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A VERY SHORT HISTORY OF ULTRAFINITISM

ROSE M. CHERUBIN AND MIRCO A. MANNUCCI

To the memory of our unforgettable friend Stanley Tennenbaum (1927-2005), Mathematician, Educator, Free Spirit.

In this first of a series of papers on ultrafinitistic themes, we offer a short history and a conceptual pre-history of ultrafinistism. While the ancient Greeks did not have a theory of the ultrafinite, they did have two words, *murios* and *apeiron*, that express an awareness of crucial and often underemphasized features of the ultrafinite, viz. feasibility, and transcendence of limits within a context. We trace the flowering of these insights in the work of Van Dantzig, Parikh, Nelson and others, concluding with a summary of requirements which we think a satisfactory general theory of the ultrafinite should satisfy.

First papers often tend to take on the character of manifestos, road maps, or both, and this one is no exception. It is the revised version of an invited conference talk, and was aimed at a general audience of philosophers, logicians, computer scientists, and mathematicians. It is therefore not meant to be a detailed investigation. Rather, some proposals are advanced, and questions raised, which will be explored in subsequent works of the series.

Our chief hope is that readers will find the overall flavor somewhat "Tennenbaumian".

§1. Introduction: The radical Wing of constructivism. In their *Constructivism in Mathematics*[1], A. Troelstra and D. Van Dalen dedicate only a small section to Ultrafinitism (UF in the following). This is no accident: as they themselves explain therein, there is no consistent model theory for ultrafinitistic mathematics. It is well-known that there is a plethora of models for intuitionist logic and mathematics: realizability models, Kripke models and their generalizations based on category theory, for example. Thus, a skeptical mathematician who does not feel moved to embrace the intuitionist faith (and most do not), can still understand and enjoy the intuitionist's viewpoint while remaining all along within the confines of classical mathematics. Model theory creates, as it were, the bridge between quite different worlds.

[1] [27].

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It is hoped that something similar were available for the more radical positions that go under the common banner of Ultrafinitism. To be sure, in the fifteen years since the publication of the above-cited book, some proposals have emerged to fill the void. It is our opinion, though, that nothing comparable to the sturdy structure of model theory for intuitionism is available thus far. This article is the first in a series which aims at proposing several independent but related frameworks for UF.

Before embarking on this task, though, an obvious question has to be addressed first: what is Ultrafinitism, really? As it turns out, a satisfactory answer has proved to be somewhat elusive. A simple answer: all positions in foundations of mathematics that are more radical than traditional constructivism (in its various flavors). But this begs the question. So, what is it that makes a foundational program that radical?

There is at least one common denominator for ultrafinitists, namely the deep-seated *mistrust of the infinite, both actual and potential*. Having said that, it would be tempting to conclude that UF is quite simply the rejection of infinity in favor of the study of finite structures (finite sets, finite categories), a program that has been partly carried out in some quarters[2].

Luckily (or unluckily, depending on reader's taste), things are not that straightforward, for two substantive reasons:

- First, the rejection of infinitary methods, even the ones based on the so-called potential infinite, must be applied at *all levels*, including that of the meta-mathematics and that of the logical rules. Both syntax and semantics must fit the ultrafinitistic paradigm. Approaches such as Finite Model Theory are simply not radical enough for the task at hand, as they are still grounded in a semantics and syntax that are saturated with infinite concepts[3].
- Second, barring one term in the dichotomy finite-infinite, is, paradoxically, an admission of guilt: the denier implicitly agrees that the dichotomy itself is valid. But is it? Perhaps what is here black and white should be replaced with various shades of grey.

These two points must be addressed by a convincing model theory of Ultrafinitism. This means that such a model theory, assuming that anything like it can be produced, must be able to generate classical (or intuitionistic) structures, let us call them ultrafinitistic universes, in which an ultrafinitist mathematician can happily live. What this has to mean in practice, if one takes a moment to think about it, is that denizens of those universes should be allowed to treat some finite objects as, *de facto*, infinite. And, indeed, logicians are quite used to the "inside versus outside" pattern of thought; regarding the

[2]See, for instance, [28].

[3]For example, Trakhtenbrot's Theorem states that first order validity for finite models is not even recursively enumerable. On the other hand the theory of finite *fields* is decidable.

minimal model of ZF, for example, inside of it countable ordinals look and feel like enormous cardinals. One way of stating our ultimate goal is this: if we could somehow "squeeze" the minimal model below $\aleph_0$, we could get what we are looking for.

Only one major obstacle stands in the way: *the apparently absolute character of the natural number series*.

But it is now time for a bit of history ...

§2. Short history and prehistory of ultrafinitism.

—*The trouble with eternity is that one never knows when it will end.*
Tom Stoppard, *Rosenkrantz and Guildenstern Are Dead*.

Ultrafinitism has, ironically, a very long prehistory, encroaching even upon the domains of cultural anthropology and child cognitive psychology. For instance we know that some "primitive" cultures, and also children of a certain age, do not seem to have a notion of arbitrarily large numbers. To them, the natural number series looks a bit like: One, two, three, ... many! An exploration of these alluring territories would bring us too far afield, so we shall restrict our tale to the traditional beginning of Western culture, the Greeks.

