ON THE THEORY OF GAMES OF STRATEGY[1]

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INTRODUCTION

1. The present paper is concerned with the following question:

> n players $S_1, S_2, \ldots, S_n$ are playing a given game of strategy, $\mathfrak{G}$. How must one of the participants, S_m, play in order to achieve a most advantageous result?

The problem is well known, and there is hardly a situation in daily life into which this problem does not enter. Yet, the meaning of this question is not unambiguous. For, as soon as $n > 1$ (i.e., $\mathfrak{G}$ is a game of strategy in the proper sense), the fate of each player depends not only on his own actions but also on those of the others, and their behavior is motivated by the same selfish interests as the behavior of the first player. We feel that the situation is inherently circular.

Hence we must first endeavor to find a clear formulation of the question. What, exactly, is a game of strategy? A great many different things come under this heading, anything from roulette to chess, from baccarat to bridge. And after all, any event - given the external conditions and the participants in the situation (provided the latter are acting of their own free will) - may be regarded as a game of strategy if one looks at the effect it has on the participants.[2] What element do all these things have in common?

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[1] A shortened version of this paper has been presented to the Goettingen Mathematical Society on December 7, 1926.

[2] This is the principal problem of classical economics: how is the absolutely selfish "homo economicus" going to act under given external circumstances?

We may assume that it is the following:

A game of strategy consists of a certain series of events each of which may have a finite number of distinct results. In some cases, the outcome depends on chance, i.e., the probabilities with which each of the possible results will occur are known, but nobody can influence them. All other events depend on the free decision of the players S_1, $S_2, \ldots, S_n$. In other words, for each of these events it is known which player, S_m, determines its outcome and what is his state of information with respect to the results of other ("earlier") events at the time when he makes his decision. Eventually, after the outcome of all events is known, one can calculate according to a fixed rule what payments the players $S_1, S_2, \ldots, S_n$ must make to each other.

It is easy to bring this somewhat qualitative explanation into a precise form. The definition of a game of strategy would then be the following:

For a complete description of a game $\mathfrak{G}$, the following data are necessary, which in their entirety are the "rules of the game."

α) The number of events or "draws" depending on chance, and the number of events or "steps" depending on the free decision of the individual players must be specified. Let these numbers be z and s respectively, and let us denote the "draws" by $E_1, E_2, \ldots, E_z$, the "steps" by $F_1, F_2, \ldots, F_s$.

β) The number of possible results of each "draw", E_μ, and of each "step", F_ν, must be specified. Let these numbers be M_μ and N_ν respectively. ($\mu = 1, 2, \ldots, z$, $\nu = 1, 2, \ldots, s$.) We shall denote the results, for short, by their numbers $1, 2, \ldots, M_\mu$ and $1, 2, \ldots, N_\nu$ respectively.

γ) For each "draw" E_μ the probabilities $\alpha_\mu^{(1)}$, $\alpha_\mu^{(2)}, \ldots, \alpha_\mu^{(M_\mu)}$ of the different results $1, 2, \ldots, M_\mu$ must be given. Obviously, we have

$$\alpha_\mu^{(1)} \geq 0,\ \alpha_\mu^{(2)} \geq 0,\ \ldots,\ \alpha_\mu^{(M_\mu)} \geq 0$$

$$\alpha_\mu^{(1)} + \alpha_\mu^{(2)} + \ldots + \alpha_\mu^{(M_\mu)} = 1\ .$$

δ) For every "step" F_ν, the player S_m who determines the outcome of this "step" ("whose step" F_ν is) must be specified. In addition, the numbers of all "draws and steps" must be specified of whose outcome the player is informed at the time he makes his decision concerning F_ν. (These "draws" and "steps" we shall call "earlier" than F_ν.)

In order that this whole scheme be consistent and permit of a temporal-causal interpretation, there must be no cycles $F_{\nu_1}, F_{\nu_2}, \ldots, F_{\nu_p}, F_{\nu_{p+1}} = F_{\nu_1}$, and F_{ν_q} must always be "earlier" than $F_{\nu_{q+1}}$ $(q = 1, 2, \ldots, p)$.

ε) Finally, n functions $f_1, f_2, \ldots, f_n$ must be given. Each of them depends on $z + s$ variables which take on the values

$$1, 2, \ldots, M_1; \quad 1, 2, \ldots, M_2; \quad \ldots; \quad 1, 2, \ldots, M_z;$$
$$1, 2, \ldots, N_1; \quad 1, 2, \ldots, N_2; \quad \ldots; \quad 1, 2, \ldots, N_s$$

respectively. These functions are real-valued and

$$f_1 + f_2 + \ldots + f_n \equiv 0$$

holds identically. If in the course of a play which has been completed the results of the z "draws" and s "steps" were $x_1, x_2, \ldots, x_z, y_1, y_2, \ldots, y_s$ respectively, $(x_\mu = 1, 2, \ldots, M_\mu, y_\nu = 1, 2, \ldots, N_\nu;$ $\mu = 1, 2, \ldots, z, \nu = 1, 2, \ldots, s)$, the players S_1, $S_2, \ldots, S_n$ obtain from each other[3] the amounts

$$f_1(x_1, \ldots, x_z, y_1, \ldots, y_s), f_2(x_1, \ldots, x_z, y_1, \ldots, y_s), \ldots, f_n(x_1, \ldots, x_z, y_1, \ldots, y_s).$$

(On closer inspection it can be seen that in spite of this somewhat lengthy description we are dealing here with quite simple matters. Actually, in several respects our definition might have been somewhat more

3 The identity

$$f_1 + f_2 + \ldots + f_n \equiv 0$$

expresses that the players make payments to each other only, but collectively they neither gain nor lose.

general. We could, e.g., have included the case that the M_μ, N_ν and $\alpha_1^{(\mu)}, \alpha_2^{(\mu)}, \ldots, \alpha_{M_\mu}^{(\mu)}$ depend on the results of the "earlier" "draws" and "steps", and the like. It is easy to see, however, that such generalizations would be inessential.)

2. With this definition the concept of a game of strategy is precisely defined. But it also becomes clear that, as we indicated at the beginning of 1., the expression "S_m tries to achieve a result as advantageous as possible" is rather obscure. What constitutes the most advantageous result for the player S_m is obviously the largest possible value of f_m, but how can any value of f_m be "achieved" by S_m? By himself, S_m is in no position to fix the value of f_m! The value of f_m depends on the variables $x_1, \ldots, x_z, y_1, \ldots, y_s$, only part of which are determined by S_m's decision (viz. those y_ν for which S_m has the "step" F_ν, i.e., those for which $S_{(F_\nu)} = S_m$). All other variables y_ν depend on the decisions of the participants and all variables x_μ depend on chance.

In our case, the "unforeseeable" chance event is actually the factor which it is easiest to deal with. In fact, assume that a particular f_m depends only on those y_ν which are decided on by S_m ($S_{(F_\nu)} = S_m$), and in addition on the x_μ (which are determined by chance). In that case, S_m can at least anticipate this much: If I make certain moves, I can expect such and such results (i.e., values of f_m) with such and such probabilities (since the probabilities $\alpha_1^{(\mu)}, \alpha_2^{(\mu)m}, \ldots, \alpha_{M_\mu}^{(\mu)}$ are given) - regardless of how the other players act! If we now assume that "a most advantageous result" is the highest possible value of the expectation (and this assumption or a similar one has to be made in order to apply the methods of the theory of probability)[4] we have, in principle, solved our problem. For, we have here a simple maximum problem: The values of those variables y_ν which S_m has to determine must be so chosen by him that the expected value of f_m (which depends only on these variables y_ν) becomes as large as possible.

It is this type of game which in the theory of probability is treated in the so-called "theory of games of chance." A typical example is roulette: Let the number of players be $k + 1$ ($S_1, \ldots, S_k$ are the "pointers", S_{k+1} is the "banker"), S_{k+1} has no influence whatsoever on the game,[5] and the result achieved by S_ℓ, f_ℓ, ($\ell = 1, 2, \ldots, k$) only

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[4] We shall not enter on the well-known objections to the use of the expected value (and the ensuing attempts to replace the latter by the so-called moral expectation or similar concepts). The difficulties that form the subject of our considerations are of a different nature.

[5] Anyway, he has no need to, since according to the rules of the game his gain after each play is 2.70% of the turn-over.

depends on chance and his own actions.[6]

The name alone, "game of chance", indicates that the main emphasis is put on the variables x_μ, which are dependent on chance, and not on the variables y_ν, which are subject to the decisions of the players. But this is exactly what we are interested in. We shall try to investigate the effects which the players have on each other, the consequences of the fact (so typical of all social happenings!) that each player influences the results of all other players, even though he is only interested in his own.

§1. GENERAL SIMPLIFICATIONS

1. The definition of a game of strategy given in the Introduction is rather complicated, which may appear justified in view of the fact that games of strategy may be arbitrarily complex. Nevertheless it is possible to bring all games falling under this definition into a much simpler normal form, in a way, into the simplest form that is at all conceivable. We contend that it is sufficient to consider games of the following kind:

$z = 1$ (i.e., only one "draw" takes place).

$s = n$, the ν-th "step" being that of the player S_ν $(S_{(F_\nu)} = S_\nu)$.

The relation "earlier" is never realized (i.e., each player must make his dispositions without knowing anything about the other participants or about the "draw").

