DIFFERENCE EQUATION AND PERMUTATION GROUP FOR $\zeta(4)$[1]

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Abstract. We present a hypergeometric construction of linear forms in 1 and $\zeta(4)=\pi^4/90$ that leads to a second-order Apéry-like recursion as well as to a certain permutation group yielding a conditional upper bound for the irrationality measure of $\zeta(4)$. We also give a new 'elementary' proof of the irrationality of $\zeta(3)$ based on Zeilberger's algorithm of creative telescoping.

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1. Introduction

In this work, we deal with the values of Riemann's zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at integral points $s=2,3,4,\dots$. Lindemann's proof of the transcendence of π as well as Euler's formula for even zeta values, summarized by the inclusions $\zeta(2n)\in\mathbb{Q}\pi^{2n}$ for $n=1,2,\dots$, yield the irrationality of $\zeta(2),\zeta(4),\zeta(6),\dots$. The story for odd zeta values is not so complete, we know only that:

- $\zeta(3)$ is irrational (R. Apéry [Ap], 1978);
- infinitely many of the numbers $\zeta(3),\zeta(5),\zeta(7),\dots$ are irrational (T. Rivoal [Ri1], [BR], 2000);
- at least one of the four numbers $\zeta(5),\zeta(7),\zeta(9),\zeta(11)$ is irrational (this author [Zu4], [Zu5], 2001).

[1] A preliminary version of the paper for *Actes des 12èmes rencontres arithmétiques de Caen.*

After remarkable Apéry's proof [Ap] of the irrationality of both $\zeta(2)$ and $\zeta(3)$, there have appeared several other explanations of why it is so; we are not able to indicate here the complete list of such publications and mention the most known approaches:

- orthogonal polynomials [Be1] and Padé-type approximations [Be2], [So1], [So2];
- multiple Euler-type integrals [Be1], [Hat], [RV2];
- hypergeometric-type series [Gu], [Ne1];
- modular interpretation [Be3].

In his proof of the irrationality of $\zeta(3)$, Apéry consider the sequences u_n and v_n of rationals satisfying the difference equation

$$(n+1)^3u_{n+1}-(2n+1)(17n^2+17n+5)u_n+n^3u_n=0, \tag{1}$$
$$u_0=1, \quad u_1=5, \qquad v_0=0, \quad v_1=6.$$

A priori, the recursion (1) implies the obvious inclusions $n!^3u_n, n!^3v_n \in \mathbb{Z}$, but a miracle happens and one can deduce (at least experimentally) the inclusions

$$u_n \in \mathbb{Z}, \quad D_n^3v_n \in \mathbb{Z}$$

for each $n=1,2,\dots$; here and later, by D_n we denote the least common multiple of the numbers $1,2,\dots,n$ (and $D_0=1$ for completeness), thanks to the prime number theorem

$$\lim_{n\to\infty}\frac{\log D_n}{n}=1. \tag{2}$$

The sequence

$$u_n\zeta(3)-v_n, \qquad n=0,1,2,\dots,$$

is also a solution of the difference equation (1), and it exponentially tends to 0 as $n\to\infty$ (even after multiplying it by D_n^3), since

$$\lim_{n\to\infty}\frac{v_n}{u_n}=\zeta(3).$$

A similar approach has been put forward for proving the irrationality of $\zeta(2)$ (see [Ap], [Po]), and several other Apéry-like difference equations have been discovered later (see, e.g., [Za]). But no second-order recursion for $\zeta(4)$ and/or further zeta values has been known.

A part of this work is devoted to difference equations satisfied by (small) linear forms in zeta values with 'almost integral' coefficients. For instance, we present the difference equation

$$(n+1)^5u_{n+1}-b(n)u_n-3n^3(3n-1)(3n+1)u_{n-1}=0, \tag{3}$$

where

$$\begin{aligned} b(n) &= 3(2n+1)(3n^2+3n+1)(15n^2+15n+4) \\ &= 270n^5+675n^4+702n^3+378n^2+105n+12, \end{aligned} \tag{4}$$

with the initial data

$$u_0 = 1, \quad u_1 = 12, \qquad v_0 = 0, \quad v_1 = 13 \tag{5}$$

for its two independent solutions u_n and v_n, and prove the following result.

Theorem 1. *For each $n = 0, 1, 2, \dots$, the numbers u_n and v_n are positive rationals satisfying the inclusions*

$$6D_n u_n \in \mathbb{Z}, \qquad 6D_n^5 v_n \in \mathbb{Z}, \tag{6}$$

and there holds the limit relation[2]

$$\lim_{n\to\infty} \frac{v_n}{u_n} = \frac{\pi^4}{90} = \zeta(4). \tag{7}$$

Application of Poincaré's theorem (see also [Zu6, Proposition 2]) then yields the asymptotic relations

$$\lim_{n\to\infty} \frac{\log u_n}{n} = \lim_{n\to\infty} \frac{\log v_n}{n} = 3\log(3+2\sqrt{3}\,) = 5.59879212\dots$$

and

$$\lim_{n\to\infty} \frac{\log|u_n\zeta(4) - v_n|}{n} = 3\log|3-2\sqrt{3}\,| = -2.30295525\dots,$$

since the characteristic polynomial $\lambda^2 - 270\lambda - 27$ of the equation (3) has zeros $135 \pm 78\sqrt{3} = (3 \pm 2\sqrt{3}\,)^3$. Thus, we can consider v_n/u_n as convergents of a continued fraction for $\zeta(4)$ and making the equivalent transform of the fraction [JT, Theorems 2.2 and 2.6] we obtain

Theorem 2. *There holds the following continued-fraction expansion:*

$$\zeta(4) = \frac{13|}{|b(0)} + \frac{1^7\cdot 2\cdot 3\cdot 4|}{|b(1)} + \frac{2^7\cdot 5\cdot 6\cdot 7|}{|b(2)} + \dots + \frac{n^7(3n-1)(3n)(3n+1)|}{|b(n)} + \cdots,$$

[2] During the preparation of this article, we have known that the difference equation (3), in slightly different normalization, and the limit relation (7) had been stated independently by V. N. Sorokin [So3] by means of certain explicit Padé-type approximations. We underline that our approach differs from that of [So3].

where the polynomial $b(n)$ is defined in (4).

Unfortunately, the linear forms

$$6D_n^5(u_n\zeta(4)-v_n)\in\mathbb{Z}\zeta(4)+\mathbb{Z}$$

do not tend to 0 as $n\to\infty$...[3]

We prove Theorem 1 in Section 3 and devote Section 2 to an 'elementary' proof of Apéry's theorem on irrationality of $\zeta(3)$. The idea of such proof is due to T. Rivoal [Ri2], [Ri3], who mixed the ideas of Yu. Nesterenko [Ne1] and K. Ball, and our contribution here is to make a use of Zeilberger's algorithm of creative telescoping in the most elementary manner. Therefore, Section 3 reads a natural generalization of Section 2 and arguments presented in Section 3 are quite elementary.

Our proof of Theorem 1 is deeply related to a certain general hypergeometric construction of linear forms in 1 and $\zeta(4)$, which we present in Section 4, proposed in general by Yu. Nesterenko [Ne2], [Ne3] and exploited by us in study of arithmetic properties of odd zeta values [Zu1]–[Zu5]. Section 5 is devoted to solution of asymptotic problems for linear forms so constructed. Another (nice and important) ingredient of the hypergeometric construction is the existence of a non-trivial transformation group for $\zeta(4)$ (Section 6), which is based on classical Bailey's integral transform. G. Rhin and C. Viola have shown [RV1], [RV2], how such groups work in p-adic study of coefficients of linear forms, and have obtained by these means the best known upper estimates for the irrationality measures of $\zeta(2)$ and $\zeta(3)$ (see also [Zu5] for a 'hypergeometric' interpretation of their results). Unfortunately, an 'obvious' arithmetic of our linear forms in 1 and $\zeta(4)$ does not lead to any qualitative result for the irrationality measure of $\zeta(4)$, and we present in Section 7 a conjecture supported by our numerical calculations that would produce the arithmetic result if somebody proved it. Finally, Section 8 is devoted to a discussion of possible generalizations of Theorem 1 and the hypergeometric construction to the case of linear forms in higher zeta values.

2. Elementary proof of Apéry's theorem

Our starting point is repetition of [Ne1, Section 1]. For each integer $n=0,1,2,\dots$ define the rational function

$$R_n(t):=\left(\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)}\right)^2. \tag{8}$$

[3] For a simple explanation why $\zeta(4)$ is irrational, see [Han].

Lemma 1 (cf. [Ne1, Lemma 1]). *There holds the equality*

$$F_n := -\sum_{t=1}^{\infty} R_n'(t) = u_n\zeta(3) - v_n, \tag{9}$$

where $u_n \in \mathbb{Z}$, $D_n^3 v_n \in \mathbb{Z}$.

Proof. Taking square of the partial-fraction expansion

$$\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)} = \sum_{k=0}^{n} \frac{(-1)^{n-k}\binom{n+k}{n}\binom{n}{k}}{t+k}$$

we arrive at the formula

$$R_n(t) = \sum_{k=0}^{n}\biggl(\frac{A_{2k}^{(n)}}{(t+k)^2} + \frac{A_{1k}^{(n)}}{t+k}\biggr),$$

with $A_{jk} = A_{jk}^{(n)}$ satisfying the inclusions

$$A_{2k} = \binom{n+k}{n}^2\binom{n}{k}^2 \in \mathbb{Z} \quad \text{and} \quad D_n A_{1k} \in \mathbb{Z}, \qquad k = 0, 1, \dots, n. \tag{10}$$

Furthermore,

$$\sum_{k=0}^{n} A_{1k} = \sum_{k=0}^{n} \operatorname{Res}_{t=-k} R_n(t) = -\operatorname{Res}_{t=\infty} R_n(t) = 0 \tag{11}$$

since $R_n(t) = O(t^{-2})$ as $t \to \infty$, hence the quantity

$$\begin{aligned} F_n &= \sum_{t=1}^{\infty}\sum_{k=0}^{n}\biggl(\frac{2A_{2k}}{(t+k)^3} + \frac{A_{1k}}{(t+k)^2}\biggr) = \sum_{k=0}^{n}\sum_{l=k+1}^{\infty}\biggl(\frac{2A_{2k}}{l^3} + \frac{A_{1k}}{l^2}\biggr) \\ &= 2\sum_{k=0}^{n} A_{2k}\biggl(\sum_{l=1}^{\infty} - \sum_{l=1}^{k}\biggr)\frac{1}{l^3} + \sum_{k=0}^{n} A_{1k}\biggl(\sum_{l=1}^{\infty} - \sum_{l=1}^{k}\biggr)\frac{1}{l^2} \end{aligned}$$

has the desired form (9), with

$$u_n = 2\sum_{k=0}^{n} A_{2k}, \qquad v_n = 2\sum_{k=0}^{n} A_{2k}\sum_{l=1}^{k}\frac{1}{l^3} + \sum_{k=0}^{n} A_{1k}\sum_{l=1}^{k}\frac{1}{l^2}. \tag{12}$$

Finally, using the inclusions (10) and

$$D_n^j \cdot \sum_{l=1}^{k}\frac{1}{l^j} \in \mathbb{Z} \qquad \text{for} \quad k = 0, 1, \dots, n, \quad j = 2, 3, 4, \dots, \tag{13}$$

we deduce that $u_n \in \mathbb{Z}$ and $D_n^3 v_n \in \mathbb{Z}$ as required.

Since

$$R_0(t)=\frac{1}{t^2}, \qquad R_1(t)=\frac{1}{t^2}+\frac{4}{(t+1)^2}-\frac{4}{t}+\frac{4}{t+1},$$

in accordance with formulae (12) we find that

$$F_0=2\zeta(3) \qquad \text{and} \qquad F_1=10\zeta(3)-12. \tag{14}$$

Now, with a help of Zeilberger's algorithm of creative telescoping [PWZ, Chapter 6] we get the rational function $S_n(t):=s_n(t)R_n(t)$, where

$$s_n(t):=4(2n+1)(-2t^2+t+(2n+1)^2), \tag{15}$$

satisfying the following property.