Ancient Greek mathematics does not explicitly treat the ultrafinite. It is therefore all the more interesting to note that early Greek poetry, philosophy, and historical writing incorporate two notions that are quite relevant for the study of the ultrafinite. These notions are epitomized by the two words: *murios* (μυρίος) and *apeirōn* (ἀπείρων).

2.1. Murios. The word *murios*, root of the English "myriad", has two basic senses in ancient Greek writing. These senses are "very many" or "a lot of"; and "ten thousand". The first sense denotes an aggregate or quantity whose exact number is either not known or not relevant; the second denotes a precise number. With some exceptions, to be given below, the syntax and context make clear which sense is intended in each case. It is part of the aim of this paper to draw attention to the importance of contextualized usage in understanding the ultrafinite.

The earliest occurrences of the term *murios* appear in the oldest extant Greek writing, viz., Homer's *Iliad* and *Odyssey*. In Homer, all 32 instances of forms of *murios* have the sense "very many" or "a lot of". Translations often render the word as "numberless", "countless", or "without measure". But what exactly does this mean? Does *murios* refer to an indefinite number or quantity, to an infinite number or quantity, to a number or quantity that is finite and well-defined but that is not feasibly countable for some reason, or to a number or quantity that the speaker deems large but unnecessary to count? Our investigation reveals that Homer tends to use the term in the last two ways, that is, to refer to numbers or quantities for which a count or measure would be unfeasible, unnecessary, or not to the point. In general, Homer uses the word in situations where it is not important to know the exact number of things in a large group, or the exact quantity of some large mass.

Some representative examples of Homer's usages of *murios*:

(a) At *Iliad* 2.468[4], the Achaeans who take up a position on the banks of the Scamander are *murioi* (plural adjective), "such as grow the leaves and flowers in season". The leaves and flowers are certainly not infinite in number, nor are they indefinite in number, but they are not practicably countable, and there is no reason to do so—it is enough to know that there are very many all over.

(b) In the previous example, the Achaeans must have numbered at least in the thousands. *Murios* can, however, be used to refer to much smaller groups. At *Iliad* 4.434, the clamoring noise made by the Trojans is compared to the noise made by *muriai* ewes who are being milked in the courtyard of a very wealthy man (the ewes are bleating for their lambs). The number of ewes owned by a man of much property would certainly be many more than the number owned by someone of more moderate means, but that rich man's ewes—especially if they all fit in a courtyard—must number at most in the low hundreds. This suggests that the ewes are said to be *muriai* in number because there is a comparatively large number of them; because there is no need to count them (a man who had 120 ewes and was considered very wealthy would not cease to be considered very wealthy if he lost one or even ten of them); and possibly because it might not be practicable to count them (they might be moving around, and they all look rather like one another).

Similarly, at *Odyssey* 17.422 Odysseus says he had *murioi* slaves at his home in Ithaca before he left for the Trojan war. The word for "slaves" in this case is *dmōes*, indicating that these are prisoners of war. Given what we know of archaic Greek social and economic structures, the number of slaves of this type a man in his position could have held must have been in the dozens at most. The key to Odysseus' use of the term is the context. The sentence as a whole reads: "And I had *murioi* slaves indeed, and the many other things through which one lives well and is called wealthy". That is, the quantities of slaves and of other resources that he commanded were large enough to enable

[4]All translations from the Greek are due to Cherubin. Following the standard form of reference in humanities publishing *Iliad* 2.468 refers to verse 468 of Book Two of the *Iliad*; *Odyssey* 17.422 refers to verse 422 of Book 17 of the *Odyssey*, etc.

him to be considered wealthy. The exact number of slaves might have been countable, but it would have been beside the point to count them.

(c) *Murios* can also refer to quantities that are not such as to be counted. At *Odyssey* 15.452, a kidnapped son of a king is projected to fetch a *murios* price as a slave. Here *murios* must mean "very large", "vast". This is by no means to say that the price will be infinite or indefinite, for a price could not be thus. Rather, the situation is that the exact price cannot yet be estimated, and the characters have no need to estimate it (i.e., they are not trying to raise a specific amount of money).

There are also instances of *murios* in Homer that refer to kinds of things that do not seem to be measurable or calculable. At *Iliad* 18.88, Achilles says that his mother Thetis will suffer *murios* grief (*penthos*) at the death of Achilles, which is imminent. At 20.282, *murios* distress comes over the eyes of Aeneas as he battles Achilles. The usual translation of *murios* here is "measureless". This translation may be somewhat misleading if it is taken literally, as there is no evidence that the Greeks thought that smaller amounts of grief and distress were necessarily such as to be measurable or measured. A more appropriate translation might be "vast" or "overwhelming". It is possible that Achilles means that Thetis will suffer grief so vast that she will never exhaust it nor plumb its depths even though she is immortal; but it is also possible that Homer did not consider whether grief or distress could be unending and infinite or indefinite in scope.