The play thus takes the following course: Each player S_m $(m = 1, 2, \ldots, n)$ chooses a number $1, 2, \ldots, N_m$ without knowing the choices of the others. Now a "draw" takes place in which the numbers $1, 2, \ldots, M$ will appear with the probabilities $\alpha_1, \alpha_2, \ldots, \alpha_M$. The results achieved by the players are (if the "draw" and the n "steps" have resulted in $x, y_1, y_2, \ldots, y_n$)

$$f_1(x, y_1, \ldots, y_n), f_2(x, y_1, \ldots, y_n), \ldots, f_n(x, y_1, \ldots, y_n) .$$

This apparently far-reaching restriction is actually not essential, for the following reason:

Let the "steps" of the players S_m $(S_{(F_\nu)} = S_m)$ be those with the numbers $\nu_1^{(m)}, \nu_2^{(m)}, \ldots, \nu_{\sigma_m}^{(m)}$. Obviously, it is inadmissible to make the assumption that S_m might be able to tell, before the start of the

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[6] As may be guessed, on the basis of the preceding footnote, the unambiguous result in this case for the behavior of the pointers is rather trivial: if possible they should have the turn-over zero; the closer they approximate it, so much the better!

game, what his choices for all these steps are going to be. It would mean a restriction of his free will and change his chances (for the worse). For, S_m's decision in each of these "steps" will generally be significantly influenced by the results of the "draws" and "steps" known to him at the moment of his decision.

On the other hand, it may well be assumed that before the play has started he knows how to answer the following question: What will be the outcome of the $\nu_k^{(m)}$-th "step" ($k = 1, 2, \ldots, \sigma_m$) provided the results of all "draws" and "steps" "earlier" than $\nu_k^{(m)}$ are available? In other words, the player knows beforehand how he is going to act in a precisely defined situation: he enters the play with a theory worked out in detail. Even if this may not be the case for a particular player, it is clear that such an assumption will certainly not spoil his chances.

Accordingly, we define the "strategy" of a player S_m as follows:

> In order to describe completely the "strategy" of a player S_m ($m = 1, 2, \ldots, n$) the following specifications are necessary:
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> As before, let S_m have the "steps" with the numbers $\nu_1^{(m)}, \nu_2^{(m)}, \ldots, \nu_{\sigma_m}^{(m)}$ and assume that at the moment when he decides on the $\nu_k^{(m)}$-th "step" ($k = 1, 2, \ldots, \sigma_m$) the results of the "draws" and "steps" with the numbers $\bar{\mu}_1^{(m,k)}, \bar{\mu}_2^{(m,k)}, \ldots, \bar{\mu}_{\alpha_{m,k}}^{(m,k)}$ and $\bar{\nu}_1^{(m,k)}, \bar{\nu}_2^{(m,k)}, \ldots, \bar{\nu}_{\beta_{m,k}}^{(m,k)}$ respectively are available to him, that is, they are "earlier" than $\nu_k^{(m)}$. For each possible combination of results of the "draws" and "steps" mentioned above (obviously, there is only a finite number of such combinations) it must be specified what S_m's decision with respect to the $\nu_k^{(m)}$-th "step" is going to be (i.e., what will be the outcome of this step).

One sees immediately that only a finite number of strategies is available to S_m, which we shall call $S_1^{(m)}, S_2^{(m)}, \ldots, S_{\Sigma_m}^{(m)}$.

It can now easily be shown (using the assumption on the absence of cycles in the Introduction, 1., definition of a game of strategy, (δ)) that the course of a play is described in a permissible and unambiguous manner if we specify

1. which strategies $\bar{S}^{(1)}, \bar{S}^{(2)}, \ldots, \bar{S}^{(n)}$ are being used by the players $S_1, S_2, \ldots, S_n$ respectively,

2. what are the results of the "draws" $E_1, E_2, \ldots, E_z$.

Two points should be noted here. In the first place, it is inherent in the concept of "strategy" that all the information about the actions of the participants and the outcome of "draws" a player is able to obtain or to infer is already incorporated in the "strategy." Consequently, each player must choose his strategy in complete ignorance of the choices of the rest of the players and of the results of the "draws."

Secondly, it has become entirely immaterial that the "draws" $E_1, E_2, \ldots, E_n$ are <u>separate</u> events (where for E_μ, $\mu = 1, 2, \ldots, z$, the numbers $1, 2, \ldots, M_\mu$ will occur with the respective probabilities $\alpha_\mu^{(1)}, \alpha_\mu^{(2)}, \ldots, \alpha_\mu^{(M_\mu)}$) since the players must act, i.e., choose their "strategies" without knowing the outcome of the "draws." But if this is the case nothing prevents us from combining all z "draws" into a single "draw", H, the outcome of which will be the aggregates of numbers

$$x_1, x_2, \ldots, x_z \quad (x_\mu = 1, 2, \ldots, M_\mu, \ \mu = 1, 2, \ldots, z)$$

with their respective probabilities $\alpha_1^{(x_1)} \cdot \alpha_2^{(x_2)} \cdot \ldots \cdot \alpha_z^{(x_z)}$, or, what amounts to the same thing, the numbers $1, 2, \ldots, M$ $(M = M_1 \cdot M_2 \cdot \ldots \cdot M_z)$ with their respective probabilities, which we shall call $\beta_1, \beta_2, \ldots, \beta_M$.

Thus we can modify 2. in the following way:

2'. The result of the "draw" H must be specified. (H may have the results $1, 2, \ldots, M$ with the respective probabilities $\beta_1, \beta_2, \ldots, \beta_M$.)

The choices (interpreted as "steps") of the players $S_1, S_2, \ldots, S_n$ in 1. together with the "draw" in 2'. are fully equivalent to the original game $\mathfrak{G}$ (if one takes the fact into account that each "step" is taken in complete ignorance of all other circumstances), and they evidently constitute a game $\mathfrak{G}'$ which, indeed, is of the simple form mentioned at the beginning of this secton.

2. The last element to be eliminated from the game, since from our point of view it is inessential, is the "draw". This is done by replacing the actual results for the individual players by their expected values. To be exact:

If the players $S_1, S_2, \ldots, S_n$ have chosen the "strategies"

$$S^{(1)}_{u_1}, S^{(1)}_{u_2}, \ldots, S^{(1)}_{u_n} \quad (u_m = 1, 2, \ldots, \Sigma_m, m = 1, 2, \ldots, n)$$

and if the outcome of the "draw" H has been the number v (= 1, 2, ..., M), then let the results for the players $S_1, S_2, \ldots, S_n$ be

$$f_1(v, u_1, \ldots, u_n), f_2(v, u_1, \ldots, u_n), \ldots, f_n(v, u_1, \ldots, u_n) .$$

(We may disregard the fact that we are dealing with "strategies" and not with actual "steps" and simply speak of the choices $u_1, u_2, \ldots, u_n$.)

If only the choices of $u_1, u_2, \ldots, u_n$ are known, but not yet the "draw" v, the expected values of $f_1, f_2, \ldots, f_n$ will be

$$g_m(u_1, \ldots, u_n) = \sum_{v=1}^{M} \beta_v f_m(v, u_1, \ldots, u_n) \quad (m = 1,2, \ldots, n)$$

($f_1 + f_2 + \ldots + f_n \equiv 0$ implies $g_1 + g_2 + \ldots + g_n \equiv 0$). It is entirely in the spirit of the probabilistic method to discount the "draw" altogether and to deal exclusively with the expected values $g_1, g_2, \ldots, g_n$. In doing so we obtain the following basic type of a game of strategy which is even more schematized and simplified.

> Each of the players $S_1, S_2, \ldots, S_n$ chooses a number, S_m choosing one of the numbers 1, 2, ..., Σ_m[7] (m = 1, 2, ..., n). Each player must make his decision without being informed about the choices of the other participants. After having made their choices $x_1, x_2, \ldots, x_n$ ($x_m = 1, 2, \ldots, \Sigma_m$, $m = 1, 2, \ldots, n$) the players receive the following amounts respectively:
>
> $$g_1(x_1, \ldots, x_n), g_2(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)$$
>
> (where identically $g_1 + g_2 + \ldots + g_n \equiv 0$).

The rules of the game have thus been obtained in a form which retains only those characteristics of a game of strategy which are essential to our consideration - and as we have just shown, essentially without loss

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[7] In addition, we could also make all Σ_m equal to each other by assuming a Σ which is not smaller than any Σ_m and subdividing each Σ_m-th case in $\Sigma - \Sigma_m + 1$ subcases, each of which would have the same effect as the original. But this simplification is inessential.

of generality. Nothing is left of a "game of chance". The actions of the players determine the result completely (since everything takes place as if each of the players has his eye on the expected value only). As a result, the feature which was emphasized at the end of the Introduction emerges in a particularly clear form: each g_m depends on all x_1, x_2, ..., x_n.

The standard case of probability theory that g_m depends on x_m only (which, of course, cannot hold for all m) now appears to be entirely trivial.

§2. THE CASE $n = 2$

1. Since we cannot proceed any further with the same generality, it is now appropriate to consider the simplest case for n. The case $n = 0$ is meaningless, and so is the case $n = 1$ (since $g_1 + g_2 + \dots + g_n \equiv 0$); neither involves an actual game of strategy. So we shall now investigate the case $n = 2$.

