A GENERALIZATION OF CONTINUED FRACTIONS

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ABSTRACT. We investigate a generalization of classical continued fractions, where the "numerator" 1 is replaced by an arbitrary positive integer N. We find both similarities to and surprising differences from the classical case.

Let N be an arbitrary positive integer. In this paper we consider continued fractions of the form

$$a_0 + \cfrac{N}{a_1 + \cfrac{N}{a_2 + \cfrac{N}{a_3 + \cdots}}},$$

with a_0 a nonnegative integer and $a_1, a_2, a_3, \ldots$ positive integers. We denote such a continued fraction by $[a_0, a_1, a_2, a_3, \ldots]_N$ and refer to it as a cf_N expansion. While this seems to us to be a natural generalization of classical continued fractions, i.e., the $N = 1$ case, it has not been much studied previously, though see [1, 2]. We state the main result of [1], in our language, in 2.23 below.

As we shall see, the $N > 1$ case has both a number of similarities to and some surprising differences from the $N = 1$ case.

In Section 1 of this paper, we establish foundational results on cf_N expansions. We show that every positive real number x_0 has a cf_N expansion, though for $N > 1$ it always has infinitely many. For $N > 1$, every rational number has both finite and infinite (i.e., nonterminating) cf_N expansions, and for $N > 2$ it has nonperiodic expansions. For $N > 1$, every quadratic irrationality has both periodic and nonperiodic expansions. Here we use the standard language and notation: $x_0 = [a_0, a_1, a_2, \ldots]_N$ is periodic of period k from $i = m$ if $a_{i+k} = a_i$ for all $i \geq m$, and in this case we write $x_0 = [a_0, \ldots, a_{m-1}, \overline{a_m, \ldots, a_{m+k-1}}]_N$.

We also develop a natural notion of a best cf_N expansion of the real number x_0, which we denote by $x_0 = [[a_0, a_1, a_2, \ldots]]_N$.

In Section 2 we turn our attention to quadratic irrationalities. We show that, for $N > 1$, every quadratic irrationality has periodic cf_N expansions, and that in many cases the best cf_N expansion of a quadratic irrationality is periodic, but, on the grounds of extensive computational results, we conjecture (Conjecture 2.3) that this is *not* always the case. We focus our attention on quadratic irrationalities $\sqrt{E}$, where E is an integer that is not a perfect square. We establish here some notation and language that we will use throughout: We let $D = \lfloor \sqrt{E} \rfloor$, so that $E = D^2 + a$ with $1 \leq a \leq 2D$. We also say that N is *small* (for E) if $N \leq 2D$ and N is *large* (for E) otherwise. Note that $N = 1$ is always small. We show that if $[[\sqrt{E}]]_N$ is periodic, the period begins with $i = 1$ if N is small, as in the classical case, and with $i = 2$ if N is large. Also in the classical case the continued fraction expansion of $\sqrt{E}$ has a very special form, and we show that $[[\sqrt{E}]]_N$ has the same form for N small,

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in cases when it is periodic, but that it sometimes but not always has a similar form for N large, in cases when it is periodic.

The theory of classical continued fractions is intimately related to Pell's equation, and in Section 3 we investigate the analog in the $N > 1$ case. In the classical case there is a recursion for (p_i, q_i), where $C_i = p_i/q_i$ is the i-th convergent of $\sqrt{E}$. Setting $w_i = p_i^2 - Eq_i^2$, we have that $\{w_i\}$ is periodic and that all solutions to Pell's equation $p^2 - Eq^2 = 1$ are to be found among $\{(p_i, q_i)\}$. Part of this goes through for arbitrary N. We have a natural generalization of periodicity that we call f-periodicity (i.e., periodicity up to a factor of f). We again have a recursion for (p_i, q_i), when $C_i = p_i/q_i$ is the i-th convergent of $\sqrt{E}$, and we show that $\{w_i = p_i^2 - Eq_i^2\}$ is f-periodic whenever $[[\sqrt{E}]]_N$ is periodic. But for $N > 1$, p_i and q_i need not be relatively prime. Writing $C_i = \tilde{p}_i/\tilde{q}_i$, a fraction in lowest terms, we consider $\{\tilde{w}_i = \tilde{p}_i^2 - E\tilde{q}_i^2\}$. We conjecture (Conjecture 3.11) that $\{\tilde{w}_i\}$ is f-periodic whenever $[[\sqrt{E}]]_N$ is periodic. We show this is true in a number of cases, where we obtain precise information, and we give computational results that indicate the possibilities that appear.

In this paper, we give three sorts of results: completely general results, results on $[[\sqrt{E}]]_N$ that hold for general families of E and N, and results on $[[\sqrt{E}]]_N$ for particular values of E and N. The behavior of $[[\sqrt{E}]]_N$ is far more varied and intricate for $N > 1$ than it is in the classical case of $N = 1$, and so we have made a point of giving many examples to illustrate the wide sort of behavior that can occur.

1. General results

Lemma 1.1. *Let b_0 be a nonnegative real number and let $b_1, \ldots, b_n$ be positive real numbers.*

(a) $[b_0, b_1, \ldots, b_n]_N = [b_0, b_1, \ldots, b_{k-1}, [b_k, b_{k+1}, \ldots, b_n]_N]_N$.
(b) $[b_0, b_1, \ldots, b_n]_N = [b_0, b_1, \ldots, b_{n-1} + N/b_n]_N$.
(c) *for any positive integer m,*

$$\left[b_0, mb_1, b_2, mb_3, \ldots, kb_n\right]_{mN} = \left[b_0, b_1, \ldots, b_n\right]_N,$$

where $k = 1$ if n is even and $k = m$ if n is odd.

Proof. (a) and (b) are immediate and (c) is an easy inductive computation. □

Theorem 1.2. *Define sequences $\{p_n\}$ and $\{q_n\}$ inductively by*

$$\begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_n &= b_n p_{n-1} + p_{n-2}N \quad n \geq 0 \\ q_{-2} &= 1/N, & q_{-1} &= 0, & q_n &= b_n q_{n-1} + q_{n-2}N \quad n \geq 0. \end{aligned}$$

Let $C_n = p_n/q_n$ for $n \geq 0$. Then for every $n \geq 0$,

$$C_n = \left[b_0, b_1, \ldots, b_n\right]_N.$$

Proof. Well-known for $N = 1$ and easily generalized. □

Theorem 1.3. *In the situation of Theorem 1.2,*

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1} N^n, \quad \text{for } n \geq 1.$$

Proof. This is a special case of [4, page 8, formula (30)] and easily follows from an inductive argument. □

Theorem 1.4. *Let a_0 be a nonnegative integer and let $a_1, a_2, \ldots$ be positive integers. Then*

$$\left[a_0, a_1, a_2, \ldots\right]_N = \lim_{n\to\infty} \left[a_0, a_1, a_2, \ldots, a_n\right]_N$$

exists.

Proof. By Lemma 1.1(c), for each n,

$$\left[a_0, a_1, \ldots, a_n\right]_N = \left[b_0, b_1, \ldots, b_n\right]_1$$

with $b_i = a_i$ for i even and $b_i = a_i/N$ for i odd. Let $C_n = \left[b_0, b_1, \ldots, b_n\right]_1$. The sequence $\{C_0, C_2, C_4, \ldots\}$ is strictly increasing and the sequence $\{C_1, C_3, C_5, \ldots\}$ is strictly decreasing, and every term in the first sequence is less than every term in the second sequence. Thus the first sequence converges to its least upper bound L_e and the second sequence converges to its lower bound L_o, with $L_e \le L_o$. By [4, page 237, Satz 8] we have that $L_e = L_o$, i.e., that the sequence $\{C_0, C_1, C_2, \ldots\}$ converges, if and only if the series $\sum_{n=0}^{\infty} b_i$ diverges. But since each a_i is an integer, $b_i \ge 1/N$ for $i \ge 1$, so this is certainly the case.

In our situation it is easy to show convergence of $\{C_0, C_1, C_2, \ldots\}$ directly. We have that $|L_o - L_e| = L_o - L_e < C_{2n+1} - C_{2n}$ for every n, and from Theorem 1.3 we have that $C_{2n+1} - C_{2n} = 1/q_{2n+1}q_{2n}$. Then, since also $C_n = \left[a_0, a_1, \ldots, a_n\right]_N$, an inductive argument shows that $q_{2n+1} \ge (a_1/N)(1+1/N)^n$ and $q_{2n} \ge (1+1/N)^n$, so $1/q_{2n+1}q_{2n} \to 0$ as $n \to \infty$. □

We now present an algorithm to produce cf_N expansions.

