Journal of Mathematical Physics

RESEARCH ARTICLE | SEPTEMBER 01 1962

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*J. Math. Phys.* 3, 1040–1043 (1962)
https://doi.org/10.1063/1.1724291



JOURNAL OF MATHEMATICAL PHYSICS VOLUME 3, NUMBER 5 SEPTEMBER-OCTOBER 1962

# Proof of a Conjecture by Dyson

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(Received April 27, 1962)

We prove a mathematical conjecture by Dyson which he used in a study of the statistical distribution of energy levels in complex nuclei.

THE purpose of this paper is to prove a theorem that was proposed by Dyson in a recent paper.[1,2] This theorem (conjecture C of reference 1) may be stated as follows:

Let $z_1, \cdots, z_N$ be a set of $N$ complex variables and $a_1, \cdots, a_N$ a set of $N$ positive integers ($\geq 0$). Let $y_j$ for $1 \leq j \leq N$ be

$$y_j = \prod_{k \neq j} \left(1 - \frac{z_j}{z_k}\right). \tag{1}$$

Let

$$P(z_1, \cdots, z_N) = \prod_j (y_j)^{a_j}, \tag{2}$$

which we shall write $P(z)$ for convenience. $P$ can be expanded in positive and negative powers of the $z_j$; we are interested in the constant term $F$ given by

$$F = (2\pi i)^{-N} \int z_1^{-1}\, dz_1 \cdots \int z_N^{-1}\, dz_N P(z), \tag{3}$$

the contours being the unit circle taken counterclockwise.

*Theorem*:

$$F = (\sum_j a_j)!\, \{\prod_j a_j!\}^{-1}. \tag{4}$$

The proof depends on three lemmas, which will be stated now and proved later.

Define

$$u_j = (y_j)^{-1}. \tag{5}$$

We compute $F$ by making a change of variable from $z_1, \cdots, z_{N-1}$ to $u_2, \cdots, u_N$. Lemma 1 will state that this is possible. We note that because $P$ is homogeneous in the $z$'s, the integration over $z_N$ becomes trivial after the other integrations have been performed, and that because the $u$'s are homogeneous in the $z$'s, only $N - 1$ are independent, which is why we make $N - 1$ instead of $N$ changes of variable.

*Lemma* 1:

$$F = (2\pi i)^{-N+1} \int u_2^{-1}\, du_2 \cdots \int u_N^{-1}\, du_N \times [J(z)]^{-1} P(z) \tag{6}$$

where $J$ is the Jacobian

$$J(z) = \frac{\partial(\ln u_2, \cdots, \ln u_N)}{\partial(\ln z_1, \cdots, \ln z_{N-1})} \tag{7}$$

and the $z$'s are expressed in terms of the $u$'s by Eq. (5). The paths of integration are the circles

$$|u_i| = R_i \tag{8}$$

*taken* $i - 1$ *times* counterclockwise, where the $R_i$ are arbitrary except that they satisfy

$$R_{i+1} \ll R_i \ll 1 : 2 \leq i \leq N - 1. \tag{9}$$

Lemmas 2 and 3 will show that $J$ and $P$ are single-valued functions of the $u$'s so we do not have to specify the branch of the solution of Eq. (5).

*Lemma* 2:

$$\sum_{j=1}^{N} u_j = 1. \tag{10}$$

This is an identity in the $z$'s, when the $u_j$ are regarded as functions of the $z$'s through Eq. (5).

*Lemma* 3:

$$J(z) = (N - 1)!\, u_1. \tag{11}$$

See the note for Lemma 2.

As a result of these lemmas,

$$F = \frac{(2\pi i)^{-N+1}}{(N-1)!} \int du_2 \cdots \int du_N \prod_{j=1}^{N} (u_j)^{-a_j - 1}, \tag{12}$$

with $u_1$ given by Lemma 2. These integrals are elementary, but to save space we use a shortcut.

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[1] F. J. Dyson, J. Math. Phys., **3**, 140, 157, 166 (1962), this theorem has also been proved independently by J. Gunson, J. Math. Phys. (to be published).

[2] Conjectures A, B, and D of reference 1 are shown there to reduce to conjecture C; thus they will not be discussed here.

