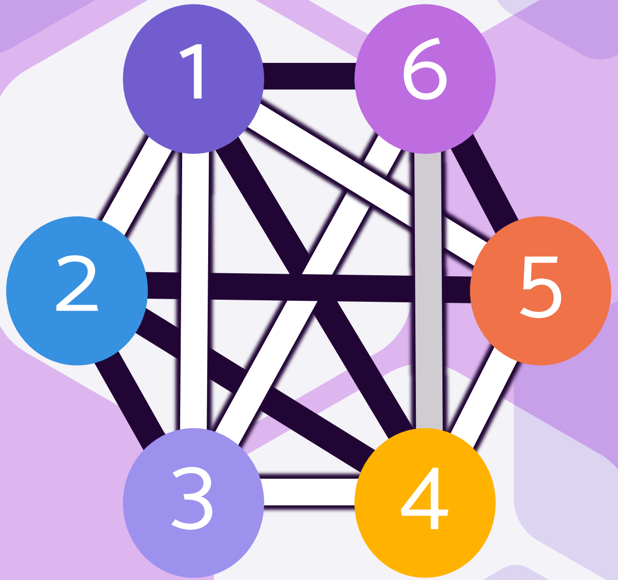
> **Canonical van der Waerden Theorem.** *Every coloring of the positive integers (with, perhaps, infinitely many colors) admits arbitrarily long arithmetic progressions that are either monochromatic or rainbow.*

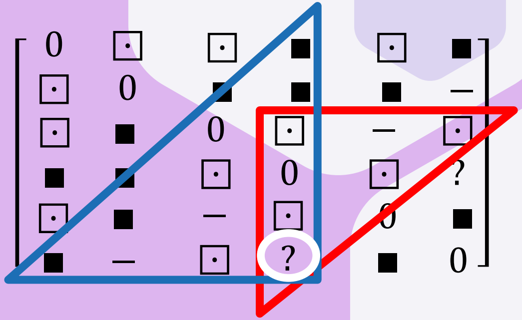
1985: A Combinatorial Approach to Matrix Algebra

- Views matrices not as linear transformations but as weights of graph edges
- A matrix is the “blueprint” of all possible edges one can draw on n given vertices
- Uses two “types” of edges – in other words, colors

Connection

➢ **Ramsey's Theorem.** *For all positive integers k and r, there exists a minimal integer n=R(k;r) such that every r-coloring of the edges of the complete graph on n vertices admits a monochromatic complete graph on k vertices.*





1985: Some Asymptotic Bijections

- Translates k-colorings to set partitions to give bijections related to the Bell numbers
- Works with signed permutations – in other words, 2-colored permutations

Connection

> **Alon** uses the Gallai-Witt Theorem to prove:

> *Every r-coloring of the edges of the complete graph on n vertices contains, for n sufficiently large, k vertices* $v_1 < v_2 < \cdots < v_k$ *such that the differences between consecutive vertices follow any prescribed permutation pattern and all edges between these vertices have the same color.*

> (

Gallai-Witt Theorem. Let $\boldsymbol{S} \subseteq \mathbb{Z}^m$ *be a finite set. Every r-coloring of the points in* $[-n,n]^m$ *admits, for n sufficiently large, a monochromatic set of the form* $\boldsymbol{a}+d\boldsymbol{S}$ *for some* $\boldsymbol{a} \in \mathbb{Z}^m$ *and* $d \in \mathbb{Z}$.

Also used to prove the **Canonical van der Waerden Theorem.**

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1987: Enumerating Totally Clean Words

- Generating function for words over a finite alphabet that do not contain a subsequence of a given set of forbidden words.

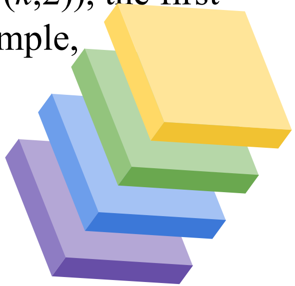
Connection

> **Hales-Jewett Theorem.** *Let k, $r \in \mathbb{Z}$. Let $W(m)$ be the set of all variable words of length m over the alphabet $\{1, 2,\ldots, k\} \cup \{x\}$. There exists a minimal positive integer $H = HJ(k;r)$ such that for any $h \geq H$, every r-coloring of the elements of $[1, k]^h$ admits $w(x) \in W(h)$ with $\{w(i) : i \in [1, k]\}$ monochromatic.*

> **Variable word**. *A word over any given alphabet that includes a variable (x). For example, 12x is a variable word, but 312 is not.*

> **Example**. Color each word over the alphabet $\{1,2\}$ of length r with one of r colors (so we have 2^{2^r} possible colorings). Consider the $r+1$ words 11…11, 11…12, 11…122, …, 22…22. Two must have the same color. Hence, there exists a variable word of the form $w(x)$=11…*xx*…*xx*..22 such that w(1) and w(2) have the same color.

> **Motivation**. To analyze multidimensional tic-tac-toe. They showed that for a board of sufficiently large dimension (HJ(k;2)), the first player can always win. For example, we know that HJ(3;2)>2 since standard tic-tac-toe does not guarantee a winner.