Theorem 1.5. *Let $x_0 \in \mathbb{R}$, $x_0 > 0$.*

(1) *Let $i = 0$*
(2) *Choose $a_i \in \mathbb{N}$ such that $x_i - N \le a_i \le \lfloor x_i \rfloor$*
(3) *Let $r_i = x_i - a_i$*
(4) *If $r_i = 0$, terminate. Otherwise let $x_{i+1} = \frac{N}{r_i}$, increment i, and go to step (2).*

Then $x_0 = [a_0, a_1, a_2, \ldots]_N$ (where there may be only finitely many a_i).

Proof. We will first verify that this algorithm can be carried out as described. The only difficulty that could arise is if $x_i < 1$ for some $i > 0$ because then we would be unable to choose a_i as the algorithm describes. We know that x_0 is a positive number and since we allow a_0 to be 0, we always have a valid choice for $i = 0$ by choosing $a_0 = \lfloor x_0 \rfloor$. Assume that we have chosen a_i satisfying the inequalities in step (2). Then we have

$$0 \le x_i - \lfloor x_i \rfloor \le x_i - a_i = r_i < x_i - \left(x_i - N\right) = N.$$

If $r_i = 0$, the algorithm terminates. Otherwise, we get $0 < r_i < N$ therefore $x_{i+1} = \frac{N}{r_i} > 1$ so we can make a valid choice for a_{i+1}. Thus, by induction, we can always choose an a_i as described in step (2) if the algorithm has not terminated yet.

The proof that this converges to x_0 is similar to the classical case and we omit it. □

Definition 1.6. If, in step (2) of the algorithm, we choose $a_i = \lfloor x_i \rfloor$, we call this the *best choice* for a_i. If we make the best choice for every a_i then we call the resulting continued fraction expansion the *best expansion* for x_0.

We denote a best cf_N expansion by $[[a_0, a_1, a_2, \ldots]]_N$. We will often use $[[x_0]]_N$ to denote the best cf_N expansion of the real number x_0.

There is an easy criterion for deciding when a cf_N expansion is a best cf_N expansion.

Lemma 1.7. *An infinite* cf_N *expansion* $[a_0, a_1, \ldots]_N$ *is a best* cf_N *expansion if and only if* $a_i \geq N$ *for all* $i \geq 1$. *A finite* cf_N *expansion* $[a_0, a_1, \ldots, a_n]_N$ *is a best* cf_N *expansion if and only if* $n = 0$, *or* $n > 0$ *and* $a_i \geq N$ *for* $1 \leq i \leq n-1$ *and* $a_n \geq N+1$.

Proof. We prove the infinite case. Suppose $[a_0, a_1, \ldots]_N$ is the best cf_N expansion of some real number x_0. Then for each $i \geq 0$, $a_i = \lfloor x_i \rfloor$ so that $r_i < 1$, and hence $a_{i+1} = \lfloor N/r_i \rfloor \geq N$. Conversely, if $a_{i+1} \geq N$, then, since the expansion does not terminate, $r_i < 1$ and so $a_i = \lfloor x_i \rfloor$. □

In the classical case, a positive irrational number has a unique continued fraction expansion, and that is a fortiori its best cf_1 expansion. A positive rational number other than 1 has two cf_1 expansions, of the form $[a_0, a_1, \ldots, a_n]_1$ with $a_n \geq 2$ and $[a_0, a_1, \ldots, a_n - 1, 1]_1$, and 1 has the two cf_1 expansions $[1]_1$ and $[0,1]_1$. In any case, the best cf_1 expansion is the first of these.

Theorem 1.8. *For* $N \geq 2$, *every positive irrational number* x_0 *has infinitely many* cf_N *expansions, and infinitely many of these expansions are nonperiodic.*

Proof. Given some expansion of x_0, $[a_0, a_1, a_2, \ldots]_N$, we modify it in the following way: choose some $k > 0$. Perform the algorithm on x_0 and create another expansion $[a'_0, a'_1, a'_2, \ldots]_N$ by choosing $a'_i = a_i$ for all $i < k$. Then choose $a'_k = \lfloor x_k \rfloor$ (a valid choice). If $a'_k \neq a_k$ we can continue choosing the a'_i in any way and we will have a new expansion for x_0. Suppose that $a_k = a'_k$. If $a_{k+1} \neq \lfloor x_{k+1} \rfloor$, choose $a'_{k+1} = \lfloor x_{k+1} \rfloor$ and we have a new expansion for x_0. Suppose that $a_{k+1} = \lfloor x_{k+1} \rfloor$. Then $r_k = x_k - \lfloor x_k \rfloor < 1$ so $x_{k+1} = \frac{N}{r_k} > N$ so $x_{k+1} - N \leq a_{k+1} - 1 \leq \lfloor x_{k+1} \rfloor$. So we can choose $a'_{k+1} = a_{k+1} - 1$ and we have a new expansion for x_0.

Every irrational number has at least one expansion (the best expansion) and the previous method allows us to acquire from that a new expansion for every $k \in \mathbb{N}$. Moreover, we can apply this method to ensure that an expansion for x_0 is nonperiodic. Fix some $s \in \mathbb{N}$ and perform the algorithm on x_0, making any valid choice for each a_i. Whenever $i+s$ is a square, find the largest $j < i$ such that $x_i = x_j$. If no such j exists, choose anything for a_i, otherwise choose $a_i \neq a_j$ or $a_{i+1} \neq a_{j+1}$ by the previously described method. This ensures that no finite sequence of choices will be repeated infinitely many times. Thus for every s we have a nonperiodic expansion for x_0. □

Lemma 1.9. *The best* cf_N *expansion of a positive rational number is finite.*

Proof. If x_0 is a rational number, then r_i is rational for all i. Let $r_i = \frac{d_i}{e_i}$ where d_i and e_i are nonnegative integers with $\gcd(d_i, e_i) = 1$. If we choose the best expansion for x_0, then $r_i < 1$ for all i. Thus

$$r_{i+1} = x_{i+1} - a_{i+1} = \frac{N}{r_i} - a_{i+1} = \frac{Ne_i - d_i a_{i+1}}{d_i} < 1.$$

Now $\gcd(Ne_i - d_i a_{i+1}, d_i)$ is not necessarily 1, but in any case d_{i+1} divides $Ne_i - d_i a_{i+1}$. Thus $d_{i+1} < d_i$, so $\{d_i\}$ is a strictly decreasing sequence of nonnegative integers. Therefore $d_j = 0$ for some j. Thus $r_j = 0$ and the algorithm terminates. □

For a positive integer m, we let $\overline{m}_k$ denote a sequence of k m's, and let $\overline{m}_\infty$ denote a sequence of infinitely many m's.

Lemma 1.10. *(a) Let* $N \geq 2$. *Then for any* $k \geq 0$,

$$N = \left[\,\overline{(N-1)}_k, N\right]_N,$$

and also

$$N = \left[\,\overline{(N-1)}_\infty\right]_N.$$

(b) Let $N \geq 4$ be even. Then

$$N = \left[N-2,(N-2)/2,N\right]_N.$$

(c) Let $N \geq 3$ be odd. Then

$$N = \left[N-2,(N-1)/2,2N-1,N\right]_N.$$

Proof. Direct computation. □

Theorem 1.11. *Let x_0 be a positive rational number.*

(a) For any $N \geq 2$, x_0 has finite cf_N expansion of arbitrarily long lengths, and at least one infinite cf_N expansion.

(b) For any $N \geq 3$, x_0 has infinitely many distinct periodic cf_N expansions and infinitely many distinct nonperiodic cf_N expansions.

(c) For $N = 2$, every infinite cf_N expansion of x_0 is of the form $[a_0,a_1,\ldots,a_k,1,1,1,\ldots]_N$ for some k and some integers $a_0,\ldots,a_k$, and there are only finitely many such expansions.

Proof. Let x_0 have best cf_N expansion

$$x_0 = \left[\left[a_0,\ldots,a_n\right]\right]_N.$$

This expansion is finite by Lemma 1.9, and $a_n \geq N+1$ by Lemma 1.7.

(a) Using Lemma 1.1 and Lemma 1.10(a), we have

$$\begin{aligned} x_0 &= \left[a_0,\ldots,a_n\right]_N = \left[a_0,\ldots,a_{n-1},a_n-1,N\right]_N \\ &= \left[a_0,\ldots,a_{n-1},\overline{(N-1)}_k,N\right]_N \quad \text{for any } k \geq 0 \end{aligned}$$

and also

$$x_0 = \left[a_0,\ldots,a_{n-1},\overline{(N-1)}_\infty\right]_N.$$

(b) In case N is even, using Lemma 1.1 and Lemma 1.10(b), we have

$$\begin{aligned} x_0 &= \left[a_0,\ldots,a_n\right]_N = \left[a_0,\ldots,a_{n-1},a_n-1,N\right]_N \\ &= \left[a_0,\ldots,a_{n-1},N-2,(N-2)/2,N\right]_N \\ &= \left[a_0,\ldots,a_{n-1},N-2,(N-2)/2,\overline{(N-1)}_k,N\right]_N \end{aligned}$$

for any $k \geq 0$.