Lemma 2. *For each $n=1,2,\dots$, there holds the identity*

$$(n+1)^3R_{n+1}(t)-(2n+1)(17n^2+17n+5)R_n(t)+n^3R_{n-1}(t)=S_n(t+1)-S_n(t). \tag{16}$$

One-line proof. Divide both sides of (16) by $R_n(t)$ and verify numerically the identity

$$(n+1)^3\left(\frac{t-n-1}{t+n+1}\right)^2-(2n+1)(17n^2+17n+5)+n^3\left(\frac{t+n}{t-n}\right)^2$$
$$=s_n(t+1)\left(\frac{t^2}{(t-n)(t+n+1)}\right)^2-s_n(t),$$

where $s_n(t)$ is given in (15).

Lemma 3. *The quantity* (9) *satisfies the difference equation* (1) *for $n=1,2,\dots$.*

Proof. Since $R_n'(t)=O(t^{-3})$ and $S_n'(t)=O(t^{-2})$, differentiating identity (16) and summing the result over $t=1,2,\dots$ we arrive at the equality

$$(n+1)^3F_{n+1}-(2n+1)(17n^2+17n+5)F_n+n^3F_{n-1}=S_n'(1).$$

It remains to note that, for $n\geqslant 1$, both functions $R_n(t)$ and $S_n(t)=s_n(t)R_n(t)$ have second-order zero at $t=1$. Thus $S_n'(1)=0$ for $n=1,2,\dots$ and we obtain the desired recurrence (1) for the quantity (9).

Consider another rational function

$$\widetilde{R}_n(t):=n!^2(2t+n)\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^4} \tag{17}$$

and the corresponding hypergeometric series

$$\widetilde{F}_n:=\sum_{t=1}^{\infty}\widetilde{R}_n(t), \tag{18}$$

proposed by K. Ball.

Lemma 4 (cf. [BR, the second proof of Lemma 3]). *For each $n = 0, 1, 2, \dots$, there holds the inequality*

$$0 < \widetilde{F}_n < 20(n+1)^4(\sqrt{2}-1)^{4n}. \tag{19}$$

Proof. Since $\widetilde{R}_n(t) = 0$ for $t = 1, 2, \dots, n$ and $\widetilde{R}_n(t) > 0$ for $t > n$ we deduce that $\widetilde{F}_n > 0$.

With a help of elementary inequality

$$\frac{1}{m} \cdot \frac{(m+1)^m}{m^{m-1}} = \left(1 + \frac{1}{m}\right)^m < e < \left(1 + \frac{1}{m}\right)^{m+1} = \frac{1}{m} \cdot \frac{(m+1)^{m+1}}{m^m}$$

that yields $(m+1)^m/m^{m-1} < em < (m+1)^{m+1}/m^m$ for $m = 1, 2, \dots$, we deduce that

$$e^{-n}\frac{(m+n)^{m+n-1}}{m^{m-1}} < m(m+1)\dots(m+n-1) < e^{-n}\frac{(m+n)^{m+n}}{m^m}.$$

Therefore, for integers $t \geqslant n+1$,

$$\begin{aligned}\widetilde{R}_n(t) \cdot \frac{(t+n)^5}{(2t+n)(t+2n)} &= n!^2 \cdot \frac{(t-1)\cdots(t-n)\cdot(t+n)\cdots(t+2n-1)}{(t(t+1)\cdots(t+n-1))^4} \\ &< (n+1)^{2(n+1)} \cdot \frac{t^{5t-4}(t+2n)^{t+2n}}{(t-n)^{t-n}(t+n)^{5(t+n)-4}}\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\widetilde{R}_n(t) \cdot \frac{t^4(t+n)}{(2t+n)(t+2n)(n+1)^2} &< (n+1)^{2n} \cdot \frac{t^{5t}(t+2n)^{t+2n}}{(t-n)^{t-n}(t+n)^{5(t+n)}} \\ &= \left(1+\frac{1}{n}\right)^{2n} \cdot e^{nf(t/n)} < e^2 \cdot \left(\sup_{\tau>1} e^{f(\tau)}\right)^n,\end{aligned} \tag{20}$$

where

$$f(\tau) := \log\frac{\tau^{5\tau}(\tau+2)^{\tau+2}}{(\tau-1)^{\tau-1}(\tau+1)^{5(\tau+1)}}.$$

The unique (real) solution τ_0 of the equation

$$f'(\tau) = \log\frac{\tau^5(\tau+2)}{(\tau-1)(\tau+1)^5} = 0$$

in the region $\tau > 1$ is the zero of the polynomial

$$\tau^5(\tau+2) - (\tau-1)(\tau+1)^5 = -\left(\tau+\frac{1}{2}\right)\left(2\left(\tau+\frac{1}{2}\right)^4 - 5\left(\tau+\frac{1}{2}\right)^2 - \frac{7}{8}\right),$$

hence we can determine it explicitly:

$$\tau_0 = -\frac{1}{2} + \sqrt{\frac{5}{4}+\sqrt{2}}.$$

Thus,

$$\sup_{\tau>1} f(\tau) = f(\tau_0) = f(\tau_0) - \tau_0 f'(\tau_0) = 2\log(\tau_0+2) + \log(\tau_0-1) - 5\log(\tau_0+1)$$
$$= 4\log(\sqrt{2}-1)$$

and we can continue the estimate (20) as follows:

$$\widetilde{R}_n(t)\cdot\frac{t^4(t+n)}{(2t+n)(t+2n)} < e^2(n+1)^2(\sqrt{2}-1)^{4n}, \tag{21}$$

Finally, we apply the inequality (21) to deduce the required estimate (19):

$$\begin{aligned}\widetilde{F}_n &= \sum_{t=n+1}^{\infty}\widetilde{R}_n(t) < e^2(n+1)^2(\sqrt{2}-1)^{4n}\sum_{t=n+1}^{\infty}\frac{(2t+n)(t+2n)}{t^4(t+n)}\\ &< e^2(n+1)^2(\sqrt{2}-1)^{4n}\sum_{t=n+1}^{\infty}\biggl(\frac{2}{t^5}+\frac{5n}{t^4}+\frac{2n^2}{t^3}\biggr)\\ &\leqslant e^2(n+1)^2\bigl(2\zeta(5)+5n\zeta(4)+2n^2\zeta(3)\bigr)(\sqrt{2}-1)^{4n} < 20(n+1)^4(\sqrt{2}-1)^{4n}.\end{aligned}$$

This completes the proof.

For the rational function (17) we obtain Zeilberger's certificate

$$\begin{aligned}\widetilde{S}_n(t) := \frac{\widetilde{R}_n(t)}{(2t+n)(t+2n-1)(t+2n)}&\cdot\bigl(-t^6-(8n-1)t^5+(4n^2+27n+5)t^4\\ &+2n(67n^2+71n+15)t^3+(358n^4+339n^3+76n^2-7n-3)t^2\\ &+(384n^5+396n^4+97n^3-29n^2-17n-2)t\\ &+n(153n^5+183n^4+50n^3-30n^2-22n-4)\bigr).\end{aligned} \tag{22}$$

Lemma 5. *For each $n=1,2,\dots$, there holds the identity*

$$(n+1)^3\widetilde{R}_{n+1}(t)-(2n+1)(17n^2+17n+5)\widetilde{R}_n(t)+n^3\widetilde{R}_{n-1}(t) = \widetilde{S}_n(t+1)-\widetilde{S}_n(t). \tag{23}$$

One-line proof. Divide both sides of (23) by $\widetilde{R}_n(t)$ and verify the reduced identity.

Lemma 6. *The quantity* (18) *satisfies the difference equation* (1) *for $n=1,2,\dots$.*

Proof. Since $\widetilde{R}_n(t)=O(t^{-5})$ and $\widetilde{S}_n(t)=O(t^{-2})$ as $t\to\infty$ for $n\geqslant 1$, summation of equalities (23) over $t=1,2,\dots$ yields the relation

$$(n+1)^3\widetilde{F}_{n+1}-(2n+1)(17n^2+17n+5)\widetilde{F}_n+n^3\widetilde{F}_{n-1} = -\widetilde{S}_n(1).$$

It remains to note that, for $n\geqslant 1$, both functions (17) and (22) have zero at $t=1$. Thus $\widetilde{S}_n(1)=0$ for $n=1,2,\dots$ and we obtain the desired recurrence (1) for the quantity (18).

Theorem 3. *For each $n = 0, 1, 2, \dots$, the quantities* (9) *and* (18) *coincide.*

Proof. Since both F_n and $\widetilde{F}_n$ satisfy the same second-order difference equation (1), we have to verify that $F_0 = \widetilde{F}_0$ and $F_1 = \widetilde{F}_1$. Direct calculations show that

$$\widetilde{R}_0(t) = \frac{2}{t^3}, \qquad \widetilde{R}_1(t) = -\frac{2}{t^4} + \frac{2}{(t+1)^4} + \frac{5}{t^3} + \frac{5}{(t+1)^3} - \frac{5}{t^2} + \frac{5}{(t+1)^2},$$

hence $\widetilde{F}_0 = 2\zeta(3)$ and $\widetilde{F}_1 = 10\zeta(3) - 12$, and comparison of this result with (14) yields the desired coincidence.

Apéry's theorem. *The number $\zeta(3)$ is irrational.*

Proof. Suppose, on the contrary, that $\zeta(3) = p/q$, where p and q are positive integers. Then, using a trivial bound $D_n < 3^n$, we deduce that, for each $n = 0, 1, 2, \dots$, the integer $qD_n^3 F_n = D_n^3 u_n p - D_n^3 v_n q$ satisfies the estimate

$$0 < qD_n^3 F_n < 20q(n+1)^4 3^{3n} (\sqrt{2} - 1)^{4n} \tag{24}$$

that is not possible since $3^3(\sqrt{2} - 1)^4 = 0.7948\ldots < 1$ and the right-hand side of (24) is less than 1 for a sufficiently large integer n.

Remark 1. Inspite of its elementary arguments, our proof of Apéry's theorem does not look simpler than original (also elementary) Apéry's proof well-explained in A. van der Poorten's informal report [Po], or (almost elementary) Beukers's proof [Be1] by means of Legendre polynomials and multiple integrals. We want to mention that our way to deduce the recursion (1) for the sequence F_n as well as for the coefficients u_n, v_n[4] slightly differs from those considered in [Po, Section 8] and [Ze, Section 13] although it is based on the same algorithm of creative telescoping.

Remark 2. The fact that $\widetilde{F}_n = \widetilde{u}_n\zeta(3) - \widetilde{v}_n$ with $D_n\widetilde{u}_n, D_n^4\widetilde{v}_n \in \mathbb{Z}$ was first discovered by K. Ball; the proof follows lines of the proof of Lemma 1 and vanishing the coefficients for $\zeta(4)$ and $\zeta(2)$ is due to well-poised origin of the series (18) (cf. Lemma 12 below). An open question of Ball and Rivoal here is to get the better inclusions $\widetilde{u}_n, D_n^3\widetilde{v}_n \in \mathbb{Z}$ by elementary means without going back to Apéry's series (9). A solution of this question accompanied with Ball's Lemma 4 can bring the 'most elementary' proof of Apéry's theorem.

[4]Hint: multiply both sides of (16) by $(t+k)^2$, substitute $t = -k$ and sum over *all* integers k to show that the sequence u_n satisfies the difference equation (1); then $v_n = u_n\zeta(3) - F_n$ also satisfies it.

3. Difference equation for $\zeta(4)$

Consider the rational function

$$R_n(t) := (-1)^n(2t+n)\biggl(\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^2}\biggr)^2 \tag{25}$$

and the corresponding series

$$F_n := -\sum_{t=1}^{\infty} R_n'(t). \tag{26}$$

In some sence, the function (25) is a mixture of the functions (8) and (17).

Lemma 7. *There holds the equality*

$$F_n = U_n\zeta(5) + U_n'\zeta(4) + U_n''\zeta(3) + U_n'''\zeta(2) - V_n, \tag{27}$$

where $U_n, D_nU_n', D_n^2U_n'', D_n^3U_n''', D_n^5V_n \in \mathbb{Z}$.

Proof. The polynomials

$$P_n^{(1)}(t) := \frac{(t-1)\cdots(t-n)}{n!} \qquad \text{and} \qquad P_n^{(2)}(t) := \frac{(t+n+1)\cdots(t+2n)}{n!} \tag{28}$$

are integral-valued and, as it is well known,

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^jP_n(t)}{\mathrm{d}t^j}\bigg|_{t=-k} \in \mathbb{Z} \qquad \text{for} \quad k\in\mathbb{Z} \quad \text{and} \quad j=0,1,2,\dots, \tag{29}$$

where $P_n(t)$ is any of the polynomials (28).