Since $g_1 + g_2 \equiv 0$, we can put $g_1 = g$, $g_2 = -g$. The description of a general two-person game is then as follows:

> The players S_1, S_2 choose arbitrary numbers among the numbers $1, 2, \dots, \Sigma_1$ and $1, 2, \dots, \Sigma_2$ respectively, each one without knowing the choice of the other. After having chosen the numbers x and y respectively, they receive the sums $g(x, y)$ and $- g(x, y)$ respectively.

$g(x, y)$ may be any function (defined for $x = 1, 2, \dots, \Sigma_1$, $y = 1, 2, \dots, \Sigma_2$!).

It is easy to picture the forces struggling with each other in such a two-person game. The value of $g(x, y)$ is being tugged at from two sides, by S_1 who wants to maximize it, and by S_2 who wants to minimize it. S_1 controls the variable x, S_2 the variable y. What will happen?

2. After S_1 has chosen the number x ($x = 1, 2, \dots, \Sigma_1$), his result $g(x, y)$ still depends on the choice y of S_2, but in any event $g(x, y) \geq \mathrm{Min}_y\, g(x, y)$. And by an appropriate choice of x this lower limit can be made equal to $\mathrm{Max}_x\, \mathrm{Min}_y\, g(x, y)$ (and not any larger!). I.e., if S_1 so wishes, he certainly can make $g(x, y)$

$$\geq \mathrm{Max}_x \mathrm{Min}_y g(x, y)$$

(irrespective of what S_2 does!). The same argument holds for S_2. If

S_2 so wishes, he certainly can make $g(x, y)$

$$\leq \mathrm{Min}_y \mathrm{Max}_x g(x, y)$$

(irrespective of what S_1 does!).

If now

$$\mathrm{Max}_x \mathrm{Min}_y g(x, y) = \mathrm{Min}_y \mathrm{Max}_x g(x, y) = M$$

it follows from the above, as well as from the fact that S_1 wants to maximize $g(x, y)$ and S_2 wants to minimize it, that $g(x, y)$ will have the value M. For, S_1 is interested in making it large and can keep it from becoming smaller than M. S_2, on the other hand, is interested in making it small and can keep it from becoming larger than M. Hence it will have the value M.

Though, in general,

$$\mathrm{Max}_x \mathrm{Min}_y g(x, y) \leq \mathrm{Min}_y \mathrm{Max}_x g(x, y)$$

it is not at all true that the $=$ sign always holds. Actually, it is easy to exhibit such $g(x, y)$ for which the $<$ sign holds, that is, for which the above consideration breaks down. The simplest example of this kind is the following:

$$\Sigma_1 = \Sigma_2 = 2, \quad g(1, 1) = 1, \quad g(1, 2) = -1,$$
$$g(2, 1) = -1, \quad g(2, 2) = 1.$$

(Evidently, Max Min $= -1$ and Min Max $= 1$.)

Another example is the so-called game of "Morra":[8]

$$\Sigma_1 = \Sigma_2 = 3, \quad g(1, 1) = 0, \quad g(1, 2) = 1, \quad g(1, 3) = -1,$$
$$g(2, 1) = -1, \quad g(2, 2) = 0, \quad g(2, 3) = 1,$$
$$g(3, 1) = 1, \quad g(3, 2) = -1, \quad g(3, 3) = 0.$$

(Here, too, Max Min $= -1$ and Min Max $= 1$.)

The fact that this difficulty comes up can also be realized in the following way:

$\mathrm{Max}_x \mathrm{Min}_y g(x, y)$ is the best result that S_1 can achieve if

[8] Also called "gangster baccarat." In the usual formulation, 1, 2, 3 are called "Paper", "Stone", "Scissors" ("Paper covers the stone, scissors cut the paper, stone grinds the scissors").

he is "found out" by S_2; if whenever S_1 plays x, S_2 plays a y such that $g(x, y) = \mathrm{Min}_y\, g(x, y)$. (According to the rules of the game S_2 was not supposed to know how S_1 was going to play, he would have to infer it in some other way. This is what we mean to indicate by the expression "finding out". In the same way, the best result that S_2 can achieve if S_1 has found him out is $\mathrm{Min}_y\, \mathrm{Max}_x\, g(x, y)$. If the two numbers are equal this means: it makes no difference which of the two players is the better psychologist, the game is so insensitive that the result is always the same. It is obvious that this is not the case for the two games just mentioned: here, everything depends on finding the adversary out, on guessing whether he is going to choose 1 or 2 (or 1 or 2 or 3).

The fact that the two quantities Max Min and Min Max are different means that it is impossible for each of the two players, S_1 and S_2, to be cleverer than the other.

3. Still, it is possible, by use of an artifice, to force the equality of the two above-mentioned expressions.

To this purpose, the possibilities of action for the two players S_1 and S_2 are extended as follows: At the beginning of the game, S_1 is not asked to choose one of the numbers $1, 2, \ldots, \Sigma_1$. He only has to specify Σ_1 probabilities

$$\xi_1, \xi_2, \ldots, \xi_{\Sigma_1} \quad (\xi_1 \geq 0,\ \xi_2 \geq 0,\ \ldots,\ \xi_{\Sigma_1} \geq 0,\ \xi_1 + \xi_2 + \ldots + \xi_{\Sigma_1} = 1)$$

and then draw the numbers $1, 2, \ldots, \Sigma_1$ from an urn containing these numbers with the probabilities $\xi_1, \xi_2, \ldots, \xi_{\Sigma_1}$. He then chooses the number drawn. This may look like a restriction of his free will: it is not he who determines x. But this is not so. Because if he really wants to get a particular x, he can specify $\xi_x = 1$, $\xi_u = 0$ (for $u \neq x$). On the other hand, he is protected against his adversary "finding him out"; for, if, e.g., $\xi_1 = \xi_2 = 1/2$, nobody (not even he himself!) can predict whether he is going to choose 1 or 2!

S_2 is supposed to act in the same way. He also chooses Σ_2 probabilities $\eta_1, \eta_2, \ldots, \eta_{\Sigma_2}$ and proceeds accordingly.

Let us denote the sequence $\xi_1, \xi_2, \ldots, \xi_{\Sigma_1}$ by ξ and the sequence $\eta_1, \eta_2, \ldots, \eta_{\Sigma_2}$ by η. If S_1 chooses ξ and S_2 chooses η, S_1 has the expected value

$$h(\xi, \eta) = \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q)\, \xi_p \eta_q$$

and S_2 has the expected value $-h(\xi, \eta)$. The new function $h(\xi, \eta)$ includes the old one, $g(x, y)$, in the following sense: if $\xi_x = \eta_y = 1$ and $\xi_u = \eta_v = 0$ (for $u \neq x$, $v \neq y$), then $h(\xi, \eta) = g(x, y)$.

We can now apply the same consideration to $h(\xi, \eta)$ we applied to $g(x, y)$. If S_1 has made the choice ξ, his expected value is at least $\text{Min}_\eta h(\xi, \eta)$. Hence, he is in a position to obtain the minimal expected value $\text{Max}_\xi \text{Min}_\eta h(\xi, \eta)$ (irrespective of what S_2 does!). In the same way, S_2 can keep the expected value of S_1 from exceeding the maximal value $\text{Min}_\eta \text{Max}_\xi h(\xi, \eta)$. Again we have

$$\text{Max}_\xi \text{Min}_\eta h(\xi, \eta) \leq \text{Min}_\eta \text{Max}_\xi h(\xi, \eta)$$

and the question is whether the equality sign always holds.

Evidently, in this case our chances are better than they were for $g(x, y)$; for, $g(x, y)$ could be any function, whereas $h(\xi, \eta)$ is a bilinear form! Even though $h(\xi, \eta)$ is essentially a generalization of $g(x, y)$, yet it is a function of a much simpler type than $g(x, y)$. In fact, we shall prove in Section 3 that the relation

$$\text{Max}_\xi \text{Min}_\eta h(\xi, \eta) \equiv \text{Min}_\eta \text{Max}_\xi h(\xi, \eta)$$

holds for all bilinear forms $h(\xi, \eta)$ (where Max_ξ is taken for all ξ fo which $\xi_1 \geq 0, \ldots, \xi_{\Sigma_1} \geq 0$, $\xi_1 + \cdots + \xi_{\Sigma_1} = 1$, and Min_η is taken for all η for which $\eta_1 \geq 0, \ldots, \eta_{\Sigma_2} \geq 0$, $\eta_1 + \cdots + \eta_{\Sigma_2} = 1$).

4. Anticipating the result we put

$$\text{Max}_\xi \text{Min}_\eta h(\xi, \eta) = \text{Min}_\eta \text{Max}_\xi h(\xi, \eta) = M \cdot$$

Let $\mathfrak{A}$ be the set of all ξ for which $\text{Min}_\eta h(\xi, \eta)$ assumes its maximal value M, and let $\mathfrak{B}$ be the set of all η for which $\text{Max}_\xi h(\xi, \eta)$ assumes its minimal value M. From these definitions the relations below follow immediately.

(1) If ξ belongs to $\mathfrak{A}$, then always $h(\xi, \eta) \geq M$
(2) If η belongs to $\mathfrak{B}$, then always $h(\xi, \eta) \leq M$
(3) If ξ does not belong to $\mathfrak{A}$, there exists an η for which $h(\xi, \eta) < M$
(4) If η does not belong to $\mathfrak{B}$, there exists a ξ for which $h(\xi, \eta) > M$
(5) If ξ belongs to $\mathfrak{A}$ and η belongs to $\mathfrak{B}$, then $h(\xi, \eta) = M$.