Also for any $k \geq 0$ we have the periodic expansion of period $k+2$ given by

$$x_0 = \left[a_0,\ldots,a_{n-1},N-2,(N-2)/2,\overline{(N-1)}_k,N-2,(N-2)/2,\overline{(N-1)}_k,\ldots\right]_N$$

and for any nonperiodic sequence $k_0,k_1,\ldots$ of nonnegative integers we have the nonperiodic expansion

$$x_0 = \left[a_0,\ldots,a_{n-1},N-2,(N-2)/2,\overline{(N-1)}_{k_0},N-2,(N-2)/2,\overline{(N-1)}_{k_1},\ldots\right]_N.$$

In case N odd, a similar construction works, using Lemma 1.10(c).

(c) Write $x_0 = a/b$, a fraction in lowest terms. We prove this by complete induction on b.

Suppose $b = 1$, so that $x_0 = a$ is an integer. By inspection of our algorithm, it is easy to see that any finite cf_2 expansion of x_0 must be

$$a = [a]_2 = \left[a-1,\overline{1}_k,2\right]_2 \text{ for some } k \geq 0 = [a-2,1]_2 \text{ if } a \geq 2,$$

and that the only infinite cf_2 expansion of a is

$$a = \left[a-1, \overline{1}_\infty\right]_2.$$

Now let $x_0 = a/b$ with $b > 1$. Let $c = \lfloor a/b \rfloor$. Then the only cf_2 expansions of x_0 are of the form

$$a/b = \left[c, [x_1]_2\right]_2 \quad \text{or} \quad a/b = \left[c-1, [x_1']_2\right]_2.$$

In the first case, $x_1 = 2b/(a-bc)$ and $a - bc < b$, so by induction we are done. In the second case, $1 < x_1' < 2$ and so this expansion must be of the form

$$a/b = \left[c-1, 1, [x_2']_2\right]_2$$

with $x_2' = 2(a-b(c-1))/(2b-(a-b(c-1)))$ and $2b-(a-b(c-1)) < b$, so by induction we are done. □

Remark 1.12. There are only countably many periodic sequences $a_0, a_1, \ldots$ and a fortiori any positive number x_0 has only countably many periodic cf_N expansions (possibly none). The diagonalization argument of the proof of Theorem 1.8 shows that any irrational x_0 has uncountably many nonperiodic cf_N expansions for any $N \geq 2$, and the construction in the proof of Theorem 1.11 shows that any rational x_0 has uncountably many nonperiodic cf_N expansions for any $N \geq 3$.

2. Quadratic irrationalities

In this section we investigate cf_N expansions of quadratic irrationalities.

Definition 2.1. Consider an arbitrary cf_N expansion $[a_0, a_1, \ldots]_N$. The *m-inflation* of this expansion is the cf_{mN} expansion

$$I_m\left(\left[a_0, a_1, a_2, a_3, \ldots\right]_N\right) = \left[a_0, ma_1, a_2, ma_3, \ldots\right]_{mN}.$$

Note that, by Lemma 1.1(c), if $x_0 = [a_0, a_1, \ldots]_N$, then also $x_0 = I_m([a_0, a_1, \ldots]_N)$ for any m.

Theorem 2.2. *Let x_0 be a quadratic irrationality. Then for any N, x_0 has a periodic* cf_N *expansion.*

Proof. From the classical theory we know that x_0 has a periodic cf_1 expansion of some period k. Then the N-inflation of this expansion is a cf_N expansion of x_0, periodic of period k (or, in exceptional cases, $k/2$) if k is even and periodic of period $2k$ (in all cases) if k is odd. □

We observe that there is no reason to expect in general that the cf_N expansion of x_0 obtained in this way will be the best cf_N expansion of x_0. Indeed from Lemma 1.7 we see that this will never be the case if N is sufficiently large.

We will exhibit a number of families of periodic best cf_N expansions of quadratic irrationalities below, and a number of specific examples of periodic best cf_N examples of quadratic irrationalities, but we make the following conjecture.

Conjecture 2.3. For $N \geq 2$, the best cf_N expansion of a quadratic irrationality is not always periodic.

As evidence for this conjecture we have the computation that the best cf_2 expansion of $\sqrt{124}$ is not periodic within its first 6,000 terms, and that the best cf_7 expansion of $\sqrt{8}$ is not periodic within its first 6,000 terms. (Such examples abound.)

We remind the reader of our conventions: E is a positive integer that is not a perfect square, $D = \lfloor\sqrt{E}\rfloor$, and $a = E - D^2$, so that $E = D^2 + a$ with $1 \le a \le 2D$. Also, N is said to be *small* (for E) if $N \le 2D$ and *large* (for E) otherwise. (Note that $N = 1$ is always small.)

Lemma 2.4. *Suppose that a divides $2DN$. Then*

$$\sqrt{E} = \left[D, \overline{2DN/a, 2D}\right]_N,$$

periodic of period 2 if $a \ne N$ and period 1 if $a = N$. This is the best cf_N expansion of $\sqrt{E}$ if and only if a and N are both small for E.

Proof. Direct calculation shows that this is always a cf_N expansion of $\sqrt{E}$, and it follows immediately from Lemma 1.7 that it is the best cf_N expansion of $\sqrt{E}$ exactly when the given conditions are satisfied. □

Remark 2.5. Observe that if a divides $2D$, then

$$\left[D, \overline{2DN/a, 2D}\right]_N = I_N\left(\left[D, \overline{2D/a, 2D}\right]_1\right).$$

But if not, this cf_N expansion does not come from a cf_1 expansion.

The cases $a = 1$, $a = 2$, or $a = 4$ and D even are covered by Lemma 2.4. In case $a = 4$ and D odd we have the following.

Lemma 2.6. *Let $D > 1$ be odd, and let $E = D^2 + 4$. Then*

$$\sqrt{E} = \left[\left[D, \overline{(D-1)/2, 1, 1, (D-1)/2, 2D}\right]\right]_1, \quad \textit{periodic of period } 5,$$

and

$$\sqrt{E} = \left[\left[D, \overline{(D^2-1)/2, D, 2D^2+2, D, (D^2-1)/2, 2D}\right]\right]_D, \quad \textit{periodic of period } 6.$$

Proof. Direct computation and Lemma 1.7. □

Lemma 2.7. *(a) For $D > 1$, if $a = 2D - 1$, then*

$$\sqrt{E} = \left[D, \overline{1, D-1, 1, 2D}\right]_1 \quad \textit{of period } 4$$

and

$$\sqrt{E} = \left[\left[D, \overline{D+1, 2D^3+2D^2-2D, D+1, 2D}\right]\right]_D, \quad \textit{of period } 4.$$

(b) For $D \ge 4$ even, if $a = 2D - 3$, then

$$\sqrt{E} = \left[D, \overline{1, (D-2)/2, 2, (D-2)/2, 1, 2D}\right]_1 \quad \textit{of period } 6$$

and

$$\sqrt{E} = \left[\left[D, \overline{D+2, (D^2-2D)/2, D+2, 2D}\right]\right]_D, \quad \textit{of period } 4$$

for $D \ne 6$ and of period 2 for $D = 6$.

(c) For $D \ge 5$ odd, if $a = 2D - 3$, then

$$\sqrt{E} = \left[D, \overline{1, (D-3)/2, 1, 2D}\right]_1 \quad \textit{of period } 4.$$

(d) For $D \ge 3$ odd, if $a = 2D$, then

$$\sqrt{E} = \left[\left[D, 2D+2, \overline{8D^3+16D^2+6D, 2D+3}\right]\right]_{2D+1} \quad \textit{of period } 2$$

and

$$\sqrt{E} = \left[\left[D, 2D+3, \overline{4D^2+4D, 2D+4}\right]\right]_{2D+2} \quad \textit{of period } 2.$$

Proof. Direct computation and Lemma 1.7. □

Remark 2.8. (a) Lemma 2.7(c) for $D=3$ is covered by Lemma 2.4, verifying $\sqrt{12} = [3,\overline{2,6}]_1 = [3,\overline{6}]_3$.