The epic poet Hesiod (8th-7th BCE) and the historian Herodotus (5th BCE) sometimes use *murios* in the senses in which Homer does, but they also use it to mean ten thousand. With a very few exceptions, the syntax and context make clear in each instance which meaning is present. At *Works and Days* 252, Hesiod says that Zeus has *tris murioi* immortals (i.e. divinities of various kinds) who keep watch over mortals, marking the crooked and unjust humans for punishment. *Tris* means three times or thrice, and there is no parallel in Greek for understanding *tris murioi* as "three times many" or "three times a lot". There are parallels for understanding *tris* with an expression of quantity as three times a specific number; and the specific number associated with *murios* is ten thousand. Therefore *tris murios* should indicate thirty thousand.

Some instances of *murios* in Herodotus clearly refer to quantities of ten thousand; some clearly refer to large amounts whose exact quantities are unspecified; and a few are ambiguous but do not suggest any meaning other than these two.

(d) At 1.192.3, Herodotus says that the satrap Tritantaechmes had so much income from his subjects that he was able to maintain not only warhorses but eight hundred (*oktakosioi*) other breeding stallions and *hexakischiliai kai muriai* mares. *Hexakischiliai* means six times one thousand, so that the whole expression should read six thousand plus *muriai*. The next line tells us that there are twenty (*eikosi*) mares for every stallion, so that the total number of mares must be sixteen thousand, and *muriai* must mean ten thousand (it is a plural adjective to agree with the noun). The case is similar at 2.142.2-3. Here Herodotus says that three hundred (*triēkosiai*) generations of men come to *muriai* years since three generations come to one hundred (*hekaton*) years. Clearly, *muriai* means ten thousand here.

(e) In some places, Herodotus cannot be using *murios* to mean ten thousand, and it is the context that shows this. At 2.37.3, for example, describing the activities of Egyptian priests, he says that they fulfill *muriai* religious rituals, *hōs eipein logōi*. He may in fact mean that they fulfill *muriai* rituals each day, since the rest of the sentence speaks of their daily bathing routines. Herodotus does not give any details about the rituals or their number, and *hōs eipein logōi* means "so to speak". Thus Herodotus seems to be signalling that he is not giving an exact figure, and *muriai* must simply mean "a great many". At 2.148.6, Herodotus reports that the upper chambers of the Egyptian Labyrinth *thōma murion pareichonto*, furnished much wonder, so remarkably were they built and decorated. Certainly no particular amount of wonder is being specified here.

(f) Some occurrences of *murios* are ambiguous in a way that is of interest for the study of the ultrafinite. At 1.126.5, Cyrus sets the Persians the enormous task of clearing an area of eighteen or twenty stadia (2 1/4 or 2 1/2 miles) on each side in one day, and orders a feast for them the next. He tells them that if they obey him, they will have feasts and *muria* other good things without toil or slavery, but that if they do not obey him, they will have *anarithmētoi* toils like that of the previous day. That is, Cyrus is contrasting *muria* good things with *anarithmētoi* bad ones. Is he asking the Persians to consider this a choice between comparable large quantities? If so, a *murios* amount would be *anarithmētos*, which can mean either "unnumbered" or "innumerable", "numberless". It is also possible that *murios* is supposed to mean ten thousand, so that the magnitude of the undesirable consequences of defying Cyrus is greater than the great magnitude of the advantages of obeying him. If that is the meaning, Herodotus may be using *murios* in a somewhat figurative sense, as when one says that one has "ten thousand things to do today".

Murios, then, referred in the earliest recorded Greek thought to large numbers or amounts. When it did not refer to an exact figure of ten thousand, it referred to numbers or amounts for which the speaker did not have an exact count or measurement. Our analysis indicates that the speaker might lack such a count or measurement either because the mass or aggregate in question could not practicably be counted or measured under the circumstances, or because an exact count or measurement would not add anything to the point the speaker was making. In most cases it is clear that the numbers and amounts referred to as *murios* were determinate and finite, and could with appropriate technology be counted or measured. In instances where it is not clear whether that which is referred to as *murios* is supposed to be such as to admit of measuring or counting (Thetis' grief, for example; and Cyrus' *murios* good things if they are comparable to the *anarithmētos*), there is no evidence as to whether the *murios* thing or things are supposed to be infinite or indefinite in scope. Indeed, there is no evidence that these early writers thought about this point. (This is perhaps why *anarithmētos* can mean both "unnumbered" and "innumerable", and why it is often difficult to tell which might be meant and whether a writer has in mind any distinction between them.)

When *murios* does not mean "ten thousand", context determines the order of quantity to which it refers. Any number or amount that is considered to be "a lot" or "many" with respect to the circumstances in which it is found can be called *murios*. Leaves and flowers in summer near the Scamander number many more than those of other seasons, perhaps in the millions; but the rich man's ewes are *muriai* too, even if they number perhaps a hundred. They are several times more than the average farmer has, and they may fill the courtyard so much that they cannot easily be counted.

In this way groups and extents that would be acknowledged to be finite and perhaps effectively measurable or countable under some circumstances would be called *murios* when actual circumstances or purposes made counting or measuring impossible, impractical, or unnecessary. This step would be equivalent to treating finite things as *de facto* infinite. This freedom, we hold, is precisely what a convincing model theory of Ultrafinitism should allow us to do.