On the basis of the relations (1) to (5) the following statement seems justified:

> Clearly, S_1 must choose a complex ξ belonging to $\mathfrak{A}$, and S_2 must choose a complex η belonging to $\mathfrak{B}$. For any such choice, a play has the value M or $-M$ for S_1 and S_2 respectively.

Evidently, a two-person game can be called "fair" if $M = 0$; and it can be called "symmetric" if the players S_1, S_2 have the same roles. I.e., if on interchanging ξ and η (which presupposes that $\Sigma_1 = \Sigma_2$) $h(\xi, \eta)$ and $-h(\xi, \eta)$ are also interchanged, in other words, if

$$h(\xi, \eta) = -h(\eta, \xi)$$

or, equivalently,

$$g(x, y) = -g(y, x)$$

i.e., if the bilinear form $h(\xi, \eta)$, or else the matrix $g(x, y)$ is skew-symmetric. In this case, the game is, of course, also "fair", as can be seen as follows:

$$-\operatorname{Max}_\xi \operatorname{Min}_\eta h(\xi, \eta) = \operatorname{Min}_\xi \operatorname{Max}_\eta - h(\xi, \eta) = \operatorname{Min}_\xi \operatorname{Max}_\eta h(\eta, \xi)$$
$$= \operatorname{Min}_\eta \operatorname{Max}_\xi h(\xi, \eta)$$

i.e.,

$$-M = M, \quad M = 0 .$$ [9]

One can easily see that in our two examples (in 2.) $M = 0$ since $\mathfrak{A}$ contains only $\xi_1 = \xi_2 = 1/2$ and $\xi_1 = \xi_2 = \xi_3 = 1/3$ respectively, and $\mathfrak{B}$ contains only $\eta_1 = \eta_2 = 1/2$ and $\eta_1 = \eta_2 = \eta_3 = 1/3$ respectively. I.e., both games are "fair" ("Paper, Scissors, Stone" is even symmetric), and in both examples each player must choose all numbers at random, all of

[9] Use is made of the fact that Max Min = Min Max, i.e., we have applied our rather deep theorem on bilinear forms. Trivially -- i.e., from Max Min $\leq$ Min Max -- it would only follow that

$$\text{Max Min} \leq 0, \quad \text{Min Max} \geq 0 .$$

While this paper was put into its final form, I learned of the note of E. Borel in the Comptes Rendus of Jan. 10, 1927 ("Sur les systèmes de formes linéaires...et la théorie du jeu," pp. 52-55). Borel formulates the question of bilinear forms for a symmetric two-person game and states that no examples for Max Min < Min Max are known.
Our result above answers his question.

them with the same probability.

The following point should be emphasized: Although in Section 1 chance was eliminated from the games of strategy under consideration (by introducing expected values and eliminating "draws"), it has now made a spontaneous reappearance. Even if the rules of the game do not contain any elements of "hazard" (i.e., no draws from urns) - as e.g., the two examples in 2. - in specifying the rules of behavior for the players it becomes imperative to reconsider the element of "hazard". The dependence on chance (the "statistical" element) is such an intrinsic part of the game itself (if not of the world) that there is no need to introduce it artificially by way of the rules of the game: even if the formal rules contain no trace of it, it still will assert itself.

§3. PROOF OF THE THEOREM "Max Min = Min Max"

1. Let us slightly change our notation by replacing Σ_1 by $M + 1$, Σ_2 by $N + 1$ and $g(p, q)$ by α_{pq}. We then have

$$h(\xi, \eta) = \sum_{p=1}^{M+1} \sum_{q=1}^{N+1} \alpha_{pq}\xi_p\eta_q \quad .$$

Because of the conditions

$$\xi_1 + \dots + \xi_M + \xi_{M+1} = 1, \quad \eta_1 + \dots + \eta_N + \eta_{N+1} = 1$$

the complex ξ is already determined by $\xi_1, \dots, \xi_M$ and so is the complex η by $\eta_1, \dots, \eta_N$. Thus

$$h(\xi, \eta) = \sum_{p=1}^{M} \sum_{q=1}^{N} u_{pq}\xi_p\eta_q + \sum_{p=1}^{M} v_p\xi_p + \sum_{q=1}^{N} w_q\eta_q + r \; .$$

(There is no need to specify explicitly the coefficients in terms of α_{pq}.) We shall make use of only some of the properties of $h(\xi, \eta)$ and investigate continuous functions of two sets of variables $f(\xi, \eta)$ which satisfy the following conditions:

(K). If $f(\xi', \eta) \geq A$, $f(\xi'', \eta) \geq A$, then $f(\xi, \eta) \geq A$ for every $0 \leq \vartheta \leq 1$, $\xi = \vartheta\xi' + (1 - \vartheta)\xi''$ (i.e., $\xi_p = \vartheta\xi'_p + (1 - \vartheta)\xi''_p$, $p = 1, 2, \dots, M$). If $f(\xi, \eta') \leq A$, $f(\xi, \eta'') \leq A$, then $f(\xi, \eta) \leq A$ for every $0 \leq \vartheta \leq 1$, $\eta = \vartheta\eta' + (1 - \vartheta)\eta''$ (i.e., $\eta_q = \vartheta\eta'_q + (1 - \vartheta)\eta''_q$, $q = 1, 2, \dots, N$).

(It is clear that $h(\xi, \eta)$, being linear in the ξ as well as in the η, has the property (K).) For these functions $f(\xi, \eta)$ we are going to prove that

$$\mathrm{Max}_{\xi}\mathrm{Min}_{\eta}(f(\xi, \eta) = \mathrm{Min}_{\eta}\mathrm{Max}_{\xi}f(\xi, \eta)$$

where Max_{ξ} is taken over the range $\xi_1 \geq 0, \ldots, \xi_M \geq 0, \xi_1 + \ldots + \xi_M \leq 1$ and Min_{η} is taken over the range $\eta_1 \geq 0, \ldots, \eta_N \geq 0, \eta_1 + \ldots + \eta_N \leq 1$. We can also write

$$\underset{\substack{\xi_1\geq 0\\ \xi_1\leq 1}}{\mathrm{Max}_{\xi_1}}\ \underset{\substack{\xi_2\geq 0\\ \xi_1+\xi_2\leq 1}}{\mathrm{Max}_{\xi_2}} \ldots \underset{\substack{\xi_M\geq 0\\ \xi_1+\ldots+\xi_M\leq 1}}{\mathrm{Max}_{\xi_M}}\ \underset{\substack{\eta_1\geq 0\\ \eta_1\leq 1}}{\mathrm{Min}_{\eta_1}}\ \underset{\substack{\eta_2\geq 0\\ \eta_1+\eta_2\leq 1}}{\mathrm{Min}_{\eta_2}} \ldots \underset{\substack{\eta_N\geq 0\\ \eta_1+\ldots+\eta_N\leq 1}}{\mathrm{Min}_{\eta_N}} f(\xi, \eta)$$

$$= \underset{\substack{\eta_1\geq 0\\ \eta_1\leq 1}}{\mathrm{Min}_{\eta_1}}\ \underset{\substack{\eta_2\geq 0\\ \eta_1+\eta_2\leq 1}}{\mathrm{Min}_{\eta_2}} \ldots \underset{\substack{\eta_N\geq 0\\ \eta_1+\ldots+\eta_N\leq 1}}{\mathrm{Min}_{\eta_N}}\ \underset{\substack{\xi_1\geq 0\\ \xi_1\leq 1}}{\mathrm{Max}_{\xi_1}}\ \underset{\substack{\xi_2\geq 0\\ \xi_1+\xi_2\leq 1}}{\mathrm{Max}_{\xi_2}} \ldots \underset{\substack{\xi_M\geq 0\\ \xi_1+\ldots+\xi_M\leq 1}}{\mathrm{Max}_{\xi_M}} f(\xi, \eta) \quad .$$

2. We introduce the following notation:

$$M^{\xi_r}f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) = \underset{\substack{\xi_r\geq 0\\ \xi_1+\ldots+\xi_r\leq 1}}{\mathrm{Max}_{\xi_r}} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)$$

$$M^{\eta_s}f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) = \underset{\substack{\eta_s\geq 0\\ \eta_1+\ldots+\eta_s\leq 1}}{\mathrm{Min}_{\eta_s}} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)$$

Clearly, M^{ξ_r} and M^{η_s} eliminate the dependence of f on ξ_r and η_s respectively. We are going to prove: if f satisfies the condition (K) (in 1.), then

$$M^{\xi_1} M^{\xi_2} \ldots M^{\xi_p} M^{\eta_1} M^{\eta_2} \ldots M^{\eta_q} f$$
$$= M^{\eta_1} M^{\eta_2} \ldots M^{\eta_q} M^{\xi_1} M^{\xi_2} \ldots M^{\xi_p} f \quad .$$

Evidently, we need only prove the following two assertions:

α) If $f = f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)$ is continuous and has the property (K), the same holds for $M^{\xi_r} f$ and $M^{\eta_s} f$.