First we note that

$$(2\pi i)^{-1}\int u^{-n-1}e^{u}\,du = 1/n! \tag{13}$$

if the contour encloses the origin counterclockwise. Secondly, if $F(\lambda)$ is defined by Eq. (12), but with

$$u_1 = \lambda - \sum_{j=2}^{N} u_j, \tag{14}$$

where $\lambda$ is on the unit circle, then by making a change of variable to $u_j' = u_j\lambda$ we see that

$$F(\lambda) = \lambda^{-a-1}F, \tag{15}$$

where

$$a = \sum_{j=1}^{N} a_j. \tag{16}$$

Hence, by Eq. (13),

$$F = (2\pi i)^{-1}a!\int e^{\lambda}F(\lambda)\,d\lambda. \tag{17}$$

Interchanging the order of integration so that we integrate over $\lambda$ with $u_2, \cdots, u_N$ held fixed, we must evaluate the integral

$$\frac{1}{2\pi i}\int e^{\lambda}\left\{\lambda - \sum_{j=2}^{N} u_j\right\}^{-a_1-1} d\lambda = \frac{1}{a_1!}e^{u_2+\cdots+u_N} \tag{18}$$

[by Eqs. (8) and (9), the pole at $\lambda = u_2 + \cdots + u_N$ lies inside the unit circle].

Thus,

$$F = \frac{a!}{(2\pi i)^{N-1}(N-1)!\,a_1!}\int du_2 \cdots \int du_N \times \prod_{j=2}^{N} e^{u_j}u_j^{-a_j-1} = \frac{a!}{a_1!\cdots a_N!} \tag{19}$$

[the factor $(N-1)!$ is exactly compensated by the requirement of Lemma 1 that $u_j$ execute a circle $j-1$ times].

Now we prove Lemmas 1–3. We start by stating another lemma. Lemmas 2–4 have probably all occurred in other work[3] but it is easier to prove them than to locate them in the literature.

*Lemma* 4:

Let $G(x_1, \cdots, x_M)$ be a function of $M$ variables such that

1. $G$ is a symmetric function of $x_1, \cdots, x_M$,
2. $G$ is a ratio of two polynomials in the $x$'s,
3. $G$ is homogeneous of degree 0 in the $x$'s,
4. The denominator of $G$ is $\prod_{j<k}(x_j - x_k)$.

Then $G$ is a constant. This is because, since the denominator changes sign when we interchange the values of any pair $x_j$, $x_k$, the numerator must also change sign under this exchange; thus, the numerator vanishes when $x_j = x_k$. Hence the numerator has $x_j - x_k$ as a factor (for any $j$ and $k$), e.g., it has the entire denominator as a factor; thus $G$ is a polynomial. Since it is of degree 0, it must be a constant.

*Proof of Lemma* 2:

Since $\sum_j u_j$ considered as a function of the $z$'s satisfies the conditions of Lemma 4, it is constant. Putting $z_1 = 0$ we obtain $u_1 = 1$, $u_j = 0$ $(j > 1)$ so the constant is 1.

*Proof of Lemma* 3:

The Jacobian $J$ is the determinant of the matrix

$$J_{ij} = \partial \ln u_i/\partial \ln z_j \tag{20}$$

(rows numbered $i = 2$ to $N$, columns $j = 1$ to $N-1$). Without changing the value of the determinant we may add columns $j = 2$ through $N-1$ to column 1; since $\ln u_i$ is homogeneous in the $z$'s we now have

$$J_{i1} = -\partial \ln u_i/\partial \ln z_N. \tag{21}$$

Move this column to the right, calling it $J_{iN}$; thus,

$$J = (-1)^{N-2}\det|J_{ij}|, \quad \text{where} \quad 2 \le i \le N, \quad 2 \le j \le N. \tag{22}$$

Now,

$$J_{ij} = -z_i(z_j - z_i)^{-1} \qquad (i \ne j), \tag{23}$$

$$J_{ii} = \sum_{k\ne i}\frac{z_i}{z_k - z_i} \tag{24}$$

Evidently $J$ is the ratio of two polynomials in the $z$'s the denominator being a product of factors $z_j - z_i$. No such factor occurs twice; for a denominator $z_j - z_i$ occurs only in the elements $J_{ii}$, $J_{ij}$, $J_{ji}$, and $J_{jj}$ so that a term $(z_j - z_i)^2$ occurs in the denominator of $J$ only if it occurs in the $2 \times 2$ determinant $J_{ii}J_{jj} - J_{ij}J_{ji}$. However the term in $J_{ii}J_{jj}$ containing the factor $(z_i - z_j)^{-2}$ cancels the corresponding term in $J_{ij}J_{ji}$. Furthermore, $J$ has a factor $(z_2z_3 \cdots z_N)$, it is symmetric in $z_2$ through $z_N$ (but not in $z_1$), and it is homogeneous of degree 0 in the $z$'s. Using the argument of Lemma 4, we must have

