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Vincent Vatter

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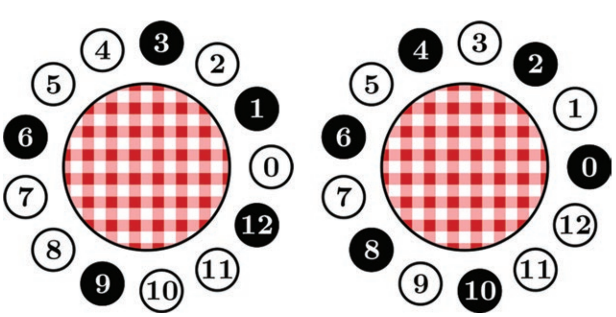
Social Distancing, Primes, and Perrin Numbers

Vincent Vatter

Harry Dweighter is a longtime restaurant owner who is famous for pondering the mathematics behind flipping pancakes (Problem E2569, *Amer. Math. Monthly*, **82** [1975], 1010). With the COVID-19 pandemic, Harry has become less harried, and he has been thinking about the possible seating arrangements at his restaurant's large round outdoor table. He loves displaying his spectacular pancake flipping abilities and hopes to have as many people at the table as possible.

The table would ordinarily seat 13, but social distancing guidelines state that no two people may sit in adjacent seats. So Harry got to wondering: in how many different ways can the table be *full*, in the sense that no additional customer is able to sit down without violating the distancing guidelines? For example, both tables shown in figure 1 are full, even though they have different numbers of customers.

Figure 1. Two full tables with 13 chairs.

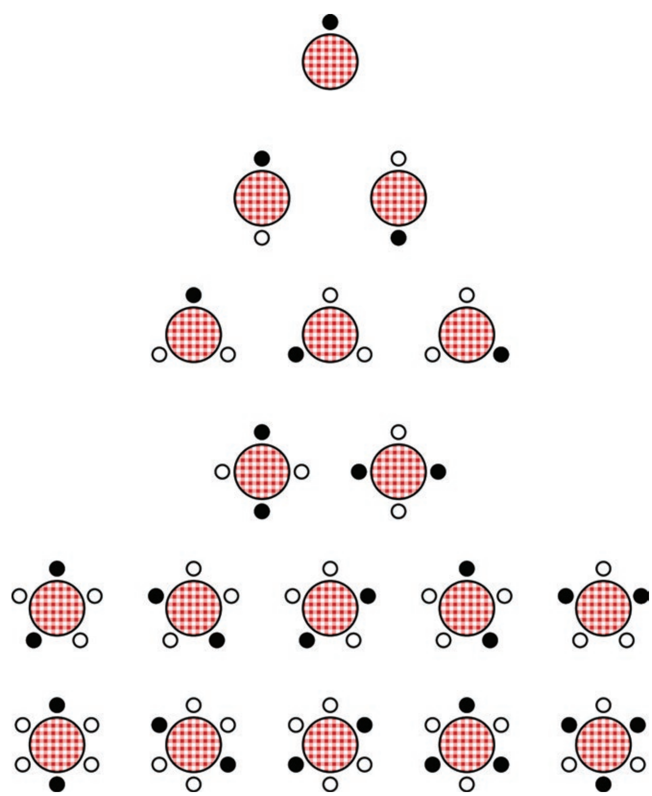


Let a_n denote the number of full tables with n seats. We compute the first six values of this sequence by constructing all of the full tables in figure 2. (We also have $a_0 = 1$ because there is precisely one way for no one to sit at a table with zero seats). Figure 2 shows that Harry considers the chairs to be distinguished (he thinks of them as numbered), but the customers are indistinguishable to him. Thus, two seating arrangements are the same if and only if they have diners in identical positions.

A Recurrence

Between any two customers in a full table, there must be either one or two empty seats—at least one to satisfy the social distancing guidelines but fewer than three because otherwise another

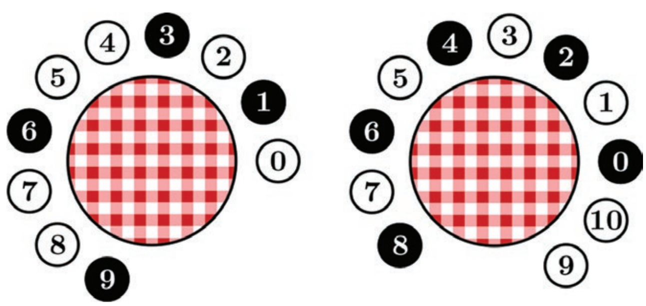
Figure 2. All full tables with one to six seats. Note that full tables with six seats have either two or three customers.



customer could safely sit in one of those seats. Thus, a full table is one for which every customer is separated from each of their neighbors by one or two seats.