β) If $f = f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)$ is continuous and has the property (K), then

$$M^{\xi_r} M^{\eta_s} f = M^{\eta_s} M^{\xi_r} f \quad .$$

We first prove (α). It is sufficient to consider $M^{\xi_r} f$; the same considerations apply to $M^{\eta_s} f$.

We have

$$M^{\xi_r} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) = f^*(\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_s) = \operatorname{Max}_{\substack{\xi_r \\ \xi_r \geq 0 \\ \xi_1 + \cdots + \xi_r \leq 1}} f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)$$

It is obvious that the continuity of f implies that of f^*. We still have to examine the two properties in (K).

First, let

$$f^*(\xi_1', \ldots, \xi_{r-1}', \eta_1, \ldots, \eta_s) \geq A, \quad f^*(\xi_1'', \ldots, \xi_{r-1}'', \eta_1, \ldots, \eta_s) \geq A \ .$$

The f^* represent maximal values of f on finite intervals. Since f is continuous, they are actually assumed, say, for ξ_r' and ξ_r'' respectively.

Then

$$f(\xi_1', \ldots, \xi_r', \eta_1, \ldots, \eta_s) \geq A, \quad f(\xi_1'', \ldots, \xi_r'', \eta_1, \ldots, \eta_s) \geq A$$

and since f satisfies (K) (we put $\xi_1 = \vartheta\xi_1' + (1 - \vartheta)\xi_1'', \ldots, \xi_{r-1} = \vartheta\xi_{r-1}' + (1 - \vartheta)\xi_{r-1}''$ and $\xi_r = \vartheta\xi_r' + (1 - \vartheta)\xi_r''$)

$$f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) \geq A \ .$$

The inequalities

$$\xi_r' \geq 0, \quad \xi_1' + \cdots + \xi_r' \leq 1, \quad \xi_r'' \geq 0, \quad \xi_1'' + \cdots + \xi_r'' \leq 1$$

imply directly

$$\xi_r \geq 0, \quad \xi_1 + \cdots + \xi_r \leq 1 \ .$$

Hence, a fortiori, for the maximum f^*

$$f^*(\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_s) \geq A \ .$$

Second, let

$$f^*(\xi_1, \ldots, \xi_{r-1}, \eta_1', \ldots, \eta_s') \leq A, \quad f^*(\xi_1, \ldots, \xi_{r-1}, \eta_1'', \ldots, \eta_s'') \leq A \ .$$

In virtue of the maximum property of f^* we have

$$f(\xi_1, \ldots, \xi_r, \eta_1', \ldots, \eta_s') \leq A, \quad f(\xi_1, \ldots, \xi_r, \eta_1'', \ldots, \eta_s'') \leq A$$

for all ξ_r for which

$$\xi_r \geq 0, \quad \xi_1 + \cdots + \xi_r \leq 1 \ .$$

Since f satisfies (K), this again implies

$$f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s) \leq A$$

$(\eta_1 = \vartheta\eta_1' + (1 - \vartheta)\eta_1'', \ldots, \eta_s = \vartheta\eta_s' + (1 - \vartheta)\eta_s'')$. And since this holds for all ξ_r mentioned above, we have

$$f^*(\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_s) \leq A \ .$$

This completes the proof of our assertion (α).

3. We will now show that always (i.e., for all $\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_{s-1}$) $M^{\xi_r} M^{\eta_s} f = M^{\eta_s} M^{\xi_r} f$. If in $f(\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_s)$ we keep the variables $\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_{s-1}$ fixed, then f, as a function of ξ_r, η_s alone, evidently still satisfies the condition (K). It remains for us to prove (writing ξ, η for ξ_r, η_s):

> If $f(\xi, \eta)$ is a continuous function and if $f(\xi', \eta) \geq A$, $f(\xi'', \eta) \geq A$ for $\xi' \leq \xi \leq \xi''$ implies that $f(\xi, \eta) \geq A$, and if $f(\xi, \eta') \leq A$, $f(\xi, \eta'') \leq A$ for $\eta' \leq \eta \leq \eta''$ implies that $f(\xi, \eta) \leq A$, then

$$\underset{0 \leq \xi \leq a}{\mathrm{Max}_\xi} \ \underset{0 \leq \eta \leq b}{\mathrm{Min}_\eta} \ f(\xi, \eta) = \underset{0 \leq \eta \leq b}{\mathrm{Min}_\eta} \ \underset{0 \leq \xi \leq a}{\mathrm{Max}_\xi} \ f(\xi, \eta)$$

(We write a and b for $1 - \xi_1 - \cdots - \xi_{r-1}$ and $1 - \eta_1 - \cdots - \eta_{s-1}$ respectively.)

The assertion can also be formulated in the following way: There exists a "saddle point" ξ_0, η_0 $(0 \leq \xi_0 \leq a,\ 0 \leq \eta_0 \leq b)$, i.e., $f(\xi_0, \eta)$ assumes its minimum for $\eta = \eta_0$ in $0 \leq \eta \leq b$ and $f(\xi, \eta_0)$ assumes its maximum for $\xi = \xi_0$ in $0 \leq \xi \leq a$.

First of all, we evidently have

$$\text{Max}_\xi \text{ Min}_\eta f(\xi, \eta) \leq \text{Min}_\eta \text{ Max}_\xi f(\xi, \eta)$$

and secondly, the assertion just formulated implies that

$$\text{Max}_\xi \text{ Min}_\eta f(\xi, \eta) \geq \text{Min}_\eta f(\xi_0, \eta) = f(\xi_0, \eta_0)$$

$$\text{Min}_\eta \text{ Max}_\xi f(\xi, \eta) \leq \text{Max}_\xi f(\xi, \eta_0) = f(\xi_0, \eta_0)$$

hence

$$\text{Max}_\xi \text{ Min}_\eta f(\xi, \eta) = \text{Min}_\eta \text{ Max}_\xi f(\xi, \eta) = f(\xi_0, \eta_0) .$$

Our task is now to find a pair ξ_0, η_0 with the desired properties.

Let ξ be fixed. For which values of η, $0 \leq \eta \leq b$, does $f(\xi, \eta)$ assume its minimum? The answer is easy. Since f is continuous, this set is closed, and because of the second assumption about f ($f(\xi, \eta') \leq A$, $f(\xi, \eta'') \leq A$ implies $f(\xi, \eta) \leq A$ for all $\eta' \leq \eta \leq \eta''$) it is convex. But the only closed and convex sets of real numbers are closed intervals. Therefore, this set is a subinterval of the interval 0, b; we call it $K'(\xi), K''(\xi)$.

If η is fixed, we conclude in the same way that those ξ, $0 \leq \xi \leq a$, for which $f(\xi, \eta)$ assumes its maximum form a closed subinterval of 0, a; we call it $L'(\eta), L''(\eta)$.

Evidently, always $K'(\xi) \leq K''(\xi)$ and $L'(\eta) \leq L''(\eta)$. Furthermore, the continuity of $f(\xi, \eta)$ implies that $K'(\xi)$, $L'(\eta)$ and $K''(\xi)$, $L''(\eta)$ are lower and upper semi-continuous functions respectively.[10]

Let now ξ^* be fixed. We form the set of all ξ^{**} with the following property: There exists an η^* such that $f(\xi^*, \eta)$ assumes its

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[10] Let us indicate the proof for $K'(\xi)$. It will be the same for the other three functions.

If $K'(\xi) = 0$, the assertion is trivial since always $K'(\zeta) \geq 0$. Let $K'(\xi) > 0$. For $0 \leq \eta \leq K'(\xi) - \epsilon\ (\epsilon > 0)$, there exists a $\delta > 0$ such that $f(\xi, \eta) \geq \text{Min}_\eta f(\xi, \eta) + \delta$, by the definition of $K'(\xi)$. Hence, if ζ is sufficiently close to ξ, we still have $f(\zeta, \eta) \geq \text{Min}_\eta f(\zeta, \eta) + \delta/2$ (because both $f(\xi, \eta)$ and $\text{Min}_\eta f(\xi, \eta)$ are continuous); i.e., $f(\zeta, \eta)$ does not assume its minimum (in η, for $0 \leq \eta \leq b$) in $0 \leq \eta \leq K'(\xi) - \epsilon$. Therefore $K'(\zeta) > K'(\xi) - \epsilon$, and $K'(\zeta)$ is lower semi-continuous as we have asserted.

minimal value (in $0 \leq \eta \leq b$) at $\eta = \eta^*$ and $f(\xi, \eta^*)$ assumes its maximal value (in $0 \leq \xi \leq a$) at $\xi = \xi^{**}$. I.e., we form the union of all intervals $L'(\eta^*) \leq \xi^{**} \leq L''(\eta^*)$ where η^* assumes all values in the interval $K'(\xi^*) \leq \eta^* \leq K''(\xi^*)$.

In the interval $K'(\xi^*) \leq \eta^* \leq K''(\xi^*)$ the lower semi-continuous function $L'(\eta^*)$ assumes its minimum and the upper semi-continuous function $L''(\eta^*)$ assumes its maximum. Hence the set of ξ^{**} contains a minimal as well as a maximal element. It also contains every intermediate element ξ', which can be demonstrated in the following way: If it were not so, every interval $L'(\eta^*)$, $L''(\eta^*)$ would lie either entirely to the left or entirely to the right of ξ', and both kinds would exist (those belonging to the smallest as well as to the largest ξ^{**}). Since η^* runs over an interval, both kinds of η^* would have a common limit-point. Since both $L'(\eta^*) \leq \xi'$ and $L''(\eta^*) \geq \xi'$ occur arbitrarily close to η' (and L', L'' are lower and upper semi-continuous respectively), it follows that $L'(\eta') \leq \xi'$, $L''(\eta'') \geq \xi'$; i.e., ξ' belongs indeed to one of the intervals, namely, to $L'(\eta')$, $L''(\eta')$.