2.2. Apeirōn. Since *murios* seems to refer overwhelmingly to determinate and finite quantities, it is useful to note that Greek had ways of referring to quantities that were indeterminate, unlimited, indefinite, or infinite. The most significant of these, for our purposes, was the word *apeiros* or *apeirōn* (m., f.)/*apeiron* (n.).

The etymology of this word is generally understood to be *peirar* or *peras*, "limit" or "boundary", plus alpha privative, signifying negation: literally, "not limited" or "lacking boundary"[5]. Etymology alone does not tell us the range of uses of the term or the ways in which it was understood, so we must again consider its occurrences in the earliest sources. The term appears as early as Homer, in whose poems it generally refers to things that are vast in extent, depth, or intensity.

[5]For an excellent review, see [21, pages 67-70]. See also [14, pages 231-239].

Homer uses *apeiros* most frequently of expanses of land or sea. In each case, the *apeiros*/*apeirōn* thing is vast in breadth or depth; whether its limits are determinable is not clear from the context, but limits do seem to be implied in these cases. Some instances may imply a surpassing of some sort of boundaries or borders (though not necessarily of all boundaries or borders). At *Iliad* 24.342 and *Odyssey* 1.98 and 5.46, a god swiftly crosses the *apeirōn* earth. Within the context, it is clear that the poet means that the divinity covers a vast distance quickly. There may be a further implication that the gods transcend or traverse boundaries (be these natural features or human institutions) with ease, so that the world has no internal borders for them. Similarly, in *Odyssey* 17.418, the expression *kat' apeirona gaian*, often translated as "through[out] the boundless earth", is used to suggest that something is spread over the whole earth. What is spread covers a vast expanse, and it also crosses all boundaries on the earth.

Two other Homeric examples are of interest. At *Odyssey* 7.286, a sleep is described as *apeirōn*, meaning either that it is very deep, or unbroken, or both. At *Odyssey* 8.340, strong bonds are *apeirōn*, surpassing limits of a god's strength, and so unbreakable.

Hesiod also uses *apeirōn* to describe things that extend all over the earth, but also uses the word once in reference to a number. In *Shield of Heracles* 472, the word refers to a large number of people from a great city involved in the funeral of a leader; the sense seems to be that there were uncountably many, and possibly that the leader's dominion had been vast.

Herodotus (5th century BCE) uses *apeirōn* in two cases where its meaning clearly derives from the privative of *peirar*[6]. In 5.9 he uses it to refer to a wilderness beyond Thracian settlements. In 1.204 a plain is *apeiron*, perhaps hugely or indeterminately vast. In both cases, Herodotus knows that the lands are finite in extent (he identifies the peoples who live beyond them). The contexts suggest that he means that these lands are vast and that their exact boundaries are not known. He may also have in mind that they cannot be easily, if at all, traversed by humans.

The first and perhaps best-known philosophical use of *apeiron* is in the reports about the work of Anaximander's in the sixth century BCE. Anaximander is reported to have held that the source of all familiar things, the fundamental generative stuff of the cosmos, was something apeiron. The testimonia report that the *apeiron* was eternal in duration, unlimited or indeterminate in extent, and qualitatively indeterminate.

All of the familiar cosmos, for Anaximander, arose from the *apeiron*.

In his [14], Kahn holds that Anaximanders *apeiron* is "primarily a huge, inexhaustible mass, stretching away endlessly in every direction". The *apeiron*

[6]Herodotus also uses a homophone word that is derived from another root, so we have only included instances where context clearly indicates that the word is the one derived from *peirar*.

surely must be at least that, but there is no reason to think that it is primarily that. As McKirahan [19] notes, the discussions of Anaximander in Aristotle and the Peripatetics make clear that the *apeiron* must also be a stuff of indefinite kind or quality. It must be this because it is supposed to be able to give rise to every kind of thing, and because (according to Aristotle in *Physics* Gamma 3) (see [3]) if any one kind of thing, e.g. fire, was *apeiron* it would overcome and destroy everything else. Clearly that has not happened. The question would then seem to arise as to why the indefinite stuff of unlimited extent does not overwhelm all specific stuffs, and result in a universe that is wholly indefinite, and so not a cosmos, i.e. an ordered universe. The answer to this question may perhaps be found in Anaximander's contention that the *apeiron* is fundamentally unstable. It is indeterminate even in its state. According to the ancient reports, the *apeiron* was supposed to be always in motion. Through this, somehow, "opposites" (hot and cold, wet and dry, light and dark, perhaps others) separate off and interact to form the world of familiar things. Eventually, according to the only apparent quotation we have from Anaximander, the things or opposites "pay penalty and restitution to one another for their injustice, according to the arrangement of time", and perish back into the *apeiron*, whence the cycle begins anew. (What the "injustice" is remains a subject of much speculation; the word used suggests that it may be some sort of imbalance or encroachment.) It is worth noting that for Anaximander the whole cosmos may be at the same stage in the cycle at any given time, but that is not the only possibility. It is also possible that different parts of the cosmos are at different stages in the cycle, so that qualitative and quantitative indeterminacy are present in some regions and not in others, and the whole therefore remains *apeiron* in some respects.