The rational function

$$Q_n(t) := \frac{n!}{t(t+1)\cdots(t+n)} \tag{30}$$

has also 'nice' arithmetic properties. Namely,

$$a_k := Q_n(t)(t+k)\big|_{t=-k} = \begin{cases} (-1)^k\binom{n}{k}\in\mathbb{Z} & \text{if } k=0,1,\dots,n, \\ 0 & \text{for other } k\in\mathbb{Z}, \end{cases} \tag{31}$$

that allow to write the following partial-fraction expansion:

$$Q_n(t) = \sum_{l=0}^{n}\frac{a_l}{t+l}.$$

Hence, for $j=1,2,\dots$ we obtain

$$\begin{aligned} \frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(Q_n(t)(t+k)\bigr)\big|_{t=-k} &= \frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\sum_{l=0}^{n}a_l\biggl(1-\frac{l-k}{t+l}\biggr)\bigg|_{t=-k} \\ &= (-1)^{j-1}D_n^j\sum_{\substack{l=0\\ l\ne k}}^{n}\frac{1}{(l-k)^j}\in\mathbb{Z}. \end{aligned} \tag{32}$$

Therefore the inclusions (29), (31), (32) and the Leibniz rule for differentiating a product imply that the numbers

$$A_{jk} = A_{jk}^{(n)} := \frac{1}{(4-j)!}\,\frac{\mathrm{d}^{4-j}}{\mathrm{d}t^{4-j}}\big(R_n(t)(t+k)^4\big)\big|_{t=-k} \tag{33}$$
$$= \frac{1}{(4-j)!}\,\frac{\mathrm{d}^{4-j}}{\mathrm{d}t^{4-j}}\big((-1)^n(2t+n)\cdot P_n^{(1)}(t)\cdot P_n^{(2)}(t)\cdot (Q_n(t)(t+k))^4\big)\big|_{t=-k}$$

satisfy the inclusions

$$D_n^{4-j}\cdot A_{jk}^{(n)} \in \mathbb{Z} \qquad \text{for} \quad k=0,1,\dots,n \quad \text{and} \quad j=1,2,3,4. \tag{34}$$

Now, writing down the partial-fraction expansion of the rational function (25),

$$R_n(t) = \sum_{j=1}^{4}\sum_{k=0}^{n}\frac{A_{jk}^{(n)}}{(t+k)^j}, \tag{35}$$

and following the proof of Lemma 1, we obtain the desired representation (27) with

$$U_n = 4\sum_{k=0}^{n} A_{4k}^{(n)}, \quad U_n' = 3\sum_{k=0}^{n} A_{3k}^{(n)}, \quad U_n'' = 2\sum_{k=0}^{n} A_{2k}^{(n)}, \quad U_n''' = \sum_{k=0}^{n} A_{1k}^{(n)}, \tag{36}$$
$$V_n = \sum_{j=1}^{4} j\sum_{k=0}^{n} A_{jk}^{(n)}\sum_{l=1}^{k}\frac{1}{l^{j+1}}. \tag{37}$$

Finally, using the inclusions (34) and (13) we deduce that $U_n, D_nU_n', D_n^2U_n'', D_n^3U_n'''$, $D_n^5V_n \in \mathbb{Z}$ as required.

For the rational function (25) we obtain Zeilberger's certificate $S_n(t) := s_n(t)R_n(t)$, where

$$\begin{aligned} s_n(t) := \frac{1}{(2t+n)(t+2n-1)^2(t+2n)^2}\cdot\big(&-(122n^2+115n+29)(t+2(5n-1))t^7 \\ &-(4796n^4+2336n^3-859n^2-459n+16)t^6 \\ &-2(4333n^5-43n^4-2645n^3-734n^2+86n+7)t^5 \\ &-(3965n^6-13782n^5-14109n^4-2207n^3+878n^2+142n+7)t^4 \\ &+2(5906n^7+17354n^6+10901n^5+329n^4-1340n^3-289n^2-15n+2)t^3 \\ &+(22774n^8+42602n^7+20740n^6-2935n^5-4922n^4-1162n^3 \\ &\quad +13n^2+44n+4)t^2 \\ &+2n(8249n^8+13764n^7+5775n^6-2178n^5-2468n^4-568n^3 \\ &\quad +94n^2+64n+8)t \\ &+n^2(4549n^8+7531n^7+2923n^6-1975n^5-2056n^4-424n^3 \\ &\quad +196n^2+112n+16)\big) \end{aligned} \tag{38}$$

with the following property.

Lemma 8. *For each $n=1,2,\dots$, there holds the identity*

$$(n+1)^5R_{n+1}(t)-b(n)R_n(t)-3n^3(3n-1)(3n+1)R_{n-1}(t)=S_n(t+1)-S_n(t), \quad (39)$$

where the polynomial $b(n)$ is given in (4).

One-line proof. Divide both sides of (39) by $R_n(t)$ and verify the identity

$$\begin{aligned}
&-(n+1)^5\cdot\frac{(2t+n+1)(t-n-1)^2(t+2n+1)^2(t+2n+2)^2}{(2t+n)(t+n+1)^6}\\
&\qquad -3(2n+1)(15n^2+15n+4)(3n^2+3n+1)\\
&\qquad +3n^3(3n-1)(3n+1)\cdot\frac{(2t+n-1)(t+n)^6}{(2t+n)(t-n)^2(t+2n-1)^2(t+2n)^2}\\
&=s_n(t+1)\frac{(2t+n+2)t^6(t+2n+1)^2}{(2t+n)(t-n)^2(t+n+1)^6}-s_n(t),
\end{aligned}$$

where $s_n(t)$ is given in (38).

Lemma 9. *The quantity* (26) *satisfies the difference equation* (3) *for $n=1,2,\dots$.*

Proof. Since $R_n(t)=O(t^{-3})$ and $S_n'(t)=O(t^{-2})$ as $t\to\infty$ for $n\geqslant 1$, summation of t-derivatives of equalities (39) over $t=1,2,\dots$ yields the relation

$$(n+1)^5F_{n+1}-b(n)F_n-3n^3(3n-1)(3n+1)F_{n-1}=S_n'(1).$$

It remains to note that, for $n\geqslant 1$, both functions $R_n(t)$ and $S_n(t)=s_n(t)R_n(t)$ have second-order zero at $t=1$. Thus $S_n'(1)=0$ for $n=1,2,\dots$ and we obtain the desired recurrence (3) for the quantity (26).

Lemma 10. *The coefficients $U_n,U_n',U_n'',U_n''',V_n$ in the representation* (27) *satisfy the difference equation* (3) *for $n=1,2,\dots$.*

Proof. Write the partial-fraction expansion (35) in the form

$$R_n(t)=\sum_{j=1}^{4}\sum_{k=-\infty}^{+\infty}\frac{A_{jk}^{(n)}}{(t+k)^j},$$

where the formulae (33) remain valid for all $k\in\mathbb{Z}$ and $j=1,2,3,4$. Multiply both sides of (39) by $(t+k)^4$, take $(4-j)$th derivative of the result, substitute $t=-k$ and sum over all $k\in\mathbb{Z}$; this procedure yields that, for each $j=1,2,3,4$, the numbers (36) written as

$$U_n=4\sum_{k=-\infty}^{+\infty}A_{4k}^{(n)}, \quad U_n'=3\sum_{k=-\infty}^{+\infty}A_{3k}^{(n)}, \quad U_n''=2\sum_{k=-\infty}^{+\infty}A_{2k}^{(n)}, \quad U_n'''=\sum_{k=-\infty}^{+\infty}A_{1k}^{(n)}$$

satisfy the difference equation (3). Finally, the sequence

$$V_n = U_n\zeta(5) + U'_n\zeta(4) + U''_n\zeta(3) + U'''_n\zeta(2) - F_n$$

also satisfies the recursion (3).

Since

$$R_0(t) = \frac{2}{t^3}, \qquad R_1(t) = -\frac{4}{t^4} + \frac{4}{(t+1)^4} + \frac{12}{t^3} + \frac{12}{(t+1)^3} - \frac{13}{t^2} + \frac{13}{(t+1)^2},$$

in accordance with (36), (37) we obtain

$$U'_0 = 6, \quad U_0 = U''_0 = U'''_0 = V_0 = 0,$$
$$U'_1 = 72, \quad V_1 = 78, \quad U_1 = U''_1 = U'''_1 = 0,$$

hence as a consequence of Lemma 10 we arrive at the following result.

Lemma 11. *There holds the equality*

$$F_n = U'_n\zeta(4) - V_n,$$

where $D_nU'_n \in \mathbb{Z}$ *and* $D_n^5V_n \in \mathbb{Z}$.

The sequences $u_n := U'_n/6$ and $v_n := V_n/6$ satisfy the difference equation (3) and initial conditions (5); the fact $|F_n| \to 0$ as $n \to \infty$, which yields the limit relation (7), will be proved in Section 5. This fact can be also derived from elementary estimates

$$|F_n| \leqslant \sum_{t=n+1}^{\infty} |R'_n(t)| < \sum_{t=n+1}^{\infty} 8(n+1)R_n(t)$$

as in the proof of Lemma 4. This completes our proof of Theorem 1.

Due to (33), (36), and (37), we can write the explicit formulae for the solutions u_n and v_n of the difference equation (3) for $n = 0, 1, 2, \ldots$:

$$u_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^4 \binom{n+k}{n}^2 \binom{2n-k}{n}^2 (1 + (n-2k)a_{nk}),$$

$$v_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^4 \binom{n+k}{n}^2 \binom{2n-k}{n}^2 \biggl(\frac{2}{3}(n-2k)\sum_{l=1}^{k}\frac{1}{l^5}$$
$$+ (1 + (n-2k)a_{nk})\sum_{l=1}^{k}\frac{1}{l^4} + \frac{1}{3}\bigl(4a_{nk} + (n-2k)(2a_{nk}^2 + b_{nk})\bigr)\sum_{l=1}^{k}\frac{1}{l^3}$$
$$+ \frac{1}{9}\bigl((6a_{nk}^2 + 3b_{nk}) + (n-2k)(2a_{nk}^3 + 3a_{nk}b_{nk} + c_{nk})\bigr)\sum_{l=1}^{k}\frac{1}{l^2}\biggr),$$

where

$$a_{nk}=-\sum_{l=1}^{n}\frac{1}{l+k}+\sum_{l=1}^{n}\frac{1}{l+n-k}-2\sum_{\substack{l=0\\ l\neq k}}^{n}\frac{1}{l-k},$$

$$b_{nk}=-\sum_{l=1}^{n}\frac{1}{(l+k)^2}-\sum_{l=1}^{n}\frac{1}{(l+n-k)^2}+2\sum_{\substack{l=0\\ l\neq k}}^{n}\frac{1}{(l-k)^2},$$

$$c_{nk}=-\sum_{l=1}^{n}\frac{1}{(l+k)^3}+\sum_{l=1}^{n}\frac{1}{(l+n-k)^3}-2\sum_{\substack{l=0\\ l\neq k}}^{n}\frac{1}{(l-k)^3}$$

for $k=0,1,\dots,n$.

Remark. The conclusion (6) of Theorem 1 is far from being precise; in fact, (experimentally) there hold the inclusions

$$u_n\in\mathbb{Z},\qquad D_n^4v_n\in\mathbb{Z},$$

and, moreover, there exists the sequence of positive integers Φ_n, $n=0,1,2,\dots$, such that

$$\Phi_n^{-1}u_n\in\mathbb{Z},\qquad \Phi_n^{-1}D_n^4v_n\in\mathbb{Z}.$$

This sequence can be determined as follows: if ν_p is the order of prime p in $(3n)!/n!^3$, then

$$\Phi_n:=\prod_p p^{\lfloor\nu_p/2\rfloor};$$

here and below $\lfloor x\rfloor$ and $\{x\}:=x-\lfloor x\rfloor$ denote respectively the integral and fractional parts of a real number x. For primes $p>\sqrt{3n}$ we obtain the explicit (simple) formula

$$\lfloor\nu_p/2\rfloor=\begin{cases}1 & \text{if } \{n/p\}\in[\frac{2}{3},1),\\ 0 & \text{otherwise,}\end{cases}$$

hence

$$\lim_{n\to\infty}\frac{\log\Phi_n}{n}=\psi(1)-\psi\Big(\frac{2}{3}\Big)=0.74101875\dots,$$

where $\psi(x):=\Gamma'(x)/\Gamma(x)$. Thus, we obtain that the linear forms

$$\Phi_n^{-1}D_n^4(u_n\zeta(4)-v_n)\overset{?}{\in}\mathbb{Z}\zeta(4)+\mathbb{Z} \qquad (40)$$

do not tend to 0 as $n\to\infty$.