(b) For $D \geq 5$ odd and $a = 2D-3$, numerical evidence suggests that the best cf_D expansion of $\sqrt{E}$ is not always (perhaps never) periodic.

(c) If $a = 2D$ and N is small, i.e., $N \leq 2D$, then $\sqrt{E}$ is covered by Lemma 2.4, so the two cases given in Lemma 2.7(d) are the first two cases for N large. There does not appear to be a similar result for $N = 2D+3$, and this may be a nonperiodic case.

Example 2.9. Here is one more family. Let $a=3$ and $N=2$. If D is divisible by 3 then $\sqrt{E}$ is covered by Lemma 2.4. Otherwise we have

$$\begin{aligned}
\sqrt{7} &= \big[\big[2,\overline{3,20,3,4}\big]\big]_2 \quad \text{of period } 4\\
\sqrt{19} &= \big[\big[4,\overline{5,3,4,34,4,3,5,8}\big]\big]_2 \quad \text{of period } 8\\
\sqrt{28} &= \big[\big[5,\overline{6,2,6,10}\big]\big]_2 \quad \text{of period } 4\\
\sqrt{52} &= \big[\big[7,\overline{9,4,9,14}\big]\big]_2 \quad \text{of period } 4\\
\sqrt{67} &= \big[\big[8,\overline{10,2,3,2,3,6,2,2,2,64,2,2,2,6,3,2,3,2,10,16}\big]\big]_2 \quad \text{of period } 20\\
\sqrt{103} &= \big[\big[10,\overline{13,4,3,9,3,4,13,20}\big]\big]_2 \quad \text{of period } 8\\
\sqrt{124} &= \big[\big[11,14,2,3,17,6,4,15,2,2,2,3,5,59,71,8,3,\ldots\big]\big]_2 \quad \text{apparently not periodic}\\
\sqrt{172} &= \big[\big[13,\overline{17,4,2,7,7,\ldots,7,7,2,4,17,26}\big]\big]_2 \quad \text{of period } 38\\
\sqrt{487} &= \big[\big[22,\overline{29,5,7,16,\ldots,16,7,5,29,44}\big]\big]_2 \quad \text{of period } 136.
\end{aligned}$$

Example 2.10. Just as when $N=1$, cases of $N>1$ when $[[\sqrt{E}]]_N$ has odd period seem to be rarer, but definitely occur. For example:

$$\begin{aligned}
\sqrt{22} &= \big[\big[4,\overline{2,2,8}\big]\big]_2 \quad \text{has period } 3\\
\sqrt{162} &= \big[\big[12,\overline{2,2,2,2,24}\big]\big]_2 \quad \text{has period } 5\\
\sqrt{241} &= \big[\big[15,\overline{3,2,4,4,2,3,20}\big]\big]_2 \quad \text{has period } 7\\
\sqrt{393} &= \big[\big[19,\overline{2,4,2,2,9,9,2,2,4,2,38}\big]\big]_2 \quad \text{has period } 11.
\end{aligned}$$

Also, $[[\sqrt{457}]]_2$ has period 9, $[[\sqrt{139}]]_3$ has period 5, $[[\sqrt{331}]]_3$ has period 9, $[[\sqrt{181}]]_4$ has period 5, $[[\sqrt{1997}]]_4$ has period 35, and $[[\sqrt{524}]]_8$ has period 3.

In fact, we have the following families of cf_N expansions with odd period.

Lemma 2.11. *(a) For any $j \geq 1$, let $D = 3j+1$, $a = 6j$, $E = D^2+a = 9j^2+12j+1$. Then*

$$\sqrt{E} = \big[\big[D,\overline{2(D-1)/3,2(D-1)/3,2D}\big]\big]_{2(D-1)/3}, \quad \textit{of period } 3.$$

(b) For any $j \geq 1$, let $D = 3j+1$, $a = 4j+2$, $E = D^2+a = 9j^2+10j+3$. Then

$$\sqrt{E} = \big[\big[D,\overline{2,2,2D}\big]\big]_2, \quad \textit{of period } 3.$$

Proof. Careful but routine computation. □

Not only is the classical continued fraction expansion of $\sqrt{E}$ periodic, it has additional structure. We investigate the analog of this structure for $[[\sqrt{E}]]_N$ in the situation where this cf_N expansion is periodic. In this situation we obtain a perfect analog to the $N=1$ case when N is small for E, but we will see different behavior when N is large for E. The arguments parallel those in the classical case, but we give them in reasonable detail to show what modifications have to be made and where the differences lie (cf. [3, Chapter 11]).

Definition 2.12. A quadratic irrationality x is N*-reduced* if $x > N$ and $-1 < \bar{x} < 0$, where $\bar{x}$ is the Galois conjugate of x.

Lemma 2.13. *(a) Let x be N-reduced. Let $A = \lfloor x \rfloor$ and $y = N/(x-A)$. Then y is N-reduced. Also, $\lfloor -N/\bar{y} \rfloor = A$.*

(b) Let x be N-reduced. Then $y = -N/\bar{x}$ is N-reduced.

Proof. Analogous to the $N = 1$ case, and routine. □

Theorem 2.14. *Let x_0 be N-reduced and suppose that $[[x_0]]_N$ is periodic of period k. Then $[[x_0]]_N = [\overline{a_0, a_1, \ldots, a_{k-1}}]_N$, i.e., the period begins with a_0.*

Proof. We have that $x_0 = [x_0]_N = [a_0, x_1]_N = [a_0, a_1, x_2]_N = \cdots$ and from Lemma 2.13 we have that x_i is N-reduced for every $i \geq 0$. Now by hypothesis we have that, for some j,

$$x_0 = \left[a_0, a_1, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{j+k-1}}\right]_N.$$

Set $z = x_j = x_{j+k}$. Then $z = x_j = N/(x_{j-1} - a_{j-1})$ and similarly $z = x_{j+k} = N/(x_{j+k-1} - a_{j+k-1})$. Thus

$$x_{j-1} = a_{j-1} + N/z, \qquad x_{j+k-1} = a_{j+k-1} + N/z$$
$$\bar{x}_{j-1} = a_{j-1} + N/\bar{z}, \qquad \bar{x}_{j+k-1} = a_{j+k-1} + N/\bar{z}$$

and hence $\bar{x}_{j-1} - \bar{x}_{j+k-1} = a_{j-1} - a_{j+k-1}$. But $-1 < x_i < 0$ for every i, so $-1 < \bar{x}_{j-1} - \bar{x}_{j+k-1} < 1$. But a_{j-1} and a_{j+k-1} are both integers, so the forces $\bar{x}_{j-1} = \bar{x}_{j+k-1}$ and hence $a_{j-1} = a_{j+k-1}$. Proceeding by downward induction we obtain $a_{j-2} = a_{j+k-2}, \ldots, a_0 = a_k$ and so the period begins with a_0. □

Corollary 2.15. *Let N be small. Suppose that $[[\sqrt{E}]]_N$ is periodic of period k. Then $[[\sqrt{E}]]_N = [a_0, \overline{a_1, \ldots, a_k}]_N$ with $a_k = 2a_0$. In particular, the period begins with a_1.*

Proof. Let $x = D + \sqrt{E}$. Then $[[x]]_N = [2a_0, a_1, a_2, \ldots]_N$. But x is N-reduced so $[[x]]_N$ is periodic beginning with $2a_0$, by Theorem 2.14. □

Corollary 2.16. *Let N be large. Suppose that $[[\sqrt{E}]]_N$ is periodic of period k. Let $h = \lfloor N/(D+\sqrt{E}) \rfloor \geq 1$. Then $[[\sqrt{E}]]_N = [a_0, a_1, \overline{a_2, \ldots, a_{k+1}}]_N$ with $a_{k+1} = a_1 + h$. In particular, the period begins with a_2.*

Proof. Let $x = \sqrt{E}$. Then $[[x]]_N = [a_0, a_1, x_2]_N$ with $a_0 = D$, $x_1 = \frac{N}{x_0 - a_0} = \frac{N}{\sqrt{E} - D}$, $a_1 = \lfloor N/(\sqrt{E} - D) \rfloor \geq N$, and $x_2 = N/(x_1 - a_1)$. Certainly $x_2 > N$.

Now $\bar{x}_2 = N/(\bar{x}_1 - a_1)$ and $\bar{x}_1 = \frac{N}{-\sqrt{E} - D} < 0$, so $\bar{x}_2 < 0$. Also, $-1/\bar{x}_2 = (a_1 - \bar{x}_1)/N > a_1/N \geq 1$, so $-1 < \bar{x}_2$. Thus x_2 is N-reduced, and so, by Theorem 2.14, $[[x_2]]_N = [a_2, a_3, \ldots]_N$ is periodic of period k beginning with a_2.