We may note that so far no instance of *apeirōn* clearly meant "infinite". Only one, Anaximander's, could possibly involve an infinite extent, and even in that case it is not clear that the extent is infinite; it may be indefinite or inexhaustible without being infinite. Anaximander's stuff is eternal, i.e. always in existence, but it is not at all clear that a sixth century Greek would have taken "always" to mean an infinite amount of time. Whether any Greek of the 8th to 5th centuries BCE conceived of quantities or magnitudes in a way that denoted what we would call infinity is not certain.

It is sometimes thought that Zeno of Elea (5th century BCE) spoke of the infinite, but there is good evidence that he had quite a different focus. It is only in the arguments concerning plurality that are preserved by Simplicius that we find what may be quotations from Zeno's work (regarding his arguments concerning motion and place we have only reports and paraphrases or interpretations)[7]. In fragments DK29 B1 and B2, Zeno argued "from saying

[7]It is possible that some of Zeno's paradoxes of motion dealt with infinitely long sequences of steps. Aristotle suggests that they did. Aristotle used the word *apeiron* to describe these

that multiple (*polla*, many) things are, saying opposite things follows". In particular, if we say that multiple things are, then we must conclude that "the same things must be so large as to be *apeira* (neuter plural) and so small as to lack magnitude (*megethos*)". Zeno was evidently interested in the claim that there are multiple things with spatial magnitude, and it appears from the fragments that he thought that the possibilities for analyzing the components of spatial magnitude were that a thing that has spatial magnitude must be composed of parts with positive spatial magnitude, parts of no magnitude, or some combination of these. If a thing had no magnitude, Zeno argued, it would not increase (in magnitude) anything to which it was added, nor decrease anything from which it was removed. Therefore it could not "be" at all (at least, it could not "be" as the spatial thing it was said to be). Nothing with magnitude could be composed entirely of such things. However, if we assume that the components of a spatial thing have positive magnitude, another problem arises. In measuring such a thing, we would try to ascertain the end of its projecting part, (i.e. the outermost part of the thing). Each such projecting part would always have its own projecting part, so that the thing would have no ultimate "extreme" (*eschaton*). That is, the outer edge of something always has some thickness, as do the lines on any ruler we might use to measure it; and this thickness itself can always be divided. Thus the magnitude of a spatial thing, and thus its exact limits, will not be determinable. There is nothing in this to suggest that Zeno thought that the claim that there are multiple spatial things led to the conclusion that such things must be infinitely large. Rather, his description suggests that the things would be indeterminable, and indeterminate or indefinite, in size. They would also be *apeira*, indeterminate or indefinite, in number.

What of Zeno's paradoxes of motion? Modern interpretations of them generally present them as dealing with infinite sequences of steps, or with distances or times that seem to be infinite. Unfortunately, our evidence concerning the paradoxes of motion does not include any quotations from Zeno, but only reports that are at best second-hand. We cannot tell whether Zeno used the term *apeiron* in any of them, much less whether he used the term to refer to the infinite. Still, Aristotle's discussion of these paradoxes in the *Physics*, our earliest report, is replete with information that is germane to our purposes here.

In *Physics* Book Zeta Chapter 2, Aristotle says that "Zeno's argument is false in taking the position that it is not possible to traverse *apeira* things or to touch each of (a collection of) *apeira* things in a *peperasmenos* time"[8]. *Apeirōn*

sequences, but it is not known whether Zeno did. See Aristotle, *Physics* Z2 [3]. See [26] for Simplicius's quotations from Zeno; some translations appear in [19].

[8]The grammar suggests that Aristotle means that according to Zeno it is not possible to traverse any group of things that is *apeiron*, not necessarily a group of things each of which is *apeiron*.

here is usually translated as "infinite", but Aristotle sometimes uses the term to mean "unlimited" or "indefinite" or "indeterminate", and he may intend those senses here. Similarly, *peperasmenon* is often translated as "finite", but it may equally well mean "limited", "definite", or "determinate". Aristotle's response to the position he attributes to Zeno is to note that "both [linear] length and time, and anything continuous, are said (spoken of) in two ways: with respect to division and with respect to extremities. Therefore while it is not possible in a *peperasmenos* time to touch things that are *apeiron* with respect to quantity, it is possible to do so if they are apeiron with respect to division, for time (or: a time, an interval of time) is itself *apeiros* in this way . . . things that are *apeiron* are touched not by a *peperasmenos* time but by *apeiros* time" (233a22-35).