$$J = C\prod_{k=2}^{N} z_k(z_k - z_1)^{-1} = Cu_1, \tag{25}$$

where $C$ is a constant. Since Eq. (4) is known to

[3] Lemma 2 appears in the theory of Lagrangian interpolation; see F. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill Book Company, Inc., New York, 1956), p. 61, Eq. (3.2.5) with $x = 0$.

be true if all the $a_j$ are 0 we shall have to have $C = (N-1)!$.

*Proof of Lemma* 1:

We shall prove Lemma 1 by introducing the new variables one at a time. For convenience we shall first change variables to $t_j$ $(2 \le j \le N)$, where

$$(t_j)^{-j+1} = y_j. \tag{26}$$

Before making this change we change the paths of integration in Eq. (3) to be the circles

$$|z_j| = r_j \qquad (1 \le j \le N), \tag{27}$$

where

$$r_j \ll r_{j+1} \qquad (1 \le j \le N-1). \tag{28}$$

We now use mathematical induction. Suppose the following propositions are true for $m < n$:

1. Equation (26) can be solved for $2 \le j \le m+1$ to give $z_1$ through $z_m$ as functions of $t_2, \cdots, t_{m+1}, z_{m+1}, \cdots, z_N$. We define functions $\omega_j$ for $1 \le j \le m$ by

$$z_j = z_{j+1} t_{j+1} (t_j)^{-(j-1)/j} e^{(i\pi/j)} \times \omega_j(t_2, \cdots, t_{j+1}, z_{j+1}, \cdots, z_N), \tag{29}$$

where we define $t_1 = 1$ and $0 \le \arg t_j \le 2\pi$. The solution can be chosen so that $\omega_j \approx 1$ in the region of interest. More specifically, $\omega_j$ is analytic as a function of $t_{j+1}$ and $z_{j+1}$ to $z_N$, and satisfies

$$|\omega_j - 1| < \tfrac{1}{2}\epsilon \tag{30}$$

when

$$|t_k| = R_k \qquad (2 \le k \le j), \tag{31}$$

$$\tfrac{1}{2}R_{j+1} < |t_{j+1}| < 2R_{j+1}, \tag{32}$$

$$\tfrac{1}{4}(1+\epsilon)^k r_k < |z_k| < 4(1-\epsilon)^k r_k \qquad (j+1 \le k \le N), \tag{33}$$

where

$$R_j = (r_1 r_2 \cdots r_{j-1})^{1/(j-1)} r_j^{-1} \tag{34}$$

and $\epsilon$ is a fixed number $\ll 1$.[4]

2.

$$F = \frac{1}{(2\pi i)^N} \int dt_2 \cdots \int dt_{m+1} \int \frac{dz_N}{z_N} \cdots \int \frac{dz_{m+1}}{z_{m+1}} \times \left\{ \frac{dt_2}{dz_1} \frac{dt_3}{dz_2} \cdots \frac{dt_{m+1}}{dz_m} \right\}^{-1} \{z_1 z_2 \cdots z_m\}^{-1} P(z), \tag{35}$$

where the integrations are carried out from right to left (e.g., $z_{m+1}$ first, holding the other $z$'s and $t$'s fixed). The contours are the circles $|z_k| = r_k$, $|t_k| = R_k$. The symbol $dt_{k+1}/dz_k$ stands for the partial derivative $\partial t_{k+1}/\partial z_k$ when $t_2, \cdots, t_k, z_{k+1}, \cdots z_N$ are held fixed.