1989: On a Conjecture of R.J. Simpson About Exact Covering Sequences

- Exact covering system: A set of non-intersecting bi-directional infinite arithmetic progressions that cover $\mathbb{Z}$. Call the common difference of terms in an arithmetic progression the *modulus*.
- Conjecture: Let D = $\{d_1, d_2, \ldots, d_n\}$ such that $\sum_{i=1}^{n} \frac{1}{d_i} = 1$. Then there exists an exact covering system with moduli set equal to D if and only if there exists a specified (p+1)-coloring of D for every prime p such that $\sum \frac{1}{d_i}$ is constant on p of the color classes.
- Mirsky-Newman result shows that not all d_i can be distinct.
- Zeilberger provides counterexamples to the conjecture.

Connection

> Exact covering systems give a well-defined finite coloring of the positive integers by infinite monochromatic arithmetic progressions.

> Addressing a related coloring converse, although van der Waerden's Theorem informs us that every finite coloring of $\mathbb{Z}^+$ contains arbitrarily long monochromatic arithmetic progressions, there is no guarantee of an infinite monochromatic arithmetic progression.

> Consider the 2-coloring of $\mathbb{Z}^+$:

$$011000011111111\ \ldots$$

i.e.,

$$0^1\ 1^2\ 0^4\ 1^8\ 0^{16}\ \ldots$$

> Any infinite arithmetic progression of modulo d with $2^{n-1} \leq d < 2^n$ has a term in a color block of length 2^n, meaning one of the next two terms has the opposite color and hence cannot be mono-chromatic.

1994: Talmudic Lattice Path Counting

- Co-authored with Jane Friedman and Ira Gessel
- Counts paths from $(0,0)$ to (a,b) and partitions these paths by the number of points below the line $y = \frac{b}{a}x$ and dealing with points on the line.
- Proof uses partial sums of sequences and what to do with equal partial sums.

Connection

> Consider this little gem:

> Let $a_1, a_2, \ldots, a_n$ be a sequence of n integers. Then there exists a subsequence whose sum is divisible by n.

> Let $s_i = a_1 + a_2 + \cdots + a_i$. Assume none of the s_i, $1 \le i \le n$, are divisible by n. Then two of them are congruent modulo n, say s_x and s_y with $y > x$.

> Then

$$s_y - s_x = a_{x+1} + a_{x+2} + \cdots + a_y$$

shows that $a_{x+1}, a_{x+2}, \ldots, a_y$ is the desired subsequence.

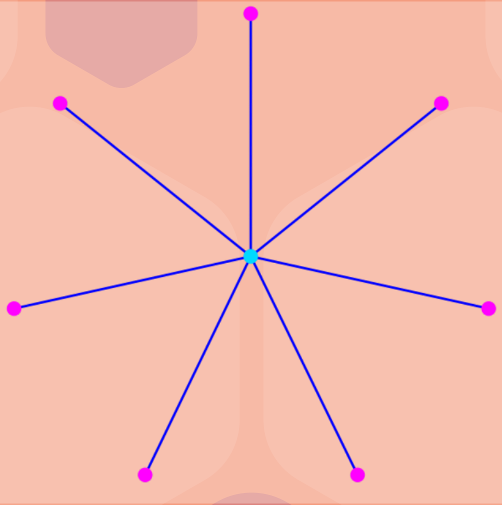
> (This is from the area of zero-sum Ramsey theory.)

1996: The Method of Undetermined Generalization and Specialization Illustrated with Fred Galvin's Amazing Proof of the Dinitz Conjecture

- The conjecture: Consider the complete bipartite graph $K_{n,n}$. Assume each edge has a list of n allowed colors from a pool of $m \geq n$ possible colors. Then there exists a color assignment for each edge such that no 2 edges with a common vertex are assigned the same color.

- This paper contains the first explicit mention of coloring (as a type of dual of the above description, coloring vertices instead of edges).

Connection



> An edge-coloring of a graph is called a **proper coloring** if no two adjacent edges have the same color. This is precisely the coloring in the conclusion of Dinitz's Conjecture.

> **Rainbow Ramsey Theorem.** Historically (Erdős, Simonovits, Sós; 1973) started as an anti-Ramsey property: AR(G,H) is the maximal number of colors allowed to color the edges of G such that every subgraph H has at least two edges of the same color.

> A proper coloring adds the restriction that the edges are adjacent.

> For the Dinitz Conjecture, creating an inflated star graph S (replace each vertex of a star graph by n vertices, using no edges instead of the usual K_n) containing all possible color combinations of edges (defined by those allowed by each edge's assigned list), we are searching for a properly colored $K_{n,n}$.

1998: A 2-coloring of [1,n] Can Contain $n^2/22 + O(n)$ Monochromatic Schur Triples, But Not Less!

- Co-authored with AR.
- Schur Triple is $\{x, y, z\}$ such that $x + y = z$.
- Prize awarded from Ron Graham (shared with Schoen, who independently solved this at the same time)

Connection

> Trivial

THANKS!

Any questions?

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