It will immediately be seen that Aristotle's argument works equally well whether *apeiron* and *peperasmenon* mean respectively "unlimited" and "limited", "infinite" and "finite", or "indeterminate" and "determinate". In Aristotle's attempt to find a coherent account of the motions and changes of distinct things, infinity, indeterminacy, and unlimitedness each pose challenges. The one perhaps most familiar to us is that of the possibility that it would take an infinite number of steps or stages, and so conceivably an infinite stretch of time, to traverse a distance of finite length. Other problems would arise for the prospects of coherence and explanation if it turned out that an indefinite number of steps would be needed to cross a definite distance, or if a determinate distance were to turn out to be composed of an unlimited number of smaller intervals (of possibly indeterminate length). For if it would take an indefinite or unlimited number of steps, and hence an indefinite or unlimited number of time intervals, to cross a defined distance, how could we tell when, and therefore if, we had completed the crossing? And if the presumed definite distance turned out to be composed of an indefinite number of intervals of positive magnitude, how could we determine where, and hence if, it began and ended? Moreover, if we could not identify exactly where objects and intervals of spatial magnitude began or ended, could we say with consistency that there are the multiple, distinct objects that are necessary (for the Greeks at least) to an account of motion?

When Aristotle returns to Zeno in Chapter 9 of Book Zeta, he describes in brief terms the four paradoxes known to us as the Dichotomy, the Achilles, the Arrow, and the Stadium or Moving Rows. Aristotle indicates that he has already discussed the Dichotomy, and his description of it matches the challenge of Zeno's that he had discussed in Chapter 2. In his descriptions of the remaining three paradoxes, the word *peperasmenon* appears only in the Achilles, and *apeiron* does not appear at all, though the concept is implicit in the discussion. The problem of the Achilles has to do with how a slower runner with a head start will be overtaken by a swifter runner, if the swifter runner must first reach each point that the slower one had reached (239b15-30). Aristotle argues that as long as the pursued runner must traverse a limited or finite (*peperasmenēn*) distance, the faster pursuer can overtake him. The defect with Zeno's alleged claim that the pursuer will not overtake the pursued, Aristotle says, is the way in which Zeno divides the magnitude (distance) between the runners. Aristotle seems to have in mind that Zeno is taking the distance between the runners as always further divisible rather than as composed of pieces of definite size that can be matched by steps of definite size. Here too, *peperasmenēn* could refer either to finitude or to limitedness, determinacy, of the distance.

Let us not lose sight, however, of Aristotle's careful locution from Chapter 2: continuous things are spoken of in two ways. With respect to division, continuous things of definite length are nonetheless *apeira*. Aristotle thus speaks of things as *apeiron* or *peperasmenon* within a certain context or in a certain respect or for a certain purpose. Where Aristotle may well have been at odds with Zeno, as we see from the remark about how Zeno looked at distance in the Achilles, was precisely over context or purpose. That is, they seem to have had different projects in mind: for Aristotle, providing an account of the things we say move and change; for Zeno, understanding whether we could have a coherent account of what is if we say that that includes discrete things of positive magnitude. It may be as well that the two philosophers differed over the question of the contexts we need to invoke in order to understand what is. One of the most important points, then, to take from the discussions of Zeno in Aristotle's *Physics* is Aristotle's care in distinguishing the aspects of a thing that are *apeiron* and those that are *peperasmenon* under each set of conditions or in each context.

The view we have presented of Zeno's concerns finds additional support in his extant fragments. In the fragments on multiplicity mentioned earlier (DK29 B1 and B2), we have seen that Zeno argued that if multiple spatial things are, they must be both so large as to be *apeira* and so small as to have no magnitude. We have already discussed why a spatial (as opposed to geometrical) object composed of parts that have no magnitude would pose problems. We have seen why Zeno might find difficulties with the prospect of spatial things that each had *apeiron* magnitude: one would not be able to establish or support the claim that there were distinct spatial things at all. It remains to be seen, then, why Zeno would say specifically that if we say that there are things with spatial magnitude, those things will turn out to have both no magnitude and *apeiron* magnitude.

Recall that on Zeno's analysis, it appeared that only if a thing had no magnitude could it have limited or finite magnitude. The only way for it to have positive magnitude, it seemed, was for it to have parts of *apeiron* magnitude. Someone might then respond that perhaps there was a way for things to have limited positive magnitude: perhaps the inner regions of such a thing would have positive and thus *apeiron* magnitude, and the surfaces or ends or edges would have no magnitude, and serve as the limits. Zeno would not accept such a solution. First, he would say, the outer parts would add no magnitude, so that the wholes of things would still have *apeiron* magnitude. Second, the two kinds of components of things would be impossible as definite spatial objects and as parts thereof.

There are two more fragments that make Zeno's concerns clearer, and that we can now see show somewhat more of an emphasis on limit, determinacy, and their opposites than on what we would term finitude and infinity. DK29 B3 supports the hypothesis that Zeno was concerned about the coherence of an account of what is that invoked distinctness. In this fragment Zeno argued that if many things are, they must be both *peperasmenon* in number, for they are as many as they are; and also *apeiron*, because something must be between any two, else those two would not be separate. That would imply that we cannot tell how many things are present in any area at any time, nor can we tell where (or thus if) any of them begins or ends.

We have no evidence that Zeno concluded from this that only one thing is. Simplicius, our main source for his fragments, claims that Zeno concluded that only one thing is, but does not furnish any quotations in which Zeno says such a thing. Moreover, in DK29 A16, Eudemus reports that Zeno said that if anyone could show him what the one is, he will be able to tell the things that are. In other words, Zeno did not think that to say that one thing is would be any more coherent or understandable than to say that many things are[9].