This observation allows us to derive a recurrence for a_n. Given a full table with n chairs and at least two customers, we can add either one or two empty chairs and an occupied chair after the last customer (the customer sitting in the occupied seat with the greatest number) to obtain a full table with either $n + 2$ or $n + 3$ chairs. For example, if we start with the full table with 10 chairs shown on the left of figure 3 and add two empty chairs and an occupied chair after the customer sitting in seat 9, then we obtain the full table shown on the left of figure 1. The same relation holds between the tables shown on the right of these figures, although in this case we only add one empty chair.

Figure 3. By adding chairs to these tables, we obtain the tables shown in figure 1.



We need to check that these new tables are still full (subject to the social distancing guidelines). By design, the new last customer is separated by one or two empty seats from the previous last customer. The new last customer is also separated from the first customer by the same number of seats as the previous last customer was. Therefore, these new tables are indeed full.

Working backward, full tables on n chairs can be obtained by adding a new customer and two or three new chairs to full tables with $n - 2$ or $n - 3$ chairs, so long as the smaller full tables have at least two customers. Because all full tables with four or more seats must have at least two diners, we conclude that $a_n = a_{n-2} + a_{n-3}$ whenever $n - 3 \geq 4$. In fact, our previous computations show that this recurrence also holds for $n = 6$.

Using the recurrence, Harry determines that $a_{13} = 39$: there are 39 different ways to fill his 13-person table (see table 1).

Table 1. The number of full tables with n diners.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	1	1	2	3	2	5	5	7	10	12	17	22	29	39

To the OEIS!

Now that he can compute terms of his sequence, Harry can look for it in the On-Line Encyclopedia of Integer Sequences (*https://oeis.org*). He knows that it is often a good idea to exclude initial terms, so Harry searches for 2, 3, 2, 5, 5, 7, 10, 12 (the second through ninth terms). The OEIS dutifully returns the entry for the *Perrin numbers*, which are defined by $b_0 = 3$, $b_1 = 0$, $b_2 = 2$, and $b_n = b_{n-2} + b_{n-3}$ for $n \geq 3$.

Table 2. Seating arrangements and Perrin numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
a_n	1	1	2	3	2	5	5	7	10	12	17	22	29	39
b_n	3	0	2	3	2	5	5	7	10	12	17	22	29	39

The sequence b_n differs from a_n on the first two values, but because a_n and b_n satisfy the same recurrence for all $n \geq 6$ and agree for $n = 2, 3, 4$, and 5, it follows by induction that the two sequences are equal for all $n \geq 2$.

Prime Tables

From the OEIS, Harry learns that the Perrin numbers have a remarkable property, first conjectured by their namesake in 1899.

Perrin's theorem. Whenever p is prime, b_p is divisible by p.

For example, $b_{13} = 39$, which is evenly divisible by the prime 13.

Perrin's theorem has several known proofs, but they are mostly algebraic. From a restauranteur's perspective, however, Perrin's theorem has a particularly natural combinatorial proof.

We call two full tables *rotationally equivalent* if one of them can be rotated (either clockwise or counterclockwise) to obtain the other. Every full table is rotationally equivalent to itself; if one full table is rotationally equivalent to another, then that second full table is also rotationally equivalent to the first; and finally, if one full table is rotationally equivalent to a second full table and that table is rotationally equivalent to a third full table, then the first table is rotationally equivalent to the third.

The previous three facts show that rotational equivalence is an equivalence relation. Consequently, the full tables are *partitioned* into rotational equivalence classes: each table is in precisely one such class. For example, an examination of figure 2 shows that all of the full tables with five seats form a single rotational equivalence class; whereas, the full tables with six seats split into two rotational equivalence classes, one consisting of the tables with three occupied seats and the other consisting of those with two occupied seats.