We now prove these propositions for $m = n$. First, we must examine the dependence of $t_{n+1}$ on $z_n$, when $t_2, \cdots, t_n, z_{n+1}, \cdots z_N$ are held fixed; $z_1$ to $z_{n-1}$ are functions of these variables and $z_n$ [through Eq. (26)], and satisfy the restrictions of proposition 1. We obtain

$$t_{n+1}(t_n)^{-(n-1)/n} = (y_n/y_{n+1})^{1/n} = (-1)^{1/n} z_n (z_{n+1})^{-1} g(z), \tag{36}$$

where

$$g(z) = \prod_{j=1}^{n-1} \left(1 - \frac{z_j}{z_n}\right)^{1/n} \left(1 - \frac{z_j}{z_{n+1}}\right)^{-1/n} \times \prod_{j=n+2}^{N} \left(1 - \frac{z_n}{z_j}\right)^{1/n} \left(1 - \frac{z_{n+1}}{z_j}\right)^{-1/n}. \tag{37}$$

If $t_2, \cdots, t_n$ and $z_n, \cdots, z_N$ satisfy the inequalities (31) and (33), one finds using Eqs. (29), (30), and (34) that Eq. (33) is satisfied also by $z_1, \cdots, z_{n-1}$. Thus (if the ratios $r_i/r_{i+1}$ are sufficiently small) we may define the $n$th roots in $g(z)$ by requiring

$$|g(z) - 1| < \tfrac{1}{4}\epsilon; \tag{38}$$

the other $n$th roots are defined as in Eq. (29).

Now consider the equation $x = t_{n+1}(z_n)$ for values of $x$ such that $\tfrac{1}{2}R_{n+1} < |x| < 2R_{n+1}$. This does not differ very much from the equation

$$x - (t_n)^{(n-1)/n}(-1)^{1/n} z_n (z_{n+1})^{-1} = 0, \tag{39}$$

which has a unique root $z_n$ for a given $x$.

Now consider the functions

$$f(z_n) = x - t_{n+1}(z_n),$$
$$g(z_n) = x - (t_n)^{(n-1)/n}(-1)^{1/n} z_n (z_{n+1})^{-1}, \tag{40}$$

where $\tfrac{1}{2}R_{n+1} < |x| < 2R_{n+1}$. On the circles $|z_n| = r_n/3$ and $|z_n| = 3r_n$, $f$ and $g$ are almost equal so that

$$|g(z_n)[f(z_n)]^{-1} - 1| < 1. \tag{41}$$

Since $g(z_n)$ has a single root between the two circles, $f(z_n)$ does also, by Rouché's theorem.[5] From the explicit formula for the inverse function[5]

$$z_n(t_{n+1}) = \frac{1}{2\pi i} \int [t_{n+1}(z_n) - t_{n+1}]^{-1} \frac{dt_{n+1}}{dz_n} z_n \, dz_n, \tag{42}$$

where the contour is taken on the two circles, we

[4] To be precise, we should choose a sufficiently small value for $\epsilon$, then choose the $r_i$ with $r_{i+1}/r_i$ sufficiently small.

[5] L. Ahlfors, *Complex Analysis* (McGraw-Hill Book Company, Inc., New York, 1953), p. 124.

see that the inverse function is analytic in $t_{n+1}$ for $\frac{1}{2}R_{n+1} < |t_{n+1}| < 2R_n$. Since $t_{n+1}(z_n)$ is analytic in $z_{n+1}, \cdots, z_N$, the inverse function is also. Since $\omega_n = 1/g(z)$, it satisfies the inequality (30).

This proves proposition 1 for $m = n$. We now change variables in Eq. (35) (with $m = n - 1$) from $z_n$ to $t_{n+1}$; the path of $t_{n+1}$ is almost the circle $|t_{n+1}| = R_{n+1}$; we change it to be exactly this circle and then interchange the order of integration with the remaining $z$'s. This proves proposition 2 for $m = n$. To complete the proof of propositions 1 and 2 they must be proved for $m = 1$; with some changes the above procedure can be used.

Now consider Eq. (35) with $m = N - 1$. The expression $I = dt_2/dz_1 \cdots, dt_N/dz_{N-1}$ is the Jacobian[6]

$$\frac{\partial(t_2, \cdots, t_N)}{\partial(z_1, \cdots, z_{N-1})};$$

by Lemma 3,

$$I = (t_2 \cdots t_N)(z_1 \cdots z_{N-1})^{-1}u_1^{-1}. \tag{43}$$

Since $u_1$ is a function only of the $t$'s, by Lemma 2, the integral over $z_N$ is trivial. Changing variables from $t_n$ to $u_n = t_n^{n-1}$ we obtain Lemma 1 [Eq. (6)].

I am indebted to Dr. Paul Federbush for suggesting this problem.

[6] R. Courant, *Differential and Integral Calculus*, (Interscience Publishers, Inc., New York, 1936), Vol. II, pp. 247–256.