We now apply the argument in the proof of Theorem 2.14 to conclude that $\bar{x}_1 - \bar{x}_{k+1} = a_1 - a_{k+1}$. Since x_{k+1} is N-reduced, $-1 < \bar{x}_{k+1} < 0$. But $x_1 = N/(\sqrt{E} - D)$ so $\bar{x}_1 = -N/(\sqrt{E} + D)$ and hence $-(h+1) < \bar{x}_1 < -h$, so we must have that $a_1 - a_{k+1} = -h$ and hence $a_{k+1} = a_1 + h$. □

The converse of Theorem 2.14 is also true.

Theorem 2.17. *Suppose that $[[x_0]]_N$ is periodic of period k beginning at a_0, $[[x_0]]_N = [\overline{a_0, a_1, \ldots, a_{k-1}}]_N$. Then x_0 is N-reduced.*

Proof. First observe that $x_0 > a_0 = a_k \geq N$.

Now $x_0 = x_k = \frac{x_0 p_{k-1} + N p_{k-2}}{x_0 q_{k-1} + N q_{k-2}}$, showing that x_0 is a root of the polynomial $f(x) = x^2 q_k + (q_{k-1}N - p_k)x - p_{k-1}N = 0$. Now $f(0) = -p_{k-1}N < 0$ and $f(-1) = q_k - q_{k-1}N + p_k - p_{k-1}N = (a_k - N)q_{k-1} + q_{k-2} + (a_k - N)p_{k-1} + p_{k-2} > 0$ as $a_k \geq N$. Hence the other root of this polynomial, which is $\bar{x}_0$, must lie between -1 and 0. □

Lemma 2.18. *Let $[[x_0]]_N = [\overline{a_0, \ldots, a_{k-1}}]_N$ be periodic of period k beginning with a_0, and let $y_0 = -N/\bar{x}_0$. Then $[[y_0]]_N = [\overline{a_{k-1}, \ldots, a_0}]_N$.*

Proof. Write $x_0 = [x_0]_N = [a_0, x_1]_N = [a_0, a_1, x_2]_N = \cdots$. Note that, by Theorem 2.17, x_0 is N-reduced, and hence by Lemma 2.13, each x_i is N-reduced. Also, by Lemma 2.13, y_0 is N-reduced. Now

$$x_0 = a_0 + N/x_1, \quad x_1 = a_1 + N/x_2, \ldots, \quad x_{k-1} = a_{k-1} + N/x_k$$

or equivalently

$$-N/\bar{x}_1 = a_0 - \bar{x}_0, \ldots, \quad -N/\bar{x}_k = a_{k-1} - \bar{x}_{k-1}.$$

Set $z_{k-i} = -N/\bar{x}_i$, $i = 0, \ldots, k$. Then we have

$$z_0 = a_{k-1} - \bar{x}_{k-1}, \quad z_1 = a_{k-2} - \bar{x}_{k-2}, \ldots, \quad z_{k-1} = a_0 - \bar{x}_0.$$

But $0 < -\bar{x}_i < 1$ and $z_{i+1} = N/(z_i - a_{k-1-i})$for each i, so we see that

$$z_0 = \left[z_0\right]_N = \left[a_{k-1}, z_1\right]_N = \left[a_{k-1}, a_{k-2}, z_2\right]_N = \cdots = \left[a_{k-1}, \ldots, a_0, z_k\right]_N.$$

But $x_k = x_0$ so $z_k = z_0$ and hence

$$z_0 = \left[\left[\overline{a_{k-1}, \ldots, a_0}\right]\right]_N,$$

this being the best expansion as $a_i \geq N$ for each i. But by definition, $y_0 = z_0$. (Also, if $y_0 = [y_0]_N = [a_{k-1}, y_1]_N = [a_{k-1}, a_{k-2}, y_2]_N = \cdots$, we have $y_i = z_i$ for each i.) □

Theorem 2.19. *Let N be small and suppose that $[[\sqrt{E}]]_N$ is periodic of period k. Then*

$$\left[[\sqrt{E}]\right]_N = \left[a_0, \overline{a_1, \ldots, a_{k-1}, 2a_0}\right]_N \quad \textit{with } a_i = a_{k-i},\ i = 1, \ldots, k-1.$$

Proof. As we have seen

$$\left[[\sqrt{E} + D]\right]_N = \left[\overline{2a_0, a_1, \ldots, a_{k-1}}\right]_N$$

so

$$\left[[\sqrt{E} - D]\right]_N = \left[0, \overline{a_1, \ldots, a_{k-1}, 2a_0}\right]_N$$

and hence

$$N/(\sqrt{E} - D) = \left[\overline{a_1, \ldots, a_{k-1}, 2a_0}\right]_N.$$

But if $x_0 = N/(\sqrt{E} - D)$, $y_0 = -N/\bar{x}_0 = \sqrt{E} + D$, so

$$\left[[\sqrt{E} + D]\right]_N = \left[\overline{2a_0, a_{k-1}, \ldots, a_1}\right]_N$$

and comparing the two expressions for $[[\sqrt{E} + D]]_N$ yields the theorem. □

Definition 2.20. A sequence of integers $c_1, \ldots, c_k$ is *palindromic* if it reads the same from right-to-left as it does from left-to-right, i.e. if $c_i = c_{k+1-i}$ for $i = 1, \ldots, k$. A sequence is *semipalindromic* of type (j, k) if it is the concatenation of a palindromic sequence of length j followed by a palindromic sequence of length k, i.e., if it is of the form $c_1, \ldots, c_j$, $d_1, \ldots, d_k$ with $c_1, \ldots, c_j$ and $d_1, \ldots, d_k$ each palindromic.

Remark 2.21. By Theorem 2.19, we see that for N small, if $[[\sqrt{E}]]_N$ is periodic of period k with periodic part given by $a_1,\ldots,a_k$ (which is always true for $N=1$), then either $k=1$ or $a_1,\ldots,a_k$ is semipalindromic of type $(k-1,1)$.

Now suppose that N is large and $[[\sqrt{E}]]_N$ is periodic of period k with periodic part given by $a_2,\ldots,a_{k+1}$. In this case the situation is more complicated.

Example 2.22. (a) The cf_N expansions in Lemma 2.7(d) are semipalindromic of type $(1,1)$.

(b) We have the semipalindromic expansions

$$\sqrt{8} = \big[\big[2,9,\overline{12,44,12,10}\big]\big]_8 \quad \text{of type } (3,1)$$
$$\sqrt{53} = \big[\big[7,399,\overline{132,132,406}\big]\big]_{112} \quad \text{of type } (2,1)$$
$$\sqrt{65} = \big[\big[8,2312,\overline{149,702,184,341,180,341,184,702,149,2320}\big]\big]_{144} \quad \text{of type } (9,1).$$

(c) We have the semipalindromic expansions

$$\sqrt{7} = \big[\big[2,15,\overline{20,17,65,17}\big]\big]_{10} \quad \text{of type } (1,3)$$
$$\sqrt{23} = \big[\big[4,55,\overline{152,60,18568,60}\big]\big]_{44} \quad \text{of type } (1,3).$$

(d) We have the semipalindromic expansions

$$\sqrt{13} = \big[\big[3,196,\overline{231,247996,231,214,7854,214}\big]\big]_{119} \quad \text{of type } (3,3)$$
$$\sqrt{129} = \big[\big[11,108,\overline{39,176,204,176,39,109,52,98,42,98,52,109}\big]\big]_{39} \quad \text{of type } (5,7).$$

(e) We have the nonsemipalindromic expansions

$$\sqrt{31} = \big[\big[5,22,\overline{14,26,56,23}\big]\big]_{13}$$
$$\sqrt{187} = \big[\big[13,85,\overline{60,63,232,84,332,87}\big]\big]_{58}$$
$$\sqrt{215} = \big[\big[14,116,\overline{480,77,128,429,112,118}\big]\big]_{77}.$$

Note that, as long as at least one of j and k is odd, a semipalindromic expansion of type (j,k) differs from a semipalindromic expansion of type $(j+k-1,1)$ only by a phase shift.

Numerical evidence seems to indicate that most periodic $[[\sqrt{E}]]_N$ expansions are semipalindromic of type $(j,1)$ or $(1,k)$, with semipalindromic expansions of type (j,k) with $j>1$ and $k>1$ being rare, and nonsemipalindromic expansions being rarer still.