4. Very-well-poised hypergeometric series

Consider the set of eight positive integral parameters

$$\begin{aligned} &\boldsymbol{h} = (h_0, h_{-1}; h_1, h_2, h_3, h_4, h_5, h_6), \\ \text{where} \quad &h_{-1} = 2 + 3h_0 - (h_1 + h_2 + h_3 + h_4 + h_5 + h_6), \end{aligned} \tag{41}$$

satisfying the conditions

$$h_0 - h_{-1} < h_j < \frac{1}{2}h_0, \qquad j = 1, 2, 3, 4, 5, 6, \tag{42}$$

and assign to $\boldsymbol{h}$ the rational function

$$\begin{aligned} R(t) = R(\boldsymbol{h}; t) &:= (-1)^{h_0}\gamma(\boldsymbol{h}) \cdot (h_0 + 2t) \cdot \frac{\prod_{j=-1}^{6} \Gamma(h_j + t)}{\prod_{j=-1}^{6} \Gamma(1 + h_0 - h_j + t)} \\ &= (-1)^{h_0} \cdot (h_0 + 2t) \\ &\quad \times \Gamma(1 + h_0 - h_1 - h_2) \frac{\Gamma(h_1 + t)}{\Gamma(1 + h_0 - h_2 + t)} \\ &\quad \times \Gamma(1 + h_0 - h_1 - h_5) \frac{\Gamma(h_5 + t)}{\Gamma(1 + h_0 - h_1 + t)} \\ &\quad \times \Gamma(1 + h_0 - h_2 - h_4) \frac{\Gamma(h_2 + t)}{\Gamma(1 + h_0 - h_4 + t)} \\ &\quad \times \Gamma(1 + h_0 - h_3 - h_6) \frac{\Gamma(h_6 + t)}{\Gamma(1 + h_0 - h_3 + t)} \\ &\quad \times \frac{1}{\Gamma(h_3)} \frac{\Gamma(h_3 + t)}{\Gamma(1 + t)} \\ &\quad \times \frac{1}{\Gamma(h_{-1} - h_0 + h_4)} \frac{\Gamma(h_4 + t)}{\Gamma(1 + h_0 - h_{-1} + t)} \\ &\quad \times \frac{1}{\Gamma(h_5)} \frac{\Gamma(h_0 + t)}{\Gamma(1 + h_0 - h_5 + t)} \\ &\quad \times \frac{1}{\Gamma(h_{-1} - h_0 + h_6)} \frac{\Gamma(h_{-1} + t)}{\Gamma(1 + h_0 - h_6 + t)}. \end{aligned} \tag{43}$$

In the last representation we pick out the rational functions

$$\begin{aligned} \Gamma(b - a) \frac{\Gamma(a + t)}{\Gamma(b + t)} &= \frac{(b - a - 1)!}{(t + a)(t + a + 1) \cdots (t + b - 1)} \quad &&\text{if } a < b, \\ \frac{1}{\Gamma(1 + a - b)} \frac{\Gamma(a + t)}{\Gamma(b + t)} &= \frac{(t + b)(t + b + 1) \cdots (t + a - 1)}{(a - b)!} \quad &&\text{if } a \geqslant b, \end{aligned}$$

of the form (30), (28), having some nice arithmetic properties [Zu5, Section 7].

It is easy to verify that, due to (41), for the rational function (43) the difference of numerator and denominator degrees is equal to 3, hence

$$R(t) = O(t^{-3}) \qquad \text{as} \quad t \to \infty. \tag{44}$$

The series

$$
\begin{gathered}
F(\boldsymbol{h}) := -\sum_{t=t_0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} R(\boldsymbol{h};t) \\
\text{with any } t_0 \in \mathbb{Z}, \quad 1 - \min_{1\leqslant j\leqslant 6}\{h_j\} \leqslant t_0 \leqslant 1 - \max\{0, h_0 - h_{-1}\},
\end{gathered}
\tag{45}
$$

produces a linear form in 1 and $\zeta(4)$.

Lemma 12. *The quantity $F(\boldsymbol{h})$ is a linear form in 1 and $\zeta(4)$ with rational coefficients.*

Proof. Order the parameters $h_1, \dots, h_6$ as $h_1^* \leqslant \dots \leqslant h_6^*$ and consider the partial-fraction expansion of the rational function (43):

$$
R(t) = \sum_{j=1}^{4} \sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*} \frac{A_{jk}}{(t+k)^j}, \tag{46}
$$

where

$$
\begin{gathered}
A_{jk} = \frac{1}{(4-j)!} \frac{\mathrm{d}^{4-j}}{\mathrm{d}t^{4-j}} \big(R(t)(t+k)^4\big)\big|_{t=-k} \in \mathbb{Q} \\
\text{for} \quad k = h_{j+2}^*, \dots, h_0 - h_{j+2}^* \quad \text{and} \quad j = 1, 2, 3, 4.
\end{gathered}
\tag{47}
$$

Then we obtain

$$
\begin{aligned}
F(\boldsymbol{h}) &= \sum_{t=1-h_1^*} \sum_{j=1}^{4} \sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*} \frac{jA_{jk}}{(t+k)^{j+1}} = \sum_{j=1}^{4} \sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*} jA_{jk} \biggl(\sum_{l=1}^{\infty} - \sum_{l=1}^{k-h_1^*} \biggr) \frac{1}{l^{j+1}} \\
&= \sum_{j=1}^{4} A_j \zeta(j+1) - A_0,
\end{aligned}
$$

with

$$
A_j = j \sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*} A_{jk}, \quad j = 1, 2, 3, 4, \qquad A_0 = \sum_{j=1}^{4} \sum_{k=h_{j+2}^*}^{h_0-h_{j+2}^*} jA_{jk} \sum_{l=1}^{k-h_1^*} \frac{1}{l^{j+1}},
$$

and the well-poised origin of the series (45) (namely, the property $R(-t-h_0) = -R(t)$, hence $A_{jk} = (-1)^{j-1} A_{j,h_0-k}$ by (47), cf. [Zu5, Section 8] with $r = 2$ and $q = 6$) yields $A_2 = A_4 = 0$, while the residue sum theorem implies $A_1 = 0$ (cf. (11)).

Remark. The question of denominators of the rational numbers A_3 and A_0 that appear as the coefficients in $F(\boldsymbol{h})$ can be solved by application of Nesterenko's denominator theorem [Ne3] (announced by Yu. Nesterenko in his Caen's talk). Namely, consider the set

$$
\begin{aligned}
\mathcal{N} := \{&h_3 - 1, h_{-1} - h_0 + h_4 - 1, h_5 - 1, h_{-1} - h_0 + h_6 - 1, h_0 - 2h_1, h_0 - h_1 - h_2, \\
&h_0 - h_1 - h_3, h_0 - h_1 - h_4, h_0 - h_1 - h_6, h_0 - 2h_2, h_0 - h_2 - h_3, \\
&h_0 - h_2 - h_5, h_0 - h_2 - h_6, h_0 - h_3 - h_5, h_0 - h_4 - h_5, h_0 - h_4 - h_6, \\
&h_0 - h_1^* - h_3^*, h_0 - h_1^* - h_3^*, h_0 - h_1^* - h_4^*, h_0 - h_1^* - h_5^*, h_0 - h_1^* - h_6^*\},
\end{aligned}
$$

then,

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4}D_{m_5}\cdot F(\boldsymbol{h})\in\mathbb{Z}\zeta(4)+\mathbb{Z}, \tag{48}$$

where $m_1\geqslant\cdots\geqslant m_5$ are the five successive maxima of the set $\mathcal{N}$.

Unfortunately, we have not succeeded in using the inclusion (48) for arithmetic applications; in reality, our experimental calculations show that the stronger inclusion holds for the linear forms $F(\boldsymbol{h})$ and we indicate the corresponding conjecture in Section 7 below.

Using standard arguments, the property (44) and the fact that $R(t)$ has second-order zeros at integers $t=1-h_1^*,\dots,-\max\{0,h_0-h_{-1}\}$, one deduces the following hypergeometric-integral representation of the series (45).

Lemma 13 (cf. [Ne1, Lemma 2]). *There holds the equality*

$$\begin{aligned}F(\boldsymbol{h})&=\frac{1}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty}R(\boldsymbol{h};t)\biggl(\frac{\pi}{\sin\pi t}\biggr)^2\mathrm{d}t\\&=\frac{(-1)^{h_{-1}}\gamma(\boldsymbol{h})}{\pi i}\int_{t_1-i\infty}^{t_1+i\infty}\frac{\begin{gathered}\Gamma(h_0+t)\,\Gamma(1+\frac12h_0+t)\,\Gamma(h_{-1}+t)\\ \times\Gamma(h_1+t)\cdots\Gamma(h_6+t)\cdot\Gamma(h_{-1}-h_0-t)\,\Gamma(-t)\end{gathered}}{\Gamma(\frac12h_0+t)\,\Gamma(1+h_0-h_1+t)\cdots\Gamma(1+h_0-h_6+t)}\,\mathrm{d}t,\end{aligned} \tag{49}$$

with any $t_1\in\mathbb{R}$, $1-h_1^*<t_1<-\max\{0,h_0-h_{-1}\}$.

The series (45) as well as the corresponding hypergeometric integral (49) are known in the theory of hypergeometric functions and integrals as very-well-poised objects, i.e., we can split their top and bottom parameters in pairs such that

$$h_0+1=(1+\tfrac12h_0)+\tfrac12h_0=h_{-1}+(1+h_0-h_{-1})=\cdots=h_6+(1+h_0-h_6)$$

and the second parameter has the special form $1+\frac12h_0$.

Remark. As it is easily seen, the sequence F_n of Section 3 corresponds (after a suitable shift of the summation parameter t) to the choice

$$h_0=h_{-1}=3n+2,\qquad h_1=h_2=h_3=h_4=h_5=h_6=n+1 \tag{50}$$

of the parameters $\boldsymbol{h}$. Hence the equalities $U_n=U_n''=U_n'''=0$ in the representation (27) can be deduced from Lemma 12.

5. Asymptotics

We take the new set of positive parameters

$$\boldsymbol{\eta}=(\eta_0,\eta_{-1};\eta_1,\dots,\eta_6) \tag{51}$$

satisfying the conditions

$$4\eta_0=\sum_{j=-1}^{6}\eta_j, \qquad \eta_0-\eta_{-1}<\eta_j<\frac12\eta_0, \quad j=1,2,3,4,5,6, \tag{52}$$

and for each $n=0,1,2,\dots$ relate them with the old parameters by the formulae

$$h_0=\eta_0n+2, \quad h_{-1}=\eta_{-1}n+2, \qquad h_j=\eta_jn+1, \quad j=1,2,\dots,6. \tag{53}$$

Then Lemma 12 yields that the quantities $F_n=F_{n,\boldsymbol{\eta}}:=F(\boldsymbol{h})$ are linear forms in 1 and $\zeta(4)$ with rational coefficients, say

$$F_n=F_{n,\boldsymbol{\eta}}=u_n\zeta(4)-v_n, \qquad n=0,1,2,\dots,$$

and the goal of this section is to determine the asymptotic behaviour of these linear forms as well as their coefficients u_n and v_n as $n\to\infty$.

To the set (51) assign the polynomial

$$\prod_{j=-1}^{6}(\tau-\eta_j)-\prod_{j=-1}^{6}(\tau-\eta_0+\eta_j) \tag{54}$$

and the function

$$f_0(\tau):=\sum_{j=-1}^{6}\eta_j\log(\eta_j-\tau)-(\eta_0-\eta_{-1})\log(\tau-\eta_0+\eta_{-1})-\sum_{j=1}^{6}(\eta_0-\eta_j)\log(\eta_0-\eta_j-\tau)$$

defined in the cut τ-plane $\mathbb{C}\setminus(-\infty,\max\{0,\eta_0-\eta_{-1}\}]\cup[\eta_1^*,+\infty)$, where $\eta_1^*\leqslant\eta_2^*\leqslant\dots\leqslant\eta_6^*$ denotes the ordered version of the set $\eta_1,\eta_2,\dots,\eta_6$.