A more extensive discussion of these matters in Zeno is beyond the scope of the present paper, and is available in [10] and [9].

In the philosophy of the fourth century BCE, and arguably as early as Zeno, an *apeirōn* quantity could not be calculated exactly, at least as long as it was regarded from the perspective according to which it was *apeirōn*. In fact, Aristotle's argument that a continuous magnitude bounded at both ends could be traversed in a finite amount of time—despite the fact that it contains,so to speak, an *apeirōn* number of points, and despite Zeno's Dichotomy argument—rests precisely on the notions that the magnitude is not composed of the *apeirōn* number of points, and that from one perspective it is bounded. Aristotle does not refute Zeno's argument, but merely argues that within the framework of his physics, the question Zeno addresses can be put differently. Thus where *murios* did not clearly refer to ten thousand, a *murios* quantity was generally recognized as definite but was not calculated exactly. An apeirōn thing or quantity in Homer or Herodotus might be definite or not, and in later thinkers, especially in philosophy, the term came to emphasize that aspect of the thing or group or quantity that was indefinite, indeterminate, or

[9] A more complete account can be found in [9].

unlimited. Thus we find the indeterminacy, indefiniteness, or unlimitedness applying with respect to some context or conditions (practical or conceptual).

This concludes our remarks on ultrafinitistic themes in Greek thought. Our subsequent reflections on the ultrafinite will orbit, for the most part, around the *murios*-*apeirōn* pair, as if around a double star.

We now skip over two thousand years of mathematical and philosophical thought—where ultrafinitistic themes do crop up from time to time—picking up the thread once again well into the twentieth century.

§3. **Recent history of UF.** The passage from the prehistory to the history of UF is difficult to trace. Perhaps a bit arbitrarily, we shall say that it begins with the criticism of Brouwer's Intutionist Programme by Van Dantzig in 1950[10].

According to this view, an infinite number is a number that surpasses any number a person can cite. One is here reminded of a game inadvertently invented by the Greek mathematician Archimedes in his *Sand Reckoner*[11]. A game that is still played to this day, it is as follows: two players, A and B, try to outdo each other at naming large numbers. The contestant who is able to construct what is essentially, in contemporary jargon, a faster growing primitive recursive function, wins the game. The winner is the (temporary) owner of the so-called infinite numbers[12].

A second major character in this story is the Russian logician Yessenin-Volpin. In a series of papers[13] he presents his views on UF. Unfortunately, in spite of their appeal, his presentation can be at times obscure[14]. In any case, one of the fundamental ideas put forth by Volpin is that there is no uniquely defined natural number series. Volpin's attack attempts to unmask the circularity behind the induction scheme, and leaves us with various non-isomorphic finite natural number series. He also argues for the idea that there is a sense in which even small finite numbers can be considered infinite.

A few years later, one morning in fall the autumn of 1976 to be precise, the Princeton mathematician Ed Nelson had what might be described as an ultra-finitistic epiphany[15], losing his "pythagorean faith" in the natural numbers. What was left was nothing more than finite arithmetic terms, and the rules to manipulate them. Nelson's *Predicative Arithmetic* (see [23]) was the result.

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[10]See [11]. Van Dantzig himself points out that some of his ideas were anticipated by the Dutch philosopher Mannoury, and by the French mathematician Emil Borel (see [7]). For example, Borel observed that large finite numbers (*les nombres inaccessibles*) present the same difficulties as the infinite.

[11]For a good translation see [2].

[12]See on this also [4].

[13]See for instance his 1970 manifesto [32].

[14]Though David Isles has made a serious and quite successful attempt to clarify some of Volpin's tenets in [13].

[15]See his [22].

That text, the product of his epiphany and an essential step toward the re-thinking of mathematics along strictly finitist lines, seems to us however to fall short of Nelson's amazing vision. For example, why stop at induction over bounded formulae? If the infinite number series is no more, and arithmetic is just a concrete manipulation of symbols (a position that could be aptly called ultra-formalist), "models" of arithmetics are conceivable, where even the successor operation is not total, and all induction is either restricted or banished altogether.

The next milestone we take note of is Rohit Parikh's 1971 "Existence and Feasibility in Arithmetics", [24]. This paper introduces a version of Peano Arithmetic enriched with a unary predicate F, where the intended meaning of the statement $F(x)$ is that x is *feasible*. Mathematical induction does not apply to formulas containing the new predicate symbol F. Moreover, a new axiom is added to Peano Arithmetic expressing that a very large number is not feasible. More precisely, the axiom says that the number 2_{1000}, where $2_0 = 1$ and $2_{k+1} = 2^{2_k}$, is not feasible. Parikh proves that the theory $PA + \neg F(2_{1000})$ is *feasibly consistent*: though inconsistent from the classical standpoint, all proofs of the inconsistency of this theory are unfeasible, in the sense that the length of any such proof is a number $n \geq 2_{1000}$.

From the point of view of these reflections, Parikh achieves at least two goals: first, he transforms some ultrafinitistic claims into concrete theorems. And secondly, he indicates the way toward an ultrafinitistic proof theory.