Remark 2.23. cf_N expansions were previously studied in [1], though the concerns of that paper are considerably different than ours. We restate the main results of [1] in our language: For any E, there exists an N such that the best cf_N expansion of $\sqrt{E}$ is periodic of period 1, and furthermore the convergents C_i of that expansion are a subset of the convergents of the classical continued fraction expansion of $\sqrt{E}$.

3. Pell's equations and related equations

Given any cf_N expansion of $x_0=\sqrt{E}$, we have its ith convergent $C_i = p_i/q_i$ where p_i and q_i are given by the recursion in Theorem 1.2. In the classical case this is intimately related to the solutions of Pell's equation $p^2 - Eq^2 = 1$.

In this section we investigate the analog for arbitrary N.

Lemma 3.1. *Let* $[\sqrt{E}]_N = [x_0]_N = [a_0,x_1]_N = [a_0,a_1,x_2]_N = \cdots$ *be any* cf_N *expansion of* $\sqrt{E}$.

Then $x_i = \frac{u_i + N^i\sqrt{E}}{v_i}$ *for integers* u_i, v_i *defined inductively by*

$$u_0 = 0, \qquad v_0 = 1$$
$$u_{i+1} = N\left(a_i v_i - u_i\right)$$
$$v_{i+1} = \frac{N^{2i+2}E - \left(u_{i+1}\right)^2}{N^2 v_i}.$$

Proof. By definition, $x_i = a_i + \frac{N}{x_{i+1}}$, i.e., $x_{i+1} = \frac{N}{x_i - a_i}$ and simple algebra shows this is equal to

$$\frac{N\left(a_i v_i - u_i\right) + N^{i+1}\sqrt{E}}{\frac{N^{2i}E - \left(a_i v_i - u_i\right)^2}{v_i}} = \frac{u_{i+1} + N^{i+1}\sqrt{E}}{v_{i+1}}.$$

Clearly u_{i+1} is an integer. We prove that v_{i+1} is an integer by induction. Note that $u_1 = Na_0$, $v_1 = E - a_0^2$ so v_0 and v_1 are integers. Then $v_{i+1} \in \mathbb{Z} \Leftrightarrow v_i \mid N^{2i}E - (a_i v_i - u_i)^2 \Leftrightarrow v_i \mid N^{2i}E - u_i^2$.

But $v_i = \frac{N^{2i}E - u_i^2}{N^2 v_{i-1}} \in \mathbb{Z}$ by induction, so $\frac{N^{2i}E - u_i^2}{v_i} = N^2 v_{i-1} \in \mathbb{Z}$ as required. □

Lemma 3.2. *Let* $[\sqrt{E}]_N = [x_0]_N = [a_0, x_1]_N = [a_0, a_1, x_2]_N = \cdots$ *be any* cf_N *expansion of* $\sqrt{E}$. *Then* $p_i^2 - Eq_i^2 = (-1)^{i+1} v_{i+1}$.

Proof. By induction on i. For $i = -1$, $p_i^2 - Eq_i^2 = (1)^2 - E(0)^2 = 1 = v_0$. For $i = 0$, $p_i^2 - Eq_i^2 = a_0^2 - E(1)^2 = -(E - a_0)^2 = -v_1$.

Assume true for i. Then

$$[\sqrt{E}]_N = \left[a_0, \ldots, a_i, x_{i+1}\right]_N$$

so

$$\sqrt{E} = \frac{x_{i+1}p_i + Np_{i-1}}{x_{i+1}q_i + Nq_{i-1}}.$$

But

$$x_{i+1} = \frac{u_{i+1} + N^{i+1}\sqrt{E}}{v_{i+1}}.$$

Substituting, we obtain

$$N^{i+1}Eq_i = u_{i+1}p_i + p_{i-1}v_{i+1}N \qquad (*)$$
$$u_{i+1}q_i + q_{i-1}v_{i+1}N = N^{i+1}p_i. \qquad (**)$$

Now $p_i(**) - q_i(*)$ gives

$$N^{i+1}\left(p_i^2 - Eq_i^2\right) = Nv_{i+1}\left(p_i q_{i-1} - p_{i-1}q_i\right).$$

But we know that $p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1} N^i$ and substituting and cancelling we obtain $p_i^2 - Eq_i^2 = (-1)^{i+1} v_i$. □

Theorem 3.3. *Let* N *be small and suppose that* $[[\sqrt{E}]]_N$ *is periodic. In this case,* $[[\sqrt{E}]]_N$ *is periodic beginning with* a_1. *Let* $[[\sqrt{E}]]_N$ *have period* k, $[[\sqrt{E}]]_N = [a_0, \overline{a_1, \ldots, a_k}]_N$. *In this case,* $a_k = 2a_0 = 2D$. *Then* $v_k = N^k$, *i.e.,* $p_{k-1}^2 - Eq_{k-1}^2 = (-N)^k$, *and* $u_k = a_0 N^k = DN^k$.

Conversely, if $v_k = N^k$ *and* u_k *is divisible by* N^k, *then* $[[\sqrt{E}]]_N$ *is periodic of period* k *beginning with* a_1, *and* $a_k = 2a_0$.

Proof. First suppose $[[\sqrt{E}]]_N$ is periodic of period k.

Then $x_1 = [\overline{a_1,\ldots,a_k}]_N$ so $[[a_0,x_1]]_N = [[a_0,a_1,\ldots,a_k,x_1]]_N = [[a_0,a_1,\ldots,a_k,x_{k+1}]]_N$ and hence $x_{k+1} = x_1$.

But $x_k = a_k + N/x_{k+1}$, $x_k - a_k = N/x_{k+1}$, and $x_0 = a_0 + N/x_1$, $x_0 - a_0 = N/x_1$, so $x_k - a_k = x_0 - a_0$, i.e., $x_k = a_k - a_0 + \sqrt{E}$.

But $x_k = \frac{u_k + N^k\sqrt{E}}{v_k}$ so we must have $v_k = N^k$ and also $u_k/v_k = a_k - a_0$, an integer. But in this case we know that $a_k = 2a_0$ so $u_k = a_0 N^k$.

Conversely, suppose that $v_k = N^k$ and that $u_k = mN^k$ for some integer m. Then $x_k = \frac{u_k + N^k\sqrt{E}}{v_k} = m + \sqrt{E}$ so $a_k = m + a_0$. But then $x_{k+1} = \frac{N}{x_k - a_k} = \frac{N}{(m+\sqrt{E})-(m+a_0)} = \frac{N}{\sqrt{E}-a_0} = \frac{N}{x_0 - a_0} = x_1$, so $[[x_{k+1}]]_N = [[x_1]]_N$, and hence $a_{k+1} = a_0$, $a_{k+2} = a_2, \ldots, a_{2k} = a_k$, $a_{2k+1} = a_{k+1} = a_1, \ldots$ so

$$\left[[\sqrt{E}]\right]_N = \left[a_0, \overline{a_1,\ldots,a_k}\right]_N$$

and we have seen that in this case we must have $a_k = 2a_0$. □

Remark 3.4. Note in case $N = 1$ the condition that u_k be divisible by N^k is automatic. But in case $N > 1$ it is not, and it is possible that $v_k = N^k$ but u_k is not divisible by N^k, so that $[[\sqrt{E}]]_N$ does not have period k. For example:

For $\left[[\sqrt{41}]\right]_4$,	$v_3 = 4^3$	but this expansion has period 6.
For $\left[[\sqrt{43}]\right]_2$,	$v_6 = 2^6$	but this expansion has period 12.
For $\left[[\sqrt{209}]\right]_3$,	$v_6 = 3^6$	but this expansion has period 30.
For $\left[[\sqrt{590}]\right]_3$,	$v_6 = 3^6$	but this expansion has period 28.
For $\left[[\sqrt{777}]\right]_{12}$,	$v_5 = 12^5$	but this expansion has period 28.
For $\left[[\sqrt{1692}]\right]_5$,	$v_4 = 5^4$	but this expansion has period 24.

We have the following generalization of periodicity.

Definition 3.5. A sequence $\{d_i\}$ is *f-periodic* of period k from $i = m$ if $d_{i+k} = f d_i$ for all $i \geq m$.

We also adopt the notation that $w_i = p_i^2 - E q_i^2$ for $i \geq -1$. (Note $w_{-1} = 1$.)

Theorem 3.6. *Suppose $[[\sqrt{E}]]_N$ is periodic of period k. Then $\{w_i\}$ is $(-N)^k$-periodic of period k. If N is small the period begins with $i = -1$, while if N is large the period begins with $i = 1$.*

Proof. By Lemma 3.2, the theorem is equivalent to the claim that $\{v_i\}$ is N^k-periodic of period k beginning with $i = 0$ if N is small and $i = 2$ if N is large.