The first condition in (52) implies that (54) is a fifth-degree polynomial; moreover, the symmetry under substitution $\tau\mapsto\eta_0-\tau$ and the second condition in (52) yield that this polynomial has zeros

$$\frac{\eta_0}{2},\ \frac{\eta_0}{2}\pm s_0,\ \text{and}\ \frac{\eta_0}{2}\pm is_1, \qquad \text{where}\quad \frac{\eta_0}{2}-s_0\in\bigl(\max\{0,\eta_0-\eta_{-1}\},\eta_1^*\bigr),\ s_1\in(0,+\infty).$$

The last four zeros can be easily determined by solving a certain biquadratic (in terms of $\eta_0/2-\tau$) equation. Set

$$\tau_0:=\frac{\eta_0}{2}-s_0 \qquad \text{and} \qquad \tau_1:=\frac{\eta_0}{2}+is_1. \tag{55}$$

Theorem 4. *The following limit relations hold:*

$$C_0 := -\lim_{n\to\infty}\frac{\log|F_n|}{n} = -f_0(\tau_0), \tag{56}$$

$$C_1 := \limsup_{n\to\infty}\frac{\log|u_n|}{n} = \limsup_{n\to\infty}\frac{\log|v_n|}{n} = \operatorname{Re} f_0(\tau_1). \tag{57}$$

Proof. The proof is based on application of the saddle-point method to the integral representation of Lemma 13 for the quantities F_n and a similar integral representation (see formula (63) below) for the coefficients u_n; the fact that both limits in (57) are equal follows immediately from the limit relation

$$\lim_{n\to\infty}\frac{v_n}{u_n} = \lim_{n\to\infty}\frac{u_n\zeta(4)-F_n}{u_n} = \zeta(4)\ne 0$$

since $-C_0 < 0 < C_1$ under the conditions (52).

Without loss of generality, we will restrict ourselves to the 'most symmetric' case (50), i.e.,

$$\eta_0 = \eta_{-1} = 3 \qquad \text{and} \qquad \eta_1 = \dots = \eta_6 = 1, \tag{58}$$

that corresponds to the linear forms in $1, \zeta(4)$ constructed in Section 3.

In the case (58), the zeros (55) of the corresponding polynomial (54) are as follows:

$$\tau_0 = \frac{3}{2} - 3^{1/4}\cos\frac{\pi}{12} = \frac{3}{2} - \sqrt{\frac{3}{4}+\frac{\sqrt{3}}{2}},$$
$$\tau_1 = \frac{3}{2} + i3^{1/4}\sin\frac{\pi}{12} = \frac{3}{2} + \sqrt{\frac{3}{4}-\frac{\sqrt{3}}{2}}.$$

By Lemma 13,

$$\begin{aligned} F_n &= \frac{(-1)^n}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty}(3n+2+2t)\frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t \\ &= \frac{(-1)^n}{2\pi i}\int_{t_1-i\infty}^{t_1+i\infty}\frac{(3n+2+2t)(3n+1+t)^2(3n+t)^2(n+t)^6}{(2n+1+t)^6(2n+t)^6} \\ &\qquad\qquad \times\frac{\Gamma(3n+t)^2\Gamma(n+t)^6\Gamma(-t)^2}{\Gamma(2n+t)^6}\,\mathrm{d}t, \end{aligned}$$

with any $t_1\in\mathbb{R}$, $-n<t_1<0$. Using the asymptotic formula

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + O\big(|z|^{-1}\big)$$

for $z\in\mathbb{C}$ with $\operatorname{Re} z = \text{const} > 0$, taking $t_1 = -n\tau_0$ and changing variables $t = -n\tau$, after necesary transformations we obtain

$$F_n = \frac{2\pi(-1)^n}{in^2}\int_{\tau_0-i\infty}^{\tau_0+i\infty}\frac{(3-2\tau)(3-\tau)^3(1-\tau)^3}{\tau(2-\tau)^9}e^{nf(\tau)}\big(1+O(n^{-1})\big)\,\mathrm{d}\tau \tag{59}$$

as $n \to \infty$, where

$$f(\tau) := 2(3-\tau)\log(3-\tau) + 6(1-\tau)\log(1-\tau) + 2\tau\log\tau - 6(2-\tau)\log(2-\tau).$$

Since

$$f'(\tau) = \log\frac{\tau^2(2-\tau)^6}{(3-\tau)^2(1-\tau)^6} \tag{60}$$

and τ_0 is a zero of the polynomial (54) (which is $(\tau-3)^2(\tau-1)^6 - \tau^2(\tau-2)^6$ in the restricted case), we conclude that $f'(\tau_0) = 0$ and τ_0 is the unique maximum of the function $\operatorname{Re} f(\tau)$ on the contour. Thus the integral (59) is determined by the contribution of the saddle-point τ_0 (see [Br, Section 5.7]):

$$F_n = \frac{(-1)^n(2\pi)^{3/2}}{n^{5/2}} \cdot \frac{(3-2\tau_0)(3-\tau_0)^3(1-\tau_0)^3}{\tau_0(2-\tau_0)^9} \cdot |f''(\tau_0)|^{-1/2} \cdot e^{nf(\tau_0)}\bigl(1+O(n^{-1})\bigr),$$

hence

$$\begin{aligned}\lim_{n\to\infty}\frac{\log|F_n|}{n} &= f(\tau_0) = f(\tau_0) - \tau_0 f'(\tau_0) =: f_0(\tau_0) \\ &= \log\frac{(3-\tau_0)^6(1-\tau_0)^6}{(2-\tau_0)^{12}} = 3\log(2\sqrt{3}-3) =: -C_0.\end{aligned} \tag{61}$$

This proves the limit relation (56).

In the neighbourhood of $t=-k$, where $k = n+1, \dots, 2n+1$, the function $R(t)$ has the expansion

$$R(t) = \frac{A_{4k}}{(t+k)^4} + \frac{A_{3k}}{(t+k)^3} + \frac{A_{2k}}{(t+k)^2} + \frac{A_{1k}}{t+k} + O(1)$$

by (46). On the other hand,

$$\biggl(\frac{\sin\pi t}{\pi}\biggr)^2 = \biggl(\frac{\sin\pi(t+k)}{\pi}\biggr)^2 = (t+k)^2 + O\bigl((t+k)^4\bigr)$$

about $t=-k$ for $k \in \mathbb{Z}$. Therefore,

$$\operatorname{Res}_{t=-k}\biggl(\biggl(\frac{\sin\pi t}{\pi}\biggr)^2 R(t)\biggr) = \begin{cases} A_{3k} & \text{if } k = n+1, \dots, 2n+1, \\ 0 & \text{for other } k \in \mathbb{Z}, \end{cases}$$

and if $\mathcal{L}$ is a closed clockwise contour surrounding points $t = -n-1, \dots, -2n-1$, then

$$\begin{aligned}\frac{1}{3}u_n = \sum_{k=n+1}^{2n+1} A_{3k} &= -\frac{1}{2\pi i}\oint_{\mathcal{L}}\biggl(\frac{\sin\pi t}{\pi}\biggr)^2 R(t)\,\mathrm{d}t \\ &= -\frac{(-1)^n}{2\pi i}\oint_{\mathcal{L}}\biggl(\frac{\sin\pi t}{\pi}\biggr)^4 (3n+2+2t)\frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t.\end{aligned} \tag{62}$$

Taking the rectangle with vertices $\pm it_2 \pm N$, for some fixed real $t_2 > 0$ and any $N > 2n+1$, as the contour $\mathcal{L}$ and using the estimates

$$\left|\frac{\sin \pi t}{t}\right| \leqslant \frac{e^{\pi t_2}}{\pi}, \qquad R(t) = O(N^{-3}) \quad \text{as } N \to \infty$$

on the lateral sides of the rectangle, from (62) we deduce that

$$\begin{aligned} u_n = -\frac{3(-1)^n}{2\pi i}\biggl(\int_{it_2-N}^{it_2+N} + \int_{-it_2+N}^{-it_2-N}\biggr)\left(\frac{\sin \pi t}{\pi}\right)^4 \\ \times (3n+2+2t)\frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t + O(N^{-2}), \end{aligned}$$

where the constant in $O(N^{-2})$ depends on t_2 only. Tending $N \to \infty$ and making the substitution $t \mapsto -t - h_0 = -t - (3n+2)$ in the first integral, we obtain

$$u_n = -\frac{3(-1)^n}{\pi i}\int_{-it_2+\infty}^{-it_2-\infty}\left(\frac{\sin \pi t}{\pi}\right)^4 (3n+2+2t)\frac{\Gamma(3n+2+t)^2\Gamma(n+1+t)^6\Gamma(-t)^2}{\Gamma(2n+2+t)^6}\,\mathrm{d}t \tag{63}$$

(cf. [Zu3, Lemma 3.1]). Finally, take $t_2 = -ns_1 = -n \operatorname{Im} \tau_1$, change the variable $t = -n\tau$ and apply the asymptotic formula

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + O\bigl(|z|^{-1}\bigr) + O(e^{-2\pi|\operatorname{Im} z|})$$
$$\text{for} \quad z \in \mathbb{C}, \quad |\operatorname{Im} z| \geqslant y_0 > 0$$

(see [Br, Section 6.5] and [Zu3, Lemma 3.2]), to get from (63) the expansion

$$\begin{aligned} u_n = \frac{12\pi(-1)^n}{in^2}\int_{is_1-\infty}^{is_1+\infty}\frac{(3-2\tau)(3-\tau)^3(1-\tau)^3}{\tau(2-\tau)^9}e^{nf(\tau)} \\ \times \left(\frac{\sin \pi n\tau}{\pi}\right)^4\bigl(1 + O(n^{-1}) + O(e^{-2\pi n s_1})\bigr)\,\mathrm{d}\tau. \end{aligned}$$

Since

$$\begin{aligned} \left|\left(\frac{\sin \pi n\tau}{\pi}\right)^4 - \frac{e^{-4\pi in\tau}}{(2\pi)^4}\right| &= \left|\frac{e^{-4\pi in\tau}}{(2\pi)^4}\right| \cdot |-4e^{2\pi in\tau} + 6e^{4\pi in\tau} - 4e^{6\pi in\tau} + e^{8\pi in\tau}| \\ &< 15e^{-2\pi ns_1} \cdot \left|\frac{e^{-4\pi in\tau}}{(2\pi)^4}\right| \end{aligned}$$

for $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau = s_1 > 0$, we obtain

$$\begin{aligned} u_n = \frac{3(-1)^n}{4\pi^3 in^2}\int_{is_1-\infty}^{is_1+\infty}\frac{(3-2\tau)(3-\tau)^3(1-\tau)^3}{\tau(2-\tau)^9}e^{n(f(\tau)-4\pi i\tau)} \\ \times \bigl(1 + O(n^{-1}) + O(e^{-2\pi ns_1})\bigr)\,\mathrm{d}\tau. \end{aligned} \tag{64}$$

By (60) and the definition of the point τ_1 (that is the zero of the polynomial (54)), hence $f'(\tau_1) - 4\pi i\tau_1 = 0$, we conclude that $\tau = \tau_1$ is the unique maximim of the function $\operatorname{Re}(f(\tau) - 4\pi i\tau)$ on the line $\operatorname{Im}\tau = s_1$. Therefore, the saddle-point method says that the asymptotics of the integral in (64) is determined by the contribution of the point $\tau = \tau_1$ that yields the desired limit relation

$$\begin{aligned}\limsup_{n\to\infty}\frac{\log|u_n|}{n} &= \operatorname{Re} f(\tau_1) = \operatorname{Re}(f(\tau_1) - \tau_1 f'(\tau_1)) =: \operatorname{Re} f_0(\tau_1)\\ &= \log\frac{|3-\tau_1|^6|1-\tau_1|^6}{|2-\tau_1|^{12}} = 3\log(2\sqrt{3}+3) =: C_1.\end{aligned}$$

The proof of Theorem 4 is complete.

Remark. The limit relation (61) yields that $|F_n| \to 0$ as $n \to \infty$, and this is the fact that we have promised to prove for Theorem 1 (see the paragraph after Lemma 11). To be honest, the fact, that the asymptotics of the linear forms and their coefficients in the case (50) is determined by the zeros $(3 \pm 2\sqrt{3}\,)^3$ of a quadratic polynomial with integral coefficients, gives us an idea to look for a second-order difference equation. Thus, the recursion (3) has appeared after Theorem 4.