Our ξ^{**} thus form a closed subinterval of $0, a$, which we shall call $H'(\xi^*)$, $H''(\xi^*)$. $H'(\xi^*)$ is the minimum of the $L'(\eta^*)$, $H''(\xi^*)$ is the maximum of the $L''(\eta^*)$, for $K'(\xi^*) \leq \eta^* \leq K''(\xi^*)$. It is easy to see that $H'(\xi^*)$ and $H''(\xi^*)$ are again lower and upper semi-continuous respectively (this is implied by the corresponding properties of $K'(\xi^*)$, $K''(\xi^*)$, and $L'(\eta^*)$, $L''(\eta^*)$).

It remains to find a $\xi^*(0 \leq \xi^* \leq a)$ which at the same time is a ξ^{**}, i.e., a ξ^* for which $H'(\xi^*) \leq \xi^* \leq H''(\xi^*)$.

If no such ξ^* existed, every interval $H'(\xi^*)$, $H''(\xi^*)$ would lie either entirely to the left or entirely to the right of ξ^*, and both kinds would exist ($\xi^* = a$ and $\xi^* = 0$). Since ξ^* runs over an interval, both kinds of ξ^* would have a common limit-point ξ'. Since arbitrarily close to ξ' both $H'(\xi^*) \leq \xi^*$ and $H''(\xi^*) \geq \xi^*$ occur (and H', H'' are lower and upper semi-continuous respectively), it follows that $H'(\xi') \leq \xi'$, $H''(\xi') \geq \xi'$, i.e., ξ' belongs indeed to the interval $H'(\xi')$, $H''(\xi')$.

We have now proved our last assertion (and hence the assertion (β)), which concludes the proof of our theorem.

§4. THE CASE n = 3

Having dealt in Sections 2 and 3 with the case $n = 2$ we now proceed to the next case, $n = 3$.

Consider a three-person game characterized, according to the description at the end of Section 1, by three functions g_1, g_2, g_3 of three variables x, y, z ($x = 1, 2, \ldots, \Sigma_1$, $y = 1, 2, \ldots, \Sigma_2$, $z = 1, 2, \ldots, \Sigma_3$), where identically

$$g_1 + g_2 + g_3 \equiv 0 \quad .$$

In the case $n = 2$ it was possible to strictly determine the value of a play for each of the players S_1 and S_2 with the results

$$\text{value for } S_1 = \text{Max}_\xi \text{Min}_\eta \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q)\xi_p\eta_q = \text{Max}_\xi \text{Min}_\eta \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g_1(p, q)\xi_p\eta_q$$

$$\text{value for } S_2 = -\text{Min}_\eta \text{Max}_\xi \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p, q)\xi_p\eta_q = \text{Max}_\eta \text{Min}_\xi \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g_2(p, q)\xi_p\eta_q$$

where

$$\text{value for } S_1 + \text{value for } S_2 = 0 \quad .$$

Let us now try in the case $n = 3$ to compute the values of a play for the three players S_1, S_2, S_3. Assume these values to be w_1, w_2, w_3 respectively. For these values to be satisfactory under any conditions and without further discussion they clearly should have the following property: No two players must be able, by forming a coalition, to achieve an expected value exceeding the sum of the "values of a play" assigned to them. Furthermore, $w_1 + w_2 + w_3 = 0$ must hold, since the players make payments only to each other.

By putting

$$\text{Max}_\xi \text{Min}_\eta \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_1(pqr) + g_2(pqr))\xi_{pq}\eta_r = M_{1,2}$$

$$\text{Max}_\xi \text{Min}_\eta \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_1(pqr) + g_3(pqr))\xi_{pr}\eta_q = M_{1,3}$$

$$\text{Max}_\xi \text{Min}_\eta \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_2(pqr) + g_3(pqr))\xi_{qr}\eta_p = M_{2,3}$$

(the ξ_{pq} form a system of probabilities, as do the η_r; similarly ξ_{pr}, η_q and ξ_{qr}, η_p), S_1 and S_2 by forming a coalition can play an ordinary two-person game against S_3, thereby procuring for themselves (in accordance with the above) the expected value $M_{1,2}$. The same holds for S_1 and S_3, and for S_2 and S_3 regarding the expected values $M_{1,3}$ and $M_{2,3}$ respectively. Hence we must have

$$w_1 + w_2 \geq M_{1,2} \quad w_1 + w_3 \geq M_{1,3} \quad w_2 + w_3 \geq M_{2,3}$$

$$w_1 + w_2 + w_3 = 0 \quad .$$

Clearly, this is possible if and only if

$$M_{1,2} + M_{1,3} + M_{2,3} \leq 0 \quad .$$

As we are going to show in 2., it is always true that

$$M_{1,2} + M_{1,3} + M_{2,3} \geq 0$$

and it is easy to give examples in which the $>$ sign holds. Such an example is provided by the following three-person game:

> $\Sigma_1 = \Sigma_2 = \Sigma_3 = 3$. If among the x_1, x_2, x_3 (i.e., among the choices of S_1, S_2, S_3, formerly also denoted by x, y, z) there are two for which $x_\mu = \nu$, $x_\nu = \mu$, then μ, ν form a "true couple". Clearly, there will be precisely one true couple, or none at all.
>
> If no "true couple" exists, let $g_1 = g_2 = g_3 = 0$. If there is a "true couple", let it be μ, ν and let the third of the numbers 1, 2, 3 be λ. Let $g_\mu = g_\nu = 1$, $g_\lambda = -2$.

In this game, evidently $M_{1,2} = M_{1,3} = M_{2,3} = 2$ (any two S_μ, S_ν in coalition can choose ν and μ respectively to form a "true couple" and thus take from the third player, S_λ, the amount 2!). Hence $M_{1,2} + M_{1,3} + M_{2,3} = 6 > 0$.

The reason that in this game any attempt at valuation is bound to fail is the following: In order to gain the amount 2, it is only necessary for any two of the three players to get together. They are then in a position to rob the third one without any ado, in spite of the fact that the rules of the game are strictly symmetrical, i.e., the game is formally fair.[11] The symmetry would imply the value for each player to be zero, but this is obviously wrong. If two players only want to, they can procure for themselves a gain of 2. How is this contradiction to be resolved?

2. Let us proceed systematically.

$$M_{1,2} + M_{1,3} + M_{2,3} \geq 0$$ [12]

always holds. For, evidently,

$$M_{1,2} = \mathrm{Max}_\xi \mathrm{Min}_\eta \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_1(pqr) + g_2(pqr))\xi_{pq}\eta_r$$

(according to our theorem on two-person games)

$$= \mathrm{Min}_\eta \mathrm{Max}_\xi \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} (g_1(pqr) + g_2(pqr))\xi_{pq}\eta_r$$

$$= - \mathrm{Max}_\eta \mathrm{Min}_\xi \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} g_3(pqr)\xi_{pq}\eta_r \quad .$$

We have to prove, therefore, that

$$\mathrm{Max}_{\eta'} \mathrm{Min}_{\xi'} \sum_{p,q,r} g_3(pqr)\xi'_{pq}\eta'_r + \mathrm{Max}_{\eta''} \mathrm{Min}_{\xi''} \sum_{p,q,r} g_2(pqr)\xi''_{pr}\eta''_q$$

$$+ \mathrm{Max}_{\eta'''} \mathrm{Min}_{\xi'''} \sum_{p,q,r} g_1(pqr)\xi'''_{qr}\eta'''_p \leq 0$$

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[11] This shows that our example is anything but a "pathological" game. Actually, it is a fairly frequent and typical case. Accordingly, we shall see in Section 4, 3 and Section 5, 1 that it is even the general case of a three-person game.

[12] Intuitively, this is immediately clear. S_2 and S_3 in coalition against S_1 can at best obtain $M_{2,3}$, hence S_1 for himself (against all others) can at best obtain $-M_{2,3}$ (because of our theorem on two-person games). Likewise, S_2 for himself can at best obtain $-M_{1,3}$. S_1 and S_2 in coalition can at best obtain $M_{1,2}$. "Unity is strength," i.e.,

$$-M_{2,3} - M_{1,3} \leq M_{1,2} \qquad M_{1,2} + M_{1,3} + M_{2,3} \geq 0 \quad .$$

i.e., for all systems $\bar{\eta}'$, $\bar{\eta}''$, $\bar{\eta}'''$

$$\mathrm{Min}_{\xi'} \sum_{p,q,r} g_3(pqr)\xi'_{pq}\bar{\eta}'_r + \mathrm{Min}_{\xi''} \sum_{p,q,r} g_2(pqr)\xi''_{pr}\bar{\eta}''_q$$

$$+ \mathrm{Min}_{\xi'''} \sum_{p,q,r} g_1(pqr)\xi'''_{qr}\bar{\eta}'''_p \leq 0 \quad .$$

This is actually the case. For, if we put

$$\xi'_{pq} = \bar{\eta}'''_q\bar{\eta}''_q, \quad \xi''_{pr} = \bar{\eta}'''_r\bar{\eta}'_r, \quad \xi'''_{qr} = \bar{\eta}''_q\bar{\eta}'_r$$

we obtain (because of $g_1 + g_2 + g_3 \equiv 0$)

$$\sum_{p,q,r} g_3(pqr)\bar{\eta}'_r\bar{\eta}''_q\bar{\eta}'''_p + \sum_{p,q,r} g_2(pqr)\bar{\eta}'_r\bar{\eta}''_q\bar{\eta}'''_p + \sum_{p,q,r} g_1(pqr)\bar{\eta}'_r\bar{\eta}''_p\bar{\eta}'''_q = 0 \quad .$$

We know already that the > sign actually occurs. The case of equality must, therefore, be regarded as a degenerate limiting case.