Suppose N is small. By Theorem 3.3, $u_k = DN^k$ and $v_k = N^k$, while $u_0 = D$ and $v_0 = 1$. For $i \geq 1$, $x_{k+1} = x_i$ by the periodicity of $[[\sqrt{E}]]_N$, which, by Corollary 2.15, begins with a_1, i.e.,

$$\frac{u_{k+1} + N^{k+i}\sqrt{E}}{v_{k+i}} = \frac{u_i + N^i\sqrt{E}}{v_i},$$

so $v_{k+i} = N^k v_i$ and then $u_{k+i} = N^k u_i$.

If N is large, the same argument works, again using the periodicity of $[[\sqrt{E}]]_N$, which, in this case, by Corollary 2.16, begins with a_2. □

Corollary 3.7. *If $C_i = p_i/q_i$ is the ith convergent of a* cf_N *expansion, then* $\gcd(p_i, q_i)$ *divides N^i for all $i \geq 0$.*

Proof. Immediate from Theorem 1.3. □

Lemma 3.8. *Let $N \leq 2D$ and suppose that N and $2D$ are relatively prime. Set $E = D^2 + N$ and consider $\sqrt{E} = [[D, \overline{2D}]]_N$. Then for all $i \geq 0$,* $\gcd(p_i, q_i) = 1$*, and $w_i = (-N)^{i+1}$.*

Proof. As easy induction, beginning with $q_0 = 1$, shows that $q_i \equiv 1 \pmod{N}$ for all $i \geq 0$, so $\gcd(p_i, q_i) = 1$ by Corollary 3.7. The second claim follows immediately from Theorem 3.6. □

Lemma 3.8 shows that p_i and q_i may be relatively prime. Here are some examples to show that the upper bound on $\gcd(p_i, q_i)$ in Corollary 3.7 is realized. Examples are plentiful for $i = 1$, so we merely give examples for $i \geq 2$.

Example 3.9.

For $[[\sqrt{13}]]_2$,	$\gcd(p_2, q_2) = 2^2$.	For $[[\sqrt{3050}]]_3$,	$\gcd(p_4, q_4) = 3^4$.
For $[[\sqrt{57}]]_2$,	$\gcd(p_3, q_3) = 2^3$.	For $[[\sqrt{499}]]_4$,	$\gcd(p_2, q_2) = 4^2$.
For $[[\sqrt{603}]]_2$,	$\gcd(p_4, q_4) = 2^4$.	For $[[\sqrt{1580}]]_4$,	$\gcd(p_3, q_3) = 4^3$.
For $[[\sqrt{3262}]]_2$,	$\gcd(p_5, q_5) = 2^5$.	For $[[\sqrt{185}]]_5$,	$\gcd(p_2, q_2) = 5^2$.
For $[[\sqrt{41}]]_3$,	$\gcd(p_2, q_2) = 3^2$.	For $[[\sqrt{1878}]]_6$,	$\gcd(p_2, q_2) = 6^2$.
For $[[\sqrt{207}]]_3$,	$\gcd(p_3, q_3) = 3^3$.	For $[[\sqrt{697}]]_7$,	$\gcd(p_2, q_2) = 7^2$.

Definition 3.10. Let $\tilde{p}_i$ and $\tilde{q}_i$ be the positive integers defined by $C_i = p_i/q_i = \tilde{p}_i/\tilde{q}_i$ where $\tilde{p}_i/\tilde{q}_i$ is in lowest terms, i.e., $\gcd(\tilde{p}_i, \tilde{q}_i) = 1$.

We may then similarly define the sequence $\{\tilde{w}_i\}$ by $\tilde{w}_i = \tilde{p}_i^2 - E\tilde{q}_i^2$. The sequence $\{\tilde{w}_i\}$ is a natural one to investigate, and of course if $\tilde{w}_i = 1$ we have a solution of Pell's equation.

Conjecture 3.11. Suppose that $[[\sqrt{E}]]_N$ is periodic. Then $\{\tilde{w}_i\}$ is f-periodic for some f.

Of course by Theorem 3.6 this is true whenever p_i and q_i are relatively prime, e.g., in the case of Lemma 3.8. Here is a more involved case.

Lemma 3.12. *For any $j \geq 1$, let $D = 3j - 1$, $a = 4j - 1$, $E = D^2 + a = 9j^2 - 2j$, and $N = 2a = 8j - 2$. Then*

$$[[\sqrt{E}]]_N = [[D, 4D+1, \overline{8D+4, 4D+2}]]_N.$$

Also,

$$p_{-1} = 1, \qquad q_{-1} = 0, \qquad w_{-1} = 1 = \tilde{w}_{-1}$$
$$p_0 = D, \qquad q_0 = 1, \qquad w_0 = -a = \tilde{w}_0$$

and for $i \geq 1$:

$$p_i p_{i-1} - E q_i q_{i-1} = -a(2D+1)(-N)^{i-1}$$

$$w_i = p_i^2 - E q_i^2 = \begin{cases} -a(-N)^i & \text{for } i \text{ even} \\ a^2(-N)^{i-1} & \text{for } i \text{ odd} \end{cases}$$

$$\gcd\left(p_i, q_i\right) = \begin{cases} a\left(2^{i/2}\right) & \text{for } i \text{ even} \\ a\left(2^{(i-1)/2}\right) & \text{for } i \text{ odd} \end{cases}$$

$$\tilde{w}_i = \tilde{p}_i^2 - E\tilde{q}_i^2 = (-a)^{i-1}.$$

In particular, $\tilde{w}_1 = 1$ and $\{\tilde{w}_i\}$ is $(-N/2)$-periodic of period 1 beginning with $i = 1$.

Proof. This follows from a careful, lengthy, but elementary inductive argument. □

Since we will be comparing cf_N expansions with cf_1 expansions, we must introduce more complicated notation. For fixed E, and any N, we let $C_{i,N} = \tilde{p}_{i,N}/\tilde{q}_{i,N}$ and $\tilde{w}_{i,N} = \tilde{p}_{i,N}^2 - E\tilde{q}_{i,N}^2$. But when N is clear from the context, we use our simpler notation.

Given the classical theory of continued fractions, there is one easy case.

Lemma 3.13. *Let $\sqrt{E} = [a_0, \overline{a_1, \ldots, a_k}]_1$ be periodic of period k.*

Let $N \leq \min(a_2, a_4, a_6, \ldots, a_k)$ if k is even, and let $N \leq \min(a_1, a_2, a_3, \ldots, a_k)$ if k is odd. Then

$$\tilde{w}_{km-1,N} = (-1)^{km} \quad \text{for every } m,$$

and every solution of $p^2 - Eq^2 = \pm 1$ in nonnegative integers arises in this way.

Proof. By Lemma 1.7, this condition on N gives

$$\left[[\sqrt{E}]\right]_N = I_N\left([\sqrt{E}]_1\right)$$

(where the N-inflation operator I_N was defined in Definition 2.1), and then in this case $C_{i,N} = C_{i,1}$ for every i. But this result for $N = 1$ is the basic relationship between classical continued fractions and solutions to Pell's equation. □

Here is another interesting general case in which we obtain all solutions from a cf_N expansion with $N > 1$, and moreover more quickly than in the classical case.

Lemma 3.14. *Let $E = D^2 + 4$ for $D > 1$ odd, and consider the best expansions given by Lemma 2.6,*

$$\sqrt{E} = \left[\left[D, \overline{(D-1)/2, 1, 1, (D-1)/2, 2D}\right]\right]_1 \quad \text{of period } 5$$

and

$$\sqrt{E} = \left[\left[D, \overline{\left(D^2-1\right)/2, D, 2D^2+2, D, \left(D^2-1\right)/2, 2D}\right]\right]_D \quad \text{of period } 6.$$

Then $\tilde{w}_{-1,D} = -1$, $\tilde{w}_{0,D} = -4$, $\tilde{w}_{1,D} = 2D^2 + 1$, and $\{\tilde{w}_{i,D}\}$ is (-1)-periodic of period 3 beginning at $i = -1$. In particular

$$\tilde{w}_{3m-1,D} = w_{5m-1,1} = (-1)^m \quad \text{for every } m,$$

and every solution of $p^2 - Eq^2 = \pm 1$ in nonnegative integers arises in this way.