Now, suppose p is prime. We will first show that every rotational equivalence class of full tables on p seats contains precisely p such seating arrangements. Suppose, to the contrary, that some rotational equivalence class has fewer full tables. Thus, in this equivalence class, there is a full table that is left unchanged after rotation by k seats, for some integer $1 \leq k < p$. Because p is prime, the greatest common divisor of k and p is 1, and an important number theoretic result known as Bézout's identity, which can be proved using Euclid's division algorithm, guarantees the existence of integers x and y with $xk + yp = 1$.

As the full table of interest is unchanged after rotation by k seats, the table is also unchanged after rotation by xk seats (rotation by k seats x times). Furthermore, *every* full table must be unchanged after rotation by p seats (rotation by p seats is identical to no rotation), and is thus unchanged after rotation by yp seats. Therefore, our assumed full table is unchanged after rotation by $xk + yp = 1$ seat. This, however, is impossible—the seats that are occupied after a rotation by one seat must have been empty before the rotation due to the social distancing guidelines.

To complete the proof, note that the a_p full tables on p seats are partitioned into some number, say m, of rotational equivalence classes. We proved above that each class has precisely p members, so that $a_p = mp$, as desired.

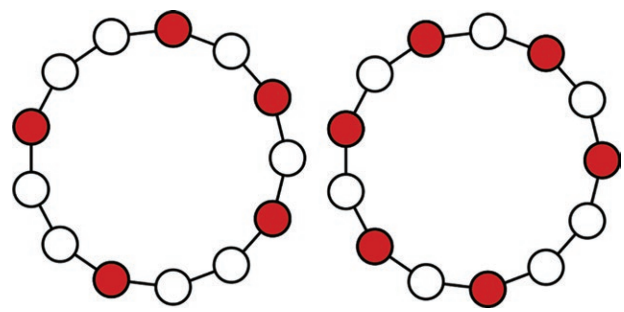
Generalizations

There are two components to our proof of Perrin's theorem: the combinatorial interpretation of the Perrin numbers and the use of rotation. Both of these aspects can be explored further by the interested reader.

Our interpretation of the Perrin numbers as counting seating arrangements during a pandemic is a thinly veiled instance of counting what are known as maximal independent sets in graphs. By a *graph*, we don't mean plots of functions from calculus, but instead the types of graphs studied in graph theory consisting of a set of *vertices* (typically represented by dots) and a set of *edges* (typically drawn as lines) connecting precisely two vertices. An *independent set* is a set of vertices, no two connected by an edge, and a *maximal independent* set is an independent set that is not a proper subset of another.

A round table with n seats can be modeled by a *cycle graph* on n vertices with edges connecting the vertices in a loop. We denote this graph by C_n. An independent set of C_n can be interpreted as a seating arrangement that satisfies the social distancing guidelines, while a maximal independent set of C_n is a full table. Figure 4 shows two examples. The fact that the Perrin numbers count the number of maximal independent sets of C_n was first observed by Z. Füredi in 1987 (The number of maximal independent sets in connected graphs, *J. Graph Theory*, **11**, 463–470).

Figure 4. The full tables of figure 1 translated to maximal independent sets of the cycle C_{13}.



As for rotational proofs of congruences, the following result encapsulates the technique.

Lemma. Let S be a finite set, $f: S \to S$, and p a prime such that $f^p(x) = x$ for all $x \in S$. Then, the number of nonfixed points of S (those not satisfying $f(x) = x$) is divisible by p.

As in our argument for full tables, the set S can be expressed as the disjoint union of *orbits*, which are sets of the form $\{x, f(x), \ldots, f^{p-1}(x)\}$. Because p is prime, each orbit has size 1 or p, and an orbit has size 1 if and only if it consists of a fixed point of f.

In our application, the set S consists of all the full tables and f is clockwise rotation by one seat; we observed that f has no fixed points. We can also apply this lemma with cycle graphs and vertices of multiple colors to prove Fermat's little theorem: for every prime p and every positive integer k, $k^p - k$ is divisible by p.

We leave it to the reader to see how our restaurateur might view Fermat's little theorem in terms of prepandemic seating arrangements in which all the seats are occupied and the customers can be distinguished by their choice of one of k sandwiches. For more applications we refer to the 2005 article "Combinatorial proofs of Fermat's, Lucas's, and Wilson's theorems" by Anderson, Benjamin, and Rouse in the *American Mathematical Monthly.*

***Vincent Vatter** is a professor in the Department of Mathematics at the University of Florida, where the weather allows him to enjoy socially distanced outdoor seating year-round.*

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