Let us assume that the player S_1 makes a claim on a gain of w_1 per play. How can he enforce it? Evidently, in two ways.

First of all, he can try to play alone. Essentially, this amounts to setting up a two-person game in which he is on one side and S_2, S_3 (in coalition) on the other. The value of a play is for him $- M_{2,3}$. This solution is acceptable only if $w_1 \leq - M_{2,3}$. Let us, therefore, assume $w_1 > - M_{2,3}$.

Then only the second possibility remains. S_1 must try to get S_2 or S_3 as an ally. In coalition with S_2 or S_3 he can win the sum $M_{1,2}$ or $M_{1,3}$ per play. Since he wants to keep w_1 for himself, he can offer the sum $M_{1,2} - w_1$ to S_2, or $M_{1,3} - w_1$ to S_3 as the price of the alliance. It is out of the question, however, that S_2 or S_3 will accept this offer if in coalition with each other they can win more than $(M_{1,2} - w_1) + (M_{1,3} - w_1)$ per play, i.e., if

$$(M_{1,2} - w_1) + (M_{1,3} - w_1) < M_{2,3}, \quad w_1 > 1/2\,(M_{1,2} + M_{1,3} - M_{2,3}) \quad .$$

Hence we can say that S_1 has no hope at all of satisfying a claim w_1 which is

$$> - M_{2,3}, \quad > 1/2\,(M_{1,2} + M_{1,3} - M_{2,3}) \quad .$$

The second number is $\geq$ the first (because $M_{1,2} + M_{1,3} + M_{2,3} \geq 0$), hence

$$w_1 \leq 1/2\ (M_{1,2} + M_{1,3} - M_{2,3}) = \bar{w}_1$$

must hold. In the same way, it can be shown that

$$w_2 \leq 1/2\ (M_{1,2} + M_{2,3} - M_{1,3}) = \bar{w}_2$$

$$w_3 \leq 1/2\ (M_{1,3} + M_{2,3} - M_{1,2}) = \bar{w}_3$$

must hold.

These upper limits for the claims $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$ of the three players can easily be obtained. If, say, S_1, S_2 enter into a coalition, they can achieve the gain $M_{1,2} = \bar{w}_1 + \bar{w}_2$ (against S_3). In the same way, S_1, S_3 or S_2, S_3 can, by entering an alliance, make sure of the gain $M_{1,3} = \bar{w}_1 + \bar{w}_3$ and $M_{2,3} = \bar{w}_2 + \bar{w}_3$ respectively. Hence, the highest possible and still completely justified claims of the three players S_1, S_2, S_3 in a play are $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$ respectively.

3. How is this valuation compatible with the impossibility we found in 1. to make a general valuation? If $M_{1,2} + M_{1,3} + M_{2,3} = 0$, no difficulty arises. In this case

$$\bar{w}_1 = -M_{2,3}, \quad \bar{w}_2 = -M_{1,3}, \quad \bar{w}_3 = M_{1,2}$$

i.e., each player can push his claims all by himself, without the help of another (and in the face of a possible coalition of his opponents). All three players can satisfy their claims simultaneously, and accordingly

$$\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = 0 \ .$$

The situation is different for $M_{1,2} + M_{1,3} + M_{2,3} > 0$. Since

$$\bar{w}_1 > -M_{2,3}, \quad \bar{w}_2 > -M_{1,3}, \quad \bar{w}_3 > -M_{1,2}$$

no player alone can satisfy his claim, and since

$$\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = 1/2\ (M_{1,2} + M_{1,3} + M_{2,3}) > 0$$

it is impossible for all three of them to obtain their desired gains at the same time. But because

$$\bar{w}_1 + \bar{w}_2 = M_{1,2} \qquad \bar{w}_1 + \bar{w}_3 = M_{1,3} \qquad \bar{w}_2 + \bar{w}_3 = M_{2,3}$$

each pair of players having entered an alliance (to rob the third) is assured of its success. Both players can completely satisfy their claims, while the third player will receive only $-M_{2,3}$, $-M_{1,3}$, $-M_{1,2}$ respectively per play, and therefore, his gain will fall short of his justified claim by the amount of $1/2\,(M_{1,2} + M_{1,3} + M_{2,3})$.

This can be formulated as follows: Each of the three players S_1, S_2, S_3 must endeavor to ally himself with another player. If he succeeds, he receives per play

$$1/2\,(M_{1,2} + M_{1,3} - M_{2,3}), \quad 1/2\,(M_{1,2} + M_{2,3} - M_{1,3}),$$
$$1/2\,(M_{1,3} + M_{2,3} - M_{1,2})$$

respectively. If he does not succeed (i.e., if the two others form a coalition), he receives only

$$-M_{2,3}, \quad -M_{1,3}, \quad -M_{1,2}$$

respectively. Still another way of describing the situation, and possibly the most concise one, would be the following:

α) A play has for the players S_1, S_2, S_3 the respective "basic values"

$$v_1 = 1/3\,(M_{1,2} + M_{1,3} - 2M_{2,3}), \quad v_2 = 1/3\,(M_{1,2} + M_{2,3} - 2M_{1,3}),$$
$$v_3 = 1/3\,(M_{1,3} + M_{2,3} - 2M_{1,2}) .$$

Since $v_1 + v_2 + v_3 = 0$, this is a proper valuation.

β) But for each of any two players entering an alliance against the third there exists the possibility to win $1/6\,D$ in excess of the above "basic values", while the third one sustains a loss of $1/3\,D$ (also in excess of his "basic value"). We have

$$D = M_{1,2} + M_{1,3} + M_{2,3} > 0 .$$[13]

(The first case, $D = M_{1,2} + M_{1,3} + M_{2,3} = 0$, can also be included in this formulation. Here, (α) is the result, and -- since $D = 0$ -- (β) is vacuous.)

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[13] Incidentally

$$v_1 = -M_{2,3} + 1/3\,D, \quad v_2 = -M_{1,3} + 1/3\,D, \quad v_3 = -M_{1,2} + 1/3\,D .$$

This solution shows immediately that the three-person game is essentially different from a game between two persons. The actual game strategy of the individual player recedes into the background. It does not offer anything new since the formation of coalitions (which is bound to take place) makes the play a two-person game. But the value of a play for the player does not only depend on the rules of the game. Rather, it is a question -- at least as soon as $D > 0$ -- of which of the three equally possible coalitions S_1, S_2; S_1, S_3; S_2, S_3 has been formed. A new element enters, which is entirely foreign to the stereotyped and well-balanced two-person game: struggle.

§5. PRELIMINARY REMARKS ON GAMES FOR $n > 3$

1. For $n > 3$ it has not yet been possible to obtain results of general validity. It may be that the best way to proceed will be in analogy to the cases $n = 2, 3$ which we have already dealt with. Let us recapitulate.

$n = 2$. We define

$$M = \text{Max}_\xi \text{ Min}_\eta \sum_{p,q} g_1(pq)\xi_p\eta_q \text{ .}^{14}$$

An individual play has for the players S_1, S_2 the values M, $-M$ respectively.

$n = 3$. We define

$$M_{1,2} = \text{Max}_\xi \text{ Min}_\eta \sum_{p,q,r} (g_1(pqr) + g_2(pqr))\xi_{pq}\eta_r$$

$$M_{1,3} = \text{Max}_\xi \text{ Min}_\eta \sum_{p,q,r} (g_1(pqr) + g_3(pqr))\xi_{pr}\eta_q$$

$$M_{2,3} = \text{Max}_\xi \text{ Min}_\eta \sum_{p,q,r} (g_2(pqr) + g_3(pqr))\xi_{qr}\eta_p \quad ^{14}$$

$$D = M_{1,2} + M_{1,3} + M_{2,3}$$

[14] The Max_ξ and Min_η are to be taken for all systems of probabilities, i.e., we require that

all $\xi_p \geq 0$, $\Sigma_p\xi_p = 1$; all $\xi_{pq} \geq 0$, $\Sigma_{pq}\xi_{pq} = 1$; etc.

and similarly,

all $\eta_p \geq 0$, $\Sigma_p\eta_p = 1$; etc.

$D \geq 0$, and we distinguish two cases, viz. $D = 0$ and $D > 0$.

$D = 0$. In this case, a play has for the players S_1, S_2, S_3 the values $-M_{2,3}$, $-M_{1,3}$, $-M_{1,2}$ respectively.