6. Group structure for $\zeta(4)$

Theorem 3 of Section 2 can be proved by specialization of Bailey's identity [Ba, Section 6.3, formula (2)]

$$\begin{aligned}&{}_7F_6\left(\begin{matrix} a, 1+\frac{1}{2}a, & b, & c, & d, & e, & f \\ & \frac{1}{2}a, 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f\end{matrix}\,\middle|\, 1\right)\\ &\quad= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d) \times\Gamma(1+a-c-d)\,\Gamma(1+a-e-f)}\\ &\quad\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(1+a-e-f+t) \times\Gamma(1+a-b-c-d-t)\,\Gamma(-t)}{\Gamma(1+a-e+t)\,\Gamma(1+a-f+t)}\,dt,\end{aligned} \tag{65}$$

provided that the series on the left-hand side converges. Namely, taking $a = 3n+2$ and $b = c = d = e = f = n+1$ and doubling both sides of (65) we obtain Ball's sequence (18) on the left and Apéry's sequence (9) on the right (for the last fact see [Ne1, Lemma 2]). Identity (65) can be put forward for an explanation how the permutation group from [RV2] for linear forms in 1 and $\zeta(3)$ appears (see [Zu5, Sections 4 and 5 for details]).

A story of number-theoretical application of Bailey's identities is not finished in [Zu5]. The following identity looks the most cool (in the sense of number of its parameters) relation among the known ones.

Lemma 14 (Bailey's integral transform [Ba, Section 6.8, formula (1)]). *There holds the identity*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+t)\,\Gamma(1+\frac12a+t)\,\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)}{\Gamma(\frac12a+t)\,\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)}\\
&\quad\times\frac{\Gamma(f+t)\,\Gamma(g+t)\,\Gamma(h+t)\,\Gamma(b-a-t)\,\Gamma(-t)}{\Gamma(1+a-f+t)\,\Gamma(1+a-g+t)\,\Gamma(1+a-h+t)}\,\mathrm{d}t\\
&\ =\frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f+b-a)\,\Gamma(g+b-a)\,\Gamma(h+b-a)}{\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)\,\Gamma(1+a-g-h)\times\Gamma(1+a-f-h)\,\Gamma(1+a-f-g)}\\
&\quad\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(k+t)\,\Gamma(1+\frac12k+t)\,\Gamma(b+t)\,\Gamma(k+c-a+t)\,\Gamma(k+d-a+t)}{\Gamma(\frac12k+t)\,\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)}\\
&\quad\times\frac{\Gamma(k+e-a+t)\,\Gamma(f+t)\,\Gamma(g+t)\,\Gamma(h+t)\,\Gamma(b-k-t)\,\Gamma(-t)}{\Gamma(1+k-f+t)\,\Gamma(1+k-g+t)\,\Gamma(1+k-h+t)}\,\mathrm{d}t,
\end{aligned}\tag{66}$$

where $k=1+2a-c-d-e$, and the parameters are connected by the relation

$$2+3a=b+c+d+e+f+g+h.$$

By Lemma 13 the transform (66) rearranges the parameters $\boldsymbol{h}$ as follows:

$$\begin{aligned}
\mathfrak{b}=\mathfrak{b}_{123}\colon \boldsymbol{h}\mapsto(&1+2h_0-h_1-h_2-h_3,h_{-1};1+h_0-h_2-h_3,1+h_0-h_1-h_3,\\
&1+h_0-h_1-h_2,h_4,h_5,h_6).
\end{aligned}\tag{67}$$

Consider the set of 27 complementary parameters $\boldsymbol{e}$,

$$\begin{gathered}
e_{jk}=h_0-h_j-h_k,\quad 1\leqslant j<k\leqslant 6,\qquad e_{0k}=h_k-1,\quad 1\leqslant k\leqslant 6,\\
\overline{e}_{0k}=h_{-1}-h_0+h_k-1=1+2h_0-(h_1+\dots+h_6)+h_k,\quad 1\leqslant k\leqslant 6,
\end{gathered}\tag{68}$$

and set

$$H(\boldsymbol{e}):=F(\boldsymbol{h}).$$

Then Bailey's transform can be written as follows:

$$H(\boldsymbol{e})=\frac{\Gamma(e_{01}+1)\,\Gamma(e_{02}+1)\,\Gamma(e_{12}+1)\,\Gamma(\overline{e}_{05}+1)}{\Gamma(e_{23}+1)\,\Gamma(e_{13}+1)\,\Gamma(e_{03}+1)\,\Gamma(e_{46}+1)}H(\mathfrak{b}\boldsymbol{e}),\tag{69}$$

where $\mathfrak{b}$ from (67) is the following second-order permutation of the parameters (68):

$$\mathfrak{b}=(e_{01}\ e_{23})(e_{02}\ e_{13})(e_{03}\ e_{12})(\overline{e}_{04}\ e_{56})(\overline{e}_{05}\ e_{46})(\overline{e}_{06}\ e_{45}).\tag{70}$$

We can also write the transform (69) in the form

$$\frac{H(\boldsymbol{e})}{\Pi_1(\boldsymbol{e})}=\frac{H(\mathfrak{b}\boldsymbol{e})}{\Pi_1(\mathfrak{b}\boldsymbol{e})},\qquad\text{where}\quad \Pi_1(\boldsymbol{e}):=e_{01}!\,e_{02}!\,e_{12}!\,\overline{e}_{05}!\,.\tag{71}$$

Further, the $\boldsymbol{h}$-trivial group (i.e., the group of permutations of the parameters $h_1, h_2, \dots, h_6$) is generated by second-order permutations of h_k, $1 \leqslant k \leqslant 5$, and h_6. The action of these five permutations on the set (68) is as follows:

$$\begin{aligned}
\mathfrak{h}_1 &= (h_1\ h_6) = (e_{01}\ e_{06})(\overline{e}_{01}\ \overline{e}_{06})(e_{12}\ e_{26})(e_{13}\ e_{36})(e_{14}\ e_{46})(e_{15}\ e_{56}),\\
\mathfrak{h}_2 &= (h_2\ h_6) = (e_{02}\ e_{06})(\overline{e}_{02}\ \overline{e}_{06})(e_{12}\ e_{16})(e_{23}\ e_{36})(e_{24}\ e_{46})(e_{25}\ e_{56}),\\
\mathfrak{h}_3 &= (h_3\ h_6) = (e_{03}\ e_{06})(\overline{e}_{03}\ \overline{e}_{06})(e_{13}\ e_{16})(e_{23}\ e_{26})(e_{34}\ e_{46})(e_{35}\ e_{56}),\\
\mathfrak{h}_4 &= (h_4\ h_6) = (e_{04}\ e_{06})(\overline{e}_{04}\ \overline{e}_{06})(e_{14}\ e_{16})(e_{24}\ e_{26})(e_{34}\ e_{36})(e_{45}\ e_{56}),\\
\mathfrak{h}_5 &= (h_5\ h_6) = (e_{05}\ e_{06})(\overline{e}_{05}\ \overline{e}_{06})(e_{15}\ e_{16})(e_{25}\ e_{26})(e_{35}\ e_{36})(e_{45}\ e_{46}),
\end{aligned} \tag{72}$$

and the quantity

$$\frac{e_{03}!\,\overline{e}_{04}!\,e_{05}!\,\overline{e}_{06}!}{e_{12}!\,e_{15}!\,e_{24}!\,e_{36}!} \cdot H(\boldsymbol{e}) \tag{73}$$

(due to the definition (43)) is stable under the action of (72). Setting

$$\mathcal{E} = \mathcal{E}(\boldsymbol{e}) := \{e_{01}, e_{02}, e_{04}, e_{06}, \overline{e}_{01}, \overline{e}_{02}, \overline{e}_{03}, \overline{e}_{05}, e_{12}, e_{15}, e_{24}, e_{36}\} \tag{74}$$

and combining the above stability results we arrive at the following fact.

Lemma 15. *The quantity*

$$\frac{H(\boldsymbol{e})}{\Pi(\boldsymbol{e})}, \qquad \textit{where} \quad \Pi(\boldsymbol{e}) := \prod_{e_{jk}\in\mathcal{E}} e_{jk}!,$$

is stable under the action of the group

$$\mathfrak{G} := \langle \mathfrak{b}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \rangle.$$

Moreover, the quantities h_{-1} and

$$\Sigma(\boldsymbol{e}) := \sum_{e_{jk}\in\mathcal{E}} e_{jk}$$

are also $\mathfrak{G}$-stable.

Proof. Routine calculations show the stability of $H(\boldsymbol{e})/\Pi(\boldsymbol{e})$ under the action of $\mathfrak{b}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5$ with a help of (71) and (73). Hence $H(\boldsymbol{e})/\Pi(\boldsymbol{e})$ is stable under the action of the $\boldsymbol{e}$-permutation group generated by these six permutations (70), (72).

The stability of h_{-1} under the action of (72) is obvious, and $\mathfrak{b}$ does not change the parameter h_{-1} by (67). Finally,

$$\Sigma(\boldsymbol{e}) = 12h_0 - 4(h_1 + h_2 + h_3 + h_4 + h_5 + h_6) = 4h_{-1} - 8$$

that yields the stability of $\Sigma(\boldsymbol{e})$ under the action of $\mathfrak{G}$. The proof is complete.

With the help of a `C++` program we have discovered that the group $\mathfrak{G}$ consists of 51840 elements, hence the left factor $\mathfrak{G}/\mathfrak{S}_6$ includes $51840/6! = 72$ left cosets; here $\mathfrak{S}_6$ is identified with the $\boldsymbol{h}$-trivial group $\langle \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \mathfrak{h}_4, \mathfrak{h}_5 \rangle$. It is interesting to mention that the group $\mathfrak{G}_0$ acting trivially on the set (74) consists of just 4 elements: $\mathfrak{g}_0 = \mathrm{id}$,

$$\begin{aligned}
\mathfrak{g}_1 &= (\mathfrak{h}_3\,\mathfrak{h}_1\,\mathfrak{h}_2\,\mathfrak{h}_5\,\mathfrak{b}\,\mathfrak{h}_1\,\mathfrak{h}_4\,\mathfrak{h}_5\,\mathfrak{b}\,\mathfrak{h}_1)^3 \\
&= (e_{01}\,\overline{e}_{02})(e_{02}\,\overline{e}_{01})(e_{03}\,\overline{e}_{06})(e_{04}\,\overline{e}_{05})(e_{05}\,\overline{e}_{04})(e_{06}\,\overline{e}_{03}) \\
&\qquad (e_{13}\,e_{26})(e_{14}\,e_{25})(e_{15}\,e_{24})(e_{16}\,e_{23})(e_{34}\,e_{56})(e_{35}\,e_{46}), \\
\mathfrak{g}_2 &= (\mathfrak{h}_1\,\mathfrak{h}_2\,\mathfrak{h}_4\,\mathfrak{h}_2\,\mathfrak{b}\,\mathfrak{h}_3\,\mathfrak{h}_5\,\mathfrak{h}_1\,\mathfrak{h}_2)^3 \\
&= (e_{01}\,e_{24})(e_{02}\,\overline{e}_{03})(e_{03}\,e_{46})(e_{04}\,\overline{e}_{05})(e_{05}\,e_{26})(e_{06}\,\overline{e}_{01}) \\
&\qquad (\overline{e}_{02}\,e_{15})(\overline{e}_{04}e_{13})(\overline{e}_{06}\,e_{35})(e_{12}\,e_{36})(e_{14}\,e_{56})(e_{25}\,e_{34}), \\
\mathfrak{g}_3 &= \mathfrak{h}_1\,\mathfrak{h}_2\,\mathfrak{b}\,\mathfrak{h}_3\,\mathfrak{h}_1\,\mathfrak{h}_5\,\mathfrak{h}_2\,\mathfrak{h}_3\,\mathfrak{b} = \mathfrak{g}_1\,\mathfrak{g}_2 \\
&= (e_{01}\,e_{15})(e_{02}\,e_{06})(e_{03}\,e_{35})(e_{05}\,e_{13})(\overline{e}_{01}\,\overline{e}_{03})(\overline{e}_{02}\,e_{24}) \\
&\qquad (\overline{e}_{04}\,e_{26})(\overline{e}_{06}\,e_{46})(e_{12}\,e_{36})(e_{14}\,e_{34})(e_{16}\,e_{23})(e_{25}\,e_{56}).
\end{aligned}$$

Remark. In the most symmetric case (50) all complementary parameters (68) are equal to n that means that any permutation from $\mathfrak{G}$ does not change the quantity $F(\boldsymbol{h})$. This fact explains why do we dub this case as 'most symmetric'.