Proof. It is easy to compute that $C_{3,D} = C_{5,1} = ((D^3+3D)/2)/((D^2+1)/2)$, giving the polynomial family of solutions

$$\left(\frac{D^3+3D}{2}\right)^2 - \left(D^2+4\right)\left(\frac{D^2+1}{2}\right)^2 = -1,$$

and that $C_{6,D} = C_{10,1} = ((D^6+6D^4+9D^2+2)/2)/((D^5+4D^3+3D)/2)$, giving the polynomial family of solutions

$$\left(\frac{D^6+6D^4+9D^6+2}{2}\right)^2 - \left(D^2+4\right)\left(\frac{D^5+4D^3+3D}{2}\right)^2 = 1,$$

and then proceed by induction. □

Lemma 3.15. *(a) For the expansion, for $D \geq 3$ odd,*

$$\sqrt{D^2+2D} = \left[\left[D, 2D+2, \overline{8D^3+16D^2+6D, 2D+3}\right]\right]_{2D+1},$$

$\tilde{w}_{-1} = 1$, $\tilde{w}_0 = -2D$, *and* $\{\tilde{w}_i\}$ *is periodic of period* 2 *from* $i = -1$.

(b) For the expansion, for $D \geq 3$ odd,

$$\sqrt{D^2+2D} = \left[\left[D, 2D+3, \overline{4D^2+4D, 2D+4}\right]\right]_{2D+2},$$

$\tilde{w}_{-1} = 1$, $\tilde{w}_0 = -2D$, $\tilde{w}_1 = 2D+4$, *and* $\{\tilde{w}_i\}$ *is periodic of period* 2 *from* $i = 0$.

(c) For the expansion

$$\sqrt{D^2+2D-1} = \left[\left[D, \overline{D+1, 2D^3+2D^2-2D, D+1, 2D}\right]\right]_D,$$

$\tilde{w}_{-1} = 1$, $\tilde{w}_0 = -(2D-1)$, *and* $\{\tilde{w}_i\}$ *is periodic of period* 2 *from* $i = -1$.

(d) For the expansion, for $D \geq 4$ even,

$$\sqrt{D^2+2D-3} = \left[\left[D, \overline{D+2, \left(D^2-2D\right)/2, D+2, 2D}\right]\right]_D,$$

$\tilde{w}_{-1} = 1$, $\tilde{w}_0 = -(2D-3)$, $\tilde{w}_1 = D+3$, $\tilde{w}_2 = -(2D-3)$, *and* $\{\tilde{w}_i\}$ *is periodic of period* 4 *from* $i = -1$.

Proof. We prove (a). The other parts are similar.

To begin with we have $p_{-1} = 1$, $q_{-1} = 0$, so $\tilde{p}_{-1} = 1$, $\tilde{q}_{-1} = 0$ and $\tilde{w}_{-1} = w_{-1} = 1$. We also have $p_0 = D$, $q_0 = 1$, so $\tilde{p}_0 = D$, $\tilde{q}_0 = 1$ and $\tilde{w}_0 = w_0 = -2D$. We then compute $p_1 = 2D^2+4D+1$, $q_1 = 2D+2$, so $\tilde{p}_1 = p_1$, $\tilde{q}_1 = q_1$, and $\tilde{w}_1 = w_1 = 1$.

We then compute inductively that, for all $k \geq 0$,

$$\begin{aligned} p_{2k+1} &\equiv p_{2k} \equiv (2D+1)^k(-1)^k D \quad \left(\mathrm{mod}\,(2D+1)^{2k+1}\right), \\ q_{2k+1} &\equiv q_{2k} \equiv (2D+1)^k(-1)^k \quad \left(\mathrm{mod}\,(2D+1)^{2k+1}\right). \end{aligned}$$

In particular this implies that g_{i+2}/g_i is divisible by $2D+1$, where $g_i = \gcd(p_i, q_i)$, and hence that w_{i+2}/w_i is divisible by $(2D+1)^2$. But by Theorem 3.6 $w_{i+2}/w_i = (2D+1)^2$. Hence $g_{i+2}/g_i = 2D+1$ for each i, and then the $(2D+1)^2$-periodicity of $\{w_i\}$ of period 2 from $i = -1$ gives the 1-periodicity (i.e., periodicity) of $\{\tilde{w}_i\}$ of period 2 from $i = -1$. □

We conclude by giving a number of illustrations of the sort of intricate and varied behavior we see. This behavior is indicated by extensive computations, but has not been proved.

Conjectural Example 3.16. (a) $[[\sqrt{335}]]_1$ is periodic of period 4, and so we obtain all nontrivial solutions of $p^2 - 335q^2 = 1$ from $(p,q) = (p_{4i-1,1}, q_{4i-1,1})$ for $i \geq 1$. $[[\sqrt{335}]]_6$ is periodic of period 26 from $i = -1$, and $\{\tilde{w}_{i,6}\}$ is periodic of period 26 from $i = -1$. We obtain solutions (p,q) of $p^2 - 335q^2 = 1$ from $(\tilde{p}_{k,6}, \tilde{q}_{k,6})$ for $k \equiv 3, 21$, or 25 $(\mathrm{mod}\, 26)$. Note these solutions are not evenly spaced among $\{\tilde{w}_{k,6}\}$. Also, for every $j \geq 0$

$$(\tilde{p}_{26j+3,6}, \tilde{q}_{26j+3,6}) = (p_{28j+3,1}, q_{28j+3,1})$$
$$(\tilde{p}_{26j+21,6}, \tilde{q}_{26j+21,6}) = (p_{28j+23,1}, q_{28j+3,1})$$
$$(\tilde{p}_{26j+25,6}, \tilde{q}_{26j+25,6}) = (p_{28j+27,1}, q_{28j+27,1})$$

so that the solutions we obtain from $[[\sqrt{335}]]_6$ are not evenly spaced among the solutions to Pell's equation.

(b) $[[\sqrt{393}]]_2$ is periodic of period 11 from $i = -1$. $\{\tilde{w}_i\}$ is (-2)-periodic of period 11 from $i = 32$. Also, $\tilde{w}_{15} = \tilde{w}_{31} = 1$, yielding two solutions to Pell's equation.

(c) $[[\sqrt{331}]]_3$ is periodic of period 9 from $i = -1$. $\{\tilde{w}_i\}$ is (-3)-periodic of period 9 from $i = 23$. Also, $\tilde{w}_{23} = 1$.

(d) $[[\sqrt{397}]]_2$ is periodic of period 10 from $i = -1$. $\{\tilde{w}_i\}$ is (-1)-periodic of period 15 from $i = -1$. Hence $\tilde{w}_k = -1$ for $k \equiv 14 \ (\mathrm{mod}\, 30)$ and $\tilde{w}_k = 1$ for $k \equiv 29 \ (\mathrm{mod}\, 30)$.

(e) $[[\sqrt{1856}]]_6$ is periodic of period 40 from $i = -1$. $\{\tilde{w}_i\}$ is periodic of period 20 from $i = -1$. Also, $\tilde{w}_k = 1$ for $k \equiv 9$ or 19 $(\mathrm{mod}\, 20)$.

(f) $[[\sqrt{118}]]_6$ is periodic of period 3 from $i = -1$. $\{\tilde{w}_i\}$ is $(-3)^8$-periodic of period 24 from $i = 2$.

(g) $[[\sqrt{61}]]_4$ is periodic of period 3 from $i = -1$. $\{\tilde{w}_i\}$ is (-1)-periodic of period 9 from $i = -1$.

(h) $[[\sqrt{407}]]_{12}$ is periodic of period 24 from $i = -1$. $\{\tilde{w}_i\}$ is periodic of period 24 from $i = -1$. Also, $\tilde{w}_i = 1$ for $i \equiv 3 \ (\mathrm{mod}\, 4)$.

(i) $[[\sqrt{283}]]_2$ is periodic of period 21 from $i = -1$. $\{\tilde{w}_i\}$ is periodic of period 42 from $i = -1$. Also, $\tilde{w}_i = 1$ for $i \equiv 13 \ (\mathrm{mod}\, 14)$.

(j) $[[\sqrt{464}]]_{30}$ is periodic of period 10 from $i = -1$. $\{\tilde{w}_i\}$ is 25-periodic of period 10 from $i = 11$. Also, $\tilde{w}_{11} = 1$.

(k) $[[\sqrt{401}]]_{50}$ is periodic of period 12 from $i = 1$. $\{\tilde{w}_i\}$ is periodic of period 12 from $i = 0$. Also, $\tilde{w}_i = -1$ for $i \equiv 0, 2$, or 10 $(\mathrm{mod}\, 12)$.

(l) $[[\sqrt{1410}]]_2$ is apparently not periodic, and $\tilde{w}_i = 1$ for $i = 3, 9, 13, 17, 25$.

Note that in all parts of this example (except part (l)) the periodicity of $[[\sqrt{E}]]_N$ is proved, and the computations of $\tilde{w}_k$ for individual values of k are correct. It is the remaining claims that are conjectural.

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