$D > 0$. In this case, a play has for the players S_1, S_2, S_3 the "basic values" $-M_{2,3} + 1/3\,D$, $-M_{1,3} + 1/3\,D$, $-M_{1,2} + 1/3\,D$ respectively. In order to obtain the correct values, however, another term has to be added to the "basic values". This is due to the fact that each of any two players who form a coalition against the third (no matter which two) can procure an additional gain of $1/6\,D$, while the third player sustains a loss of $1/3\,D$ per play (in excess of his basic value).

From this summary it becomes clear that the cases $n = 2$ and $n = 3$ with $D = 0$ are of the same type. The case $n = 3$, $D > 0$, however, as was already established at the end of Section 4, represents a new type. We shall denote these two types by the terms strictly-determined and symmetric non-strictly determined respectively, which are self-explanatory.

Is there any chance of reducing all games of strategy to these same two types, even if $n > 3$? Or are new complications to be anticipated? In particular, the possibility of asymmetrical non-strictly determined types has to be contemplated, i.e., types for which the significant possibilities of coalition-formation are not symmetrically distributed among all players. For $n = 3$ this possibility cannot arise. Any possible asymmetries of the rules of the game are entirely absorbed by the "basic values" of the three players. But all players are equally capable of forming coalitions, all three coalitions S_1, S_2; S_1, S_3; S_2, S_3 are equally possible. Let us investigate this question somewhat more closely.

2. In order to characterize a general n-person game we introduce the following constants:

$$M_{\{\mu_1,\mu_2,\ldots,\mu_k\}} = \mathrm{Max}_\xi \, \mathrm{Min}_\eta \sum_{p_1=1}^{\Sigma_1} \sum_{p_2=1}^{\Sigma_2} \cdots$$

$$\sum_{p_n=1}^{\Sigma_n} (g_{\mu_1}(p_1, \ldots, p_n) + \cdots + g_{\mu_k}(p_1, \ldots, p_n))\, \xi_{p_{\mu_1}, \ldots p_{\mu_k}} \eta_{p_{\nu_1}, \ldots, p_{\nu_{n-k}}}$$

where $\mu_1, \mu_2, \ldots, \mu_k$ are any k distinct numbers among the numbers $1, 2, \ldots, n$, and $\nu_1, \nu_2, \ldots, \nu_{n-k}$ are the remaining numbers (Max_ξ is to be taken for all $\xi_{p_{\mu_1},\ldots,p_{\mu_k}} \geq 0$, $\Sigma\xi_{p_{\mu_1},\ldots,p_{\mu_k}} = 1$, and Min_η for all $\eta_{p_{\nu_1},\ldots,p_{\nu_{n-k}}} \geq 0$, $\Sigma\eta_{p_{\nu_1},\ldots,p_{\nu_{n-k}}} = 1$). Clearly, $M_{\{\mu_1,\mu_2,\ldots,\mu_k\}}$ is the sum per play which the coalition of the players $S_{\mu_1}, \ldots, S_{\mu_k}$ is able to obtain from the coalition of the players $S_{\nu_1}, \ldots, S_{\nu_{n-k}}$ (since, in fact, the game is a two-person game).

Evidently, $M_{\{\ \}} = 0$. Our theorem on two-person games further implies that $M_{\{\mu_1,\ldots,\mu_k\}} = -M_{\{\nu_1,\ldots,\nu_{n-k}\}}$. Finally, let $\mu_1, \ldots, \mu_k$; $\nu_1, \ldots, \nu_\ell$; $\rho_1, \ldots, \rho_{n-k-\ell}$ be three subsets of $1, 2, \ldots, n$, complementary to each other. If the players $S_{\mu_1}, \ldots, S_{\mu_k}$ as well as $S_{\nu_1}, \ldots, S_{\nu_\ell}$ and $S_{\rho_1}, \ldots, S_{\rho_{n-k-\ell}}$ form fixed coalitions, this is a three-person game, and we have (putting primes on the quantities involved in this kind of game

$$M'_{1,2} = M_{\{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_\ell\}}$$

$$M'_{1,3} = M_{\{\mu_1,\ldots,\mu_k,\rho_1,\ldots,\rho_{n-k-\ell}\}} = -M_{\{\nu_1,\ldots,\nu_\ell\}}$$

$$M'_{2,3} = M_{\{\nu_1,\ldots,\nu\ ,\rho_1,\ldots,\rho_{n-k-\ell}\}} = -M_{\{\mu_1,\ \ldots,\ \mu_k\}} .$$

But according to our results for three-person games we have

$$M'_{1,2} + M'_{1,3} + M'_{2,3} \geq 0$$

i.e.,

$$M_{\{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_\ell\}} \geq M_{\{\mu_1,\ldots,\mu_k\}} + M_{\{\nu_1,\ldots,\nu_\ell\}} .$$

We recapitulate:

> A given n-person game assigns to each subset $\mu_1, \mu_2, \ldots, \mu_k$ of $1, 2, \ldots, n$ a constant $M_{\{\mu_1,\ldots,\mu_k\}}$ (that is the sum per play which the coalition of the players $S_{\mu_1}, \ldots, S_{\mu_k}$ is able to obtain from the coalition of the other players). The system of the constants $M_{\{\mu_1,\ldots,\mu_k\}}$ always satisfies

the following three conditions:

1. $M_{\{\ \}} = 0$

2. $M_{\{\mu_1,\ldots,\mu_k\}} + M_{\{\nu_1,\ldots,\nu_{n-k}\}} = 0$ if $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_{n-k}$ are complementary subsets of $1, 2, \ldots, n$.

3. $M_{\{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_\ell\}} \geq M_{\{\mu_1,\ldots,\mu_k\}} + M_{\{\nu_1,\ldots,\nu_\ell\}}$ if $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_\ell$ are disjoint subsets of $1, 2, \ldots, n$.[15]

It is not difficult to prove the converse, i.e., to specify, for each system of numbers $M_{\{\mathfrak{M}\}}$ ($\mathfrak{M}$ ranging over all 2^n subsets of $1, 2, \ldots, n$) satisfying the conditions 1. to 3., a game of strategy for which the above constants have precisely these values $M_{\{\mathfrak{M}\}}$. We refrain from discussing here such an example, which is not deep at all.

3. I venture the conjecture that the complex of valuations and coalitions in a game of strategy is determined by these 2^n constants alone. We have seen that this is true for $n = 2, 3$;[16] for $n > 3$ a general proof has yet to be found. While in the case $n = 2$ no coalition at all is possible and for $n = 3$ only one type of coalition is conceivable (i.e., "two against one"), the number of possibilities increases rapidly for $n > 3$. If $n = 4$ one must already decide whether a coalition "three against one" or "two against two" is going to be formed, i.e., which alliance offers the best chances to the participants. If $n = 4$ it is still possible to discuss the principal cases (on the basis of the $M_{\{\mathfrak{M}\}}$ alone!), but a satisfactory general theory is as yet lacking.

If our conjecture is correct, we have brought all games of strategy into a natural and final normal form. Each system of 2^n constants $M_{\{\mathfrak{M}\}}$ satisfying the conditions 1. to 3. represents a class of

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[15] Intuitively, this assertion is as clear as the one considered in the footnote 12, p. 34.

[16] For $n = 2$

$$M_{\{\ \}} = 0, \quad M_{\{1\}} = M, \quad M_{\{2\}} = -M, \quad M_{\{1,2\}} = 0$$

and for $n = 3$

$$M_{\{\ \}} = 0,\ M_{\{1\}} = -M_{2,3},\ M_{\{2\}} = -M_{1,3},\ M_{\{3\}} = -M_{1,2},\ M_{\{1,2\}} = M_{1,2},$$

$$M_{\{1,3\}} = M_{1,3},\ M_{\{2,3\}} = M_{2,3},\ M_{\{1,2,3\}} = 0 \ .$$

"tactically equivalent" games of strategy.[17]

In conclusion I would like to add that a later publication will contain numerical calculations of some well-known two-person games (Poker, though with certain schematical simplifications, Baccarat). The agreement of the results with the well-known rules of thumb of the games (e.g., proof of the necessity to "bluff" in poker) may be regarded as an empirical corroboration of the results of our theory.

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[17] Another possibility of normalizing the $M_{\{\mathfrak{M}\}}$ consists of introducing "basic values" $v_1, v_2, \ldots, v_n$ for the players $S_1, S_2, \ldots, S_n$ -- in analogy to the "values" (of a play for $n = 2$ and the "basic values" for $n = 3$. For the values exceeding the v_i one obtains, of course, the new constants

$$M^*_{\{\mathfrak{M}\}} = M_{\{\mathfrak{M}\}} - \sum_{p \text{ in } \mathfrak{M}} v_p$$

Appropriately, the v_p are chosen such that

$$M^*_{\{1\}} = M^*_{\{2\}} = \cdots = M^*_{\{n\}}, \quad v_1 + v_2 + \cdots + v_n = 0$$

i.e., all players playing for themselves are equally strong, and differences are due only to the various possibilities of coalitions.

(From 1. - 3. it follows easily that the common value of

$$M^*_{\{1\}}, M^*_{\{2\}}, \ldots, M^*_{\{n\}} \leq 0 \quad .$$

If it is zero, all $M^*_{\mathfrak{M}} = 0$, i.e., after payment of the "basic values" the play is strictly determined. Hence this common value represents a kind of measure of how non-strictly determined the game is, i.e., a measure of the tactical possibilities the game offers.)