7. Denominators of linear forms

As we have mentioned in Remark to Lemma 12, 'trivial' arithmetic (48) of the linear forms $H(\boldsymbol{e}) = F(\boldsymbol{h})$ does not lead us to a qualitative result for $\zeta(4)$. We are able to estimate the irrationality measure of $\zeta(4)$ under the following condition, which we have checked numerically for several values of $\boldsymbol{h}$ satisfying (41) and (42).

Denominator conjecture. *There holds the inclusion*[5]

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4}\cdot\Phi^{-1}(\boldsymbol{e})\cdot H(\boldsymbol{e}) \in \mathbb{Z}\zeta(4)+\mathbb{Z},$$

where $m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4$ *are the four successive maxima of the set* $\boldsymbol{e}$ *in* (68) *and*

$$\Phi(\boldsymbol{e}) := \prod_{p>\sqrt{h_{-1}}} p^{\nu_p}$$

with

$$\nu_p := \left\lfloor \frac{1}{2}\left\lfloor \frac{1}{4}\sum_{e_{jk}\in\mathcal{E}} \frac{e_{jk}}{p}\right\rfloor - \frac{1}{8}\sum_{e_{jk}\in\mathcal{E}}\left\lfloor\frac{e_{jk}}{p}\right\rfloor\right\rfloor = \left\lfloor \frac{1}{2}\left\lfloor \frac{h_{-1}-2}{p}\right\rfloor - \frac{1}{8}\sum_{e_{jk}\in\mathcal{E}}\left\lfloor\frac{e_{jk}}{p}\right\rfloor\right\rfloor.$$

[5]In the most symmetric case (50) this conjecture reduces to the conjecture (40) of Section 3.

If this conjecture is true, then taking any element $\mathfrak{g}\in\mathfrak{G}$ and writing conclusion of Lemma 15 as

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4}H(\boldsymbol{e})=D_{m_1}D_{m_2}D_{m_3}D_{m_4}\Phi^{-1}(\mathfrak{g}\boldsymbol{e})H(\mathfrak{g}\boldsymbol{e})\cdot\frac{\Pi(\boldsymbol{e})\Phi(\mathfrak{g}\boldsymbol{e})}{\Pi(\mathfrak{g}\boldsymbol{e})}$$

we deduce that, for any prime $p>\sqrt{h_{-1}}$,

$$\begin{aligned}\operatorname{ord}_p\bigl(D_{m_1}D_{m_2}D_{m_3}D_{m_4}H(\boldsymbol{e})\bigr)&\geqslant\operatorname{ord}_p\frac{\Pi(\boldsymbol{e})\Phi(\mathfrak{g}\boldsymbol{e})}{\Pi(\mathfrak{g}\boldsymbol{e})}\\&=\sum_{e_{jk}\in\mathcal{E}}\left\lfloor\frac{e_{jk}}{p}\right\rfloor-\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor+\left\lfloor\frac{1}{2}\left\lfloor\frac{h_{-1}-2}{p}\right\rfloor-\frac{1}{8}\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor\right\rfloor,\end{aligned}\tag{75}$$

where $\mathfrak{g}\mathcal{E}=\mathcal{E}(\mathfrak{g}\boldsymbol{e})$ and $\operatorname{ord}_p(u\zeta(4)-v):=\min\{\operatorname{ord}_p u,\operatorname{ord}_p v\}$ for rational numbers u,v. Finally, setting

$$\Lambda(\boldsymbol{e})=\prod_{p>\sqrt{h_{-1}}}p^{\lambda_p}$$

with

$$\lambda_p:=\max_{\mathfrak{g}\in\mathfrak{G}}\left(\sum_{e_{jk}\in\mathcal{E}}\left\lfloor\frac{e_{jk}}{p}\right\rfloor-\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor+\left\lfloor\frac{1}{2}\left\lfloor\frac{h_{-1}-2}{p}\right\rfloor-\frac{1}{8}\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\left\lfloor\frac{e'_{jk}}{p}\right\rfloor\right\rfloor\right),$$

from (75) we obtain the inclusion

$$D_{m_1}D_{m_2}D_{m_3}D_{m_4}\cdot\Lambda^{-1}(\boldsymbol{e})\cdot H(\boldsymbol{e})\in\mathbb{Z}\zeta(4)+\mathbb{Z}.\tag{76}$$

Now, to each $n=0,1,2,\dots$ assign the parameters $\boldsymbol{h}$ in accordance with (53) and set

$$\begin{gathered}e_{jk}=\eta_0-\eta_j-\eta_k,\quad 1\leqslant j<k\leqslant 6,\qquad e_{0k}=\eta_k,\quad 1\leqslant k\leqslant 6,\\\overline{e}_{0k}=\eta_{-1}-\eta_0+\eta_k=2\eta_0-(\eta_1+\dots+\eta_6)+\eta_k,\quad 1\leqslant k\leqslant 6,\end{gathered}$$

so that the set of complementary parameters $\boldsymbol{e}\cdot n$ corresponds to the set $\boldsymbol{h}$. Then, in the above notation, we can write the inclusion (76) as

$$D_{m_1n}D_{m_2n}D_{m_3n}D_{m_4n}\cdot\Lambda^{-1}(\boldsymbol{e}n)\cdot H(\boldsymbol{e}n)\in\mathbb{Z}\zeta(4)+\mathbb{Z}.$$

The asymptotic behaviour of the linear forms $H(\boldsymbol{e}n)\in\mathbb{Q}\zeta(4)+\mathbb{Q}$ and their coefficients as $n\to\infty$ is determined by Theorem 4; in addition,

$$\lim_{n\to\infty}\frac{\log(D_{m_1n}D_{m_2n}D_{m_3n}D_{m_4n})}{n}=m_1+m_2+m_3+m_4$$

by the consequence (2) of the prime number theorem, while the Chudnovsky–Rukhadze–Hata arithmetic lemma (see, e.g., [Zu3, Lemma 4.4]) yields

$$\lim_{n\to\infty}\frac{\log\Lambda(\boldsymbol{e}n)}{n}=\int_0^1\lambda(x)\,\mathrm{d}\psi(x),$$

where

$$\lambda(x) := \max_{\mathfrak{g}\in\mathfrak{G}}\biggl(\sum_{e_{jk}\in\mathcal{E}}\lfloor e_{jk}x\rfloor - \sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\lfloor e'_{jk}x\rfloor + \biggl\lfloor\frac{1}{2}\lfloor\eta_{-1}x\rfloor - \frac{1}{8}\sum_{e'_{jk}\in\mathfrak{g}\mathcal{E}}\lfloor e'_{jk}x\rfloor\biggr\rfloor\biggr)$$

is the 1-periodic non-negative integral-valued function.

Recalling the notation of Theorem 4 and combining its results with saying above, as in [RV2, the proof of Theorem 5.1], we arrive at the following statement.

Theorem 5. *Under the denominator conjecture, let*

$$C_0 = -f_0(\tau_0), \qquad C_1 = \operatorname{Re} f_0(\tau_1),$$
$$C_2 = m_1 + m_2 + m_3 + m_4 - \int_0^1 \lambda(x)\,\mathrm{d}\psi(x).$$

If $C_0 > C_2$, then the irrationality exponent of $\zeta(4)$ satisfies the estimate

$$\mu(\zeta(4)) \leqslant \frac{C_0 + C_1}{C_0 - C_2}.$$

Recall that the *irrationality exponent* $\mu = \mu(\alpha)$ of a real irrational number α is the least possible exponent such that for any $\varepsilon > 0$ the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{q^{\mu+\varepsilon}}$$

has only finitely many solutions in integers p, q with $q > 0$.

With a help of Theorem 5 we are able to state the following conditional result.

Theorem 6. *The irrationality exponent of $\zeta(4)$ satisfies the estimate*

$$\mu(\zeta(4)) \leqslant 25.38983113\ldots \tag{77}$$

provided that the denominator conjecture holds.

Proof. Taking $\boldsymbol{\eta} = (68, 57; 22, 23, 24, 25, 26, 27)$ we obtain

$$\tau_0 = 11.83684636\ldots, \qquad C_0 = -f_0(\tau_0) = 37.85606933\ldots,$$
$$\tau_1 = 34 + i6.34312459\ldots, \qquad C_1 = \operatorname{Re} f_0(\tau_1) = 104.96178579\ldots,$$

and

$$C_2 = m_1 + m_2 + m_3 + m_4 - \int_0^1 \lambda(x)\,\mathrm{d}\psi(x)$$
$$= 27 + 26 + 25 + 24 - 69.76893283\ldots = 32.23106716\ldots.$$

Thus, application of Theorem 5 yields the desired estimate (77).

The estimate (77) can be compared with the 'best known' estimate

$$\mu(\zeta(4)) \leqslant 204.94259587\ldots,$$

which follows from the general result of Yu. Aleksentsev [Al] on approximations of π by algebraic numbers.[6]

8. Further zeta values

A natural very-well-poised generalization of Ball's sequence (18),

$$F_{k,n} := n!^{k-1}\sum_{t=1}^{\infty}(2t+n)\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{t^{k+1}(t+1)^{k+1}\cdots(t+n)^{k+1}}(-1)^{(k-1)(t+n+1)}$$
$$\in\begin{cases}\mathbb{Q}\zeta(k)+\mathbb{Q}\zeta(k-2)+\cdots+\mathbb{Q}\zeta(2)+\mathbb{Q} & \text{for } k\geqslant 2 \text{ even},\\ \mathbb{Q}\zeta(k)+\mathbb{Q}\zeta(k-2)+\cdots+\mathbb{Q}\zeta(3)+\mathbb{Q} & \text{for } k\geqslant 2 \text{ odd},\end{cases}\qquad n=1,2,\ldots,$$

gives rise for searching difference equations satisfied by both linear forms $F_{k,n}$ and their rational coefficients. Applying Zeilberger's algorithm of creative telescoping in the manner of Sections 2 and 3 we deduce the following result for the linear forms

$$F_{5,n}=u_n\zeta(5)+w_n\zeta(3)-v_n. \tag{78}$$

Theorem 7. *The numbers u_n, w_n, v_n in the representation* (78) *are positive rationals satisfying the third-order difference equation*

$$(n+1)(n+2)^5b_0(n)u_{n+2}-b_1(n)u_{n+1}-b_2(n)u_n+2(2n+1)n^5b_0(n+1)u_{n-1}=0, \tag{79}$$
$$u_0=2,\quad w_0=0,\quad v_0=0,\qquad u_1=18,\quad w_1=66,\quad v_1=98,$$
$$u_2=938,\quad w_2=\frac{6125}{2},\quad v_2=\frac{74463}{16},$$

where

$$\begin{aligned}
b_0(n)&=41218n^3+48459n^2+20010n+2871,\\
b_1(n)&=2(n+1)(3874492n^8+33613836n^7+123666762n^6+250134420n^5\\
&\qquad+301587620n^4+220011738n^3+94372815n^2+21917736n+2131500),\\
b_2(n)&=2(48802112n^9+350188128n^8+1080631646n^7+1882848690n^6\\
&\qquad+2045758212n^5+1442754107n^4+663248761n^3+192486369n^2\\
&\qquad+32136756n+2360484).
\end{aligned}$$

[6] In fact, the result of [Al] is proved for approximations of π by algebraic numbers of sufficiently large degree. Yu. Nesterenko has noticed to us in private communication that this hard restriction can be omitted if one makes an accurate analytic calculation for the construction considered in [Al].

The characteristic polynomial $\lambda^3-188\lambda^2-2368\lambda+4$ of the difference equation (79) determines the asymptotic behaviour of the linear forms (78) and their coefficients as $n\to\infty$.

A similar (but quite cumbersome) fourth-order recursion with characteristic polynomial $\lambda^4-828\lambda^3-132246\lambda^2+260604\lambda-27$ has been discovered by us for the linear forms $F_{7,n}$ and their coefficients. These recursions allow us to verify the inclusions

$$D_n^5F_{5,n}\in\mathbb{Z}\zeta(5)+\mathbb{Z}\zeta(3)+\mathbb{Z},\qquad D_n^7F_{7,n}\in\mathbb{Z}\zeta(7)+\mathbb{Z}\zeta(5)+\mathbb{Z}\zeta(3)+\mathbb{Z}$$

up to $n=1000$, although we are able to prove them with just D_n^6 and D_n^8, respectively. In [Zu5, Section 9] we conjecture the equality $F_{k,n}=J_{k,n}$ for integral $k\geqslant 2$, where

$$J_{k,n}:=\int\cdots\int_{[0,1]^k}\frac{x_1^n(1-x_1)^nx_2^n(1-x_2)^n\cdots x_k^n(1-x_k)^n\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_k}{(1-(1-(\cdots(1-(1-x_1)x_2)\cdots)x_{k-1})x_k)^{n+1}}\tag{80}$$

are multiple integrals introduced by O. Vasilenko [VaO]. The integrals $J_{2,n}$ and $J_{3,n}$ has been studied by F. Beukers [Be1] in the connection with Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$. In [Zu5], we prove the coincidence of $F_{3,n}$ and $J_{3,n}$ with the help of Bailey's identity (65) and Nesterenko's integral theorem [Ne2, Theorem 2], and use similar arguments for showing that $F_{2,n}=J_{2,n}$. The cases $k=4$ and $k=5$ in (80) has been developed by D. Vasilyev [VaD]; he conjectures the inclusions

$$D_n^kJ_{k,n}\in\mathbb{Z}\zeta(k)+\mathbb{Z}\zeta(k-2)+\cdots+\mathbb{Z}\zeta(3)+\mathbb{Z}\quad\text{for } k \text{ odd},$$

and proves them if $k=5$.

There is a regular way for obtaining difference equations for the quantities (80); it is a part of general *WZ-theory* developed by H. Wilf and D. Zeilberger [WZ]. The fundamental theorem of this WZ-theory [WZ, Section 2] says that there exist effectively computable order l, polynomials $a_0(n),a_1(n),\dots,a_l(n)$ in n, and rational functions (certificates) $s_{1,n}(\boldsymbol{x}),\dots,s_{k,n}(\boldsymbol{x})$ in n and $\boldsymbol{x}=(x_1,\dots,x_k)$ such that

$$\begin{aligned}a_l(n)Q_{n+l-1}(\boldsymbol{x})&+a_{l-1}(n)Q_{n+l-2}(\boldsymbol{x})+\cdots+a_1(n)Q_n(\boldsymbol{x})+a_0(n)Q_{n-1}(\boldsymbol{x})\\&=\sum_{j=1}^k\frac{\partial}{\partial x_j}\big(s_{j,n}(\boldsymbol{x})Q_n(\boldsymbol{x})\big),\end{aligned}\tag{81}$$

where

$$Q_n(\boldsymbol{x}):=\frac{x_1^n(1-x_1)^nx_2^n(1-x_2)^n\cdots x_{k-1}^n(1-x_{k-1})^nx_k^n(1-x_k)^n}{(1-(1-(\cdots(1-(1-x_1)x_2)\cdots)x_{k-1})x_k)^{n+1}}$$

is the integrand in (80). For each fixed $k=2,3,\dots,$ discovering the polynomials $a_0(n),a_1(n),\dots,a_l(n)$ and the certificates is a 'routine matter'; then division of both sides of identity (81) by $Q_n(\boldsymbol{x})$ reduces to a 'one-line' verification as in Lemmas 2, 5, and 8. Such division also shows that the certificate $s_{j,n}(\boldsymbol{x})$ has no poles at $x_j = 0$ and $x_j = 1$, hence

$$\begin{aligned}
&\int\cdots\int_{[0,1]^k} \frac{\partial}{\partial x_j}\big(s_{j,n}(\boldsymbol{x})Q_n(\boldsymbol{x})\big)\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\
&\quad = \int\cdots\int_{[0,1]^{k-1}} \biggl(\frac{s_{j,n}(\boldsymbol{x})x_j^n(1-x_j)^n}{(1-(\cdots(1-x_1)\cdots)x_k)^{n+1}}\biggr)\bigg|_{x_j=0}^{x_j=1} \cdot \prod_{\substack{m=1\\ m\neq j}}^{k} x_m^n(1-x_m)^n\,\mathrm{d}x_m \\
&\quad = 0, \qquad j=1,\dots,k,
\end{aligned}$$

for $n \geqslant 1$, and after integration of equality (81) over the cube $[0,1]^k$ with respect to $\boldsymbol{x}$ we arrive at the desired difference equation

$$a_l(n)J_{k,n+l-1} + a_{l-1}(n)J_{k,n+l-2} + \dots + a_1(n)J_{k,n} + a_0(n)J_{k,n-1} = 0 \tag{82}$$

for $n = 1, 2, \dots$. In these terms we can extend our previous conjecture as follows:

Recurrence conjecture. *For each $k = 2, 3, 4, \dots$, the quantities $F_{k,n}$ and $J_{k,n}$ satisfy the same difference equation of order $l = \lfloor k/2 \rfloor + 1$ with polynomial coefficients in n and the same initial data $F_{k,n} = J_{k,n}$ for $n = 0, 1, \dots, \lfloor k/2 \rfloor$.*

The validity of this conjecture will prove (or disprove) our conjecture $F_{k,n} = J_{k,n}$, at least for small values of $k \geqslant 4$.

Another story deals with the quantities

$$\begin{aligned}
\widetilde{F}_n &:= \frac{1}{2}\sum_{t=1}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\biggl((2t+n)\biggl(\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^2}\biggr)^3\biggr) \\
&= \widetilde{u}_n\zeta(7) + \widetilde{w}_n\zeta(5) - \widetilde{v}_n,
\end{aligned}$$

where $\widetilde{u}_n, \widetilde{w}_n, \widetilde{v}_n$ are positive rationals. We have discovered a (quite cumbersome) fourth-order difference equation satisfied by $\widetilde{u}_n, \widetilde{w}_n, \widetilde{v}_n$; its characteristic polynomial is

$$\lambda^4 + 9264\lambda^3 - 12116166\lambda^2 - 752300\lambda - 19683 \qquad (19683 = 3^9).$$

As we have proved in [Zu3, Proposition 4.1], the following inclusions hold:

$$D_n^8 \cdot \widetilde{\Phi}_n^{-3} \cdot \widetilde{F}_n \in \mathbb{Z}\zeta(7) + \mathbb{Z}\zeta(5) + \mathbb{Z}, \qquad \text{where} \quad \widetilde{\Phi}_n := \prod_{\substack{p<n\\ \{n/p\}\in[\frac{2}{3},1)}} p,$$

while our calculations up to $n = 1000$ with a help of the recursion mentioned above show that

$$D_n^7 \cdot \widetilde{\Phi}_n^{-2} \cdot \widetilde{F}_n \in \mathbb{Z}\zeta(7) + \mathbb{Z}\zeta(5) + \mathbb{Z}.$$

What is a trick that makes arithmetic as it is?

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References

[Al] Yu. M. Aleksentsev, *On the measure of approximation for the number π by algebraic numbers*, Mat. Zametki [Math. Notes] **66** (1999), no. 4, 483–493.

[Ap] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61** (1979), 11–13.

[Ba] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Math. Tracts, vol. 32, Cambridge Univ. Press, Cambridge, 1935; 2nd reprinted edition, Stechert-Hafner, New York–London, 1964.

[BR] K. Ball and T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146** (2001), no. 1, 193–207.

[Be1] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$*, Bull. London Math. Soc. **11** (1979), no. 3, 268–272.

[Be2] F. Beukers, *Padé approximations in number theory*, Lecture Notes in Math., vol. 888, Springer-Verlag, Berlin, 1981, pp. 90–99.

[Be3] F. Beukers, *Irrationality proofs using modular forms*, Astérisque **147–148** (1987), 271–283.

[Br] N. G. de Bruijn, *Asymptotic methods in analysis*, North-Holland Publ., Amsterdam, 1958.

[Gu] L. A. Gutnik, *On the irrationality of certain quantities involving $\zeta(3)$*, Uspekhi Mat. Nauk [Russian Math. Surveys] **34** (1979), no. 3, 190; Acta Arith. **42** (1983), no. 3, 255–264.

[Han] J. Hancl, *A simple proof of the irrationality of π^4*, Amer. Math. Monthly **93** (1986), 374–375.

[Hat] M. Hata, *Legendre type polynomials and irrationality measures*, J. Reine Angew. Math. **407** (1990), no. 1, 99–125.

[JT] W. B. Jones and W. J. Thron, *Continued fractions. Analytic theory and applications*, Encyclopaedia Math. Appl. Section: Analysis, vol. 11, Addison-Wesley, London, 1980.

[Ne1] Yu. V. Nesterenko, *A few remarks on $\zeta(3)$*, Mat. Zametki [Math. Notes] **59** (1996), no. 6, 865–880.

[Ne2] Yu. V. Nesterenko, *Integral identities and constructions of approximations to zeta values*, Actes des 12èmes rencontres arithmétiques de Caen (June 29–30, 2001), J. Théorie Nombres Bordeaux, this volume (2002).

[Ne3] Yu. V. Nesterenko, *Arithmetic properties of values of the Riemann zeta function and generalized hypergeometric functions* (2002), in preparation.

[PWZ] M. Petkovšek, H. S. Wilf, and D. Zeilberger, $A = B$, A. K. Peters, Ltd., Wellesley, M.A., 1997.

[Po] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$*, An informal report, Math. Intelligencer **1** (1978/79), no. 4, 195–203.

[RV1] G. Rhin and C. Viola, *On a permutation group related to $\zeta(2)$*, Acta Arith. **77** (1996), no. 1, 23–56.

[RV2] G. Rhin and C. Viola, *The group structure for $\zeta(3)$*, Acta Arith. **97** (2001), no. 3, 269–293.

[Ri1] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 4, 267–270; E-print `math.NT/0008051`.

[Ri2] T. Rivoal, *Propriétés diophantinnes des valeurs de la fonction zêta de Riemann aux entiers impairs*, Thèse de Doctorat, Univ. de Caen, Caen, 2001.

[Ri3] T. Rivoal, *Séries hypergéométriques et irrationalité des valeurs de la fonction zêta*, Journées arithmétiques (Lille, July, 2001), 2002 (to appear).

[So1] V. N. Sorokin, *Hermite–Padé approximations for Nikishin's systems and irrationality of* $\zeta(3)$, Uspekhi Mat. Nauk [Russian Math. Surveys] **49** (1994), no. 2, 167–168.

[So2] V. N. Sorokin, *Apéry's theorem*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] (1998), no. 3, 48–52.

[So3] V. N. Sorokin, *One algorithm for fast calculation of* π^4 (2002), in preparation.

[VaO] O. N. Vasilenko, *Certain formulae for values of the Riemann zeta-function at integral points*, Number theory and its applications, Proceedings of the science-theoretical conference (Tashkent, September 26–28, 1990), Tashkent, 1990. (Russian)

[VaD] D. V. Vasilyev, *On small linear forms for the values of the Riemann zeta-function at odd points*, Preprint no. 1 (558), Nat. Acad. Sci. Belarus, Institute Math., Minsk, 2001.

[WZ] H. S. Wilf and D. Zeilberger, *An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities*, Invent. Math. **108** (1992), no. 3, 575–633.

[Za] D. Zagier, *Integral solutions of Apéry-like recurrence equations*, Preprint, 2001.

[Ze] D. Zeilberger, *Closed form (pun intended!)*, A tribute to Emil Crosswald "Number theory and related analysis", Contemporary Math. (M. Knopp and M. Sheingorn, eds.), vol. 143, Amer. Math. Soc., Providence, R.I., 1993, pp. 579–607.

[Zu1] W. V. Zudilin, *Irrationality of values of zeta function at odd integers*, Uspekhi Mat. Nauk [Russian Math. Surveys] **56** (2001), no. 2, 215–216.

[Zu2] W. Zudilin, *Irrationality of values of zeta-function*, Contemporary Research in Mathematics and Mechanics, Proceedings of the XXIII Conference of Young Scientists of the Department of Mechanics and Mathematics (Moscow State University, April 9–14, 2001), Publ. Dept. Mech. Math. MSU, Moscow, 2001, Part 2, pp. 127–135; English transl., E-print `math.NT/0104249`.

[Zu3] W. Zudilin, *Irrationality of values of the Riemann zeta function*, Izv. Ross. Akad. Nauk Ser. Mat. [Russian Acad. Sci. Izv. Math.] **66** (2002), no. 3 (to appear); available at `http://wain.mi.ras.ru/PS/zeta_main.ps.gz`.

[Zu4] W. V. Zudilin, *One of the numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational*, Uspekhi Mat. Nauk [Russian Math. Surveys] **56** (2001), no. 4, 149–150.

[Zu5] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, Invent. Math., submitted for publication.

[Zu6] W. Zudilin, *Difference equations and the irrationality measure of numbers*, Analytic number theory and applications, Collection of papers, Trudy Mat. Inst. Steklov [Proc. Steklov Inst. Math.] **218** (1997), 165–178.

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