Parikh's approach has been improved upon by several authors. Quite recently, Vladimir Sazonov in his [25] has made a serious contribution toward making explicit the structure of Ultrafinitistic Proof Theory. In the cited paper the absolute character of being a feasible number is asserted, on physicalistic grounds[16]. For our part, though, physicalistic explanations are less than convincing. As we have pointed out elsewhere in this paper, we believe that maintaining the notion of contextual feasibility is important. After all, who really knows what is the nature of the universe? Perhaps new advances in physics will show that the estimated upper bound of particles in the universe was too small. But whereas logic should be able to account for physical limitations, it should not be enslaved by them[17].

Parikh's 1971 paper, groundbreaking as it was, still leaves us with a desire for more: knowing that $PA + \neg F(2_{1000})$ is feasibly consistent, there ought to be *some* way of saying that it has a model. In other words, the suspicion arises that, were a genuine semantics for ultrafinitistic theories available, then Gödel's completeness theorem (or a finitist version thereof) should hold true

[16]Sazonov's articulated position on this issue is more subtle, as can be seen from his recent FOM postings.

[17]For further work on Parikh's approach see also [8], in which Alessandra Carbone and Stephen Semmes have investigated the consistency of $PA + \neg F(2_{1000})$ and similar theories from a novel proof theoretical standpoint, involving the combinatorial complexity of proofs.

in some form. But where to look for such a semantics? Models are structured sets, or, alternatively, objects in some category with structure. We must thus turn from proof theory to set theory and category theory.

On the set-theoretical side, there are at least two major contributions. The first one is Vopenka's proposal to reform, so to speak, Cantorian set theory, known as Alternative Set Theory, or AST (see [31]). AST has been developed for more than three decades, so even a brief exposition of it is not possible here. In broad outline, AST is a phenomenological theory of finite sets. Some sets can have subclasses that are not themselves sets, and such sets are infinite in Vopenka's sense. This calls to mind one of the senses of the word *apeiron*, as previously described: some sets are (or appear) infinite because they live outside of our perceptual horizon. It should be pointed out that AST is not, per se, a UF framework. However, Vopenka envisioned the possibility of "witnessed universes", i.e. universes where infinite (in his sense) semisets contained in finite sets do exist. Such witnessed universes would turn AST into a universe of discourse for ultrafinitism. To our knowledge, though, witnessed AST has not been developed beyond its initial stage.

Other variants of set theory with some finitist flavor have been suggested. Andreev and Gordon in their [1], for example, describe a theory of Hyperfinite Sets (THS) which, unlike Vopenka's, is not incompatible with classical set theory. Interestingly, both AST and THS produce as a by-product a natural model of non-standard analysis[18], a result which should be of interest to mainstream mathematicians.

The second set-theoretical approach of which we are aware, is that described in Shaughan Lavine's [15]. Here, a finitistic variant of Zermelo-Frankel set theory is introduced, where the existence of a large number, the Zillion, is posited. The reader may recall an idea which we hope is, by now, a familiar one, namely that of of *murios*. Here the number Zillion replaces the missing $\aleph_0$.

We move finally, to category theory. From our point of view this is, with one notable exception, an uncharted, but very promising, area. The single exception is the work of the late Jon M. Beck, involving the use of simplicial and homotopic methods to model finite, concrete analysis (see for instance [5, 6]). As we understand it, Beck's core idea is to use the simplicial category $\triangle$, truncated at a certain level $\triangle[n]$, to replace the role of the natural number series—or, because we are here in a categorical framework, the so-called natural number object that several topoi possess. The truncated simplicial category has enough structure to serve as a framework for some finitistic version of recursion; moreover, its homotopy theory provides new tools to model finite flow diagrams. As has been pointed out by Michael Barr, addition for the

[18]The very large and the very small are indeed intimately related: if one has a consistent notion of large, unfeasible number n, one automatically gets the infinitesimal $\frac{1}{n}$, via the usual construction of the field of fractions $\mathcal{Q}$.

finite calculator is not an associative operation. But homotopy repairs the lack of associativity by providing associativity up to homotopy via coherence rules.

As is well known, topoi have an internal logic which is intuitionistic. One hopes that by isolating feasible objects in the realizability topos via a suitable notion of *feasible realizability*, a categorical universe of discourse for UF could be, as it were, carved out.

In conclusion:

- First, the notion of feasibility should be *contextual*. An object such as a term, a number, or a set is feasible only within a specific context, namely one which specifies the type of resources available (functions, memory, time, etc). Thus a full-blown model theory of UF should provide the framework for a dynamic notion of feasibility.
- As the context changes, so does the notion of feasibility. What was unfeasible before, may become feasible now. Perhaps our notion of potential infinity came as the realization, or belief, that *any* contest can be transcended.
- Degrees of feasibility, so to speak, are not necessarily linearly ordered. One can imagine contexts in which what is feasible for A is not feasible for B, and vice versa.
- Last, but not least, as to the *murios*-*apeiron* pair: Every convincing approach to UF should be broad enough to encompass both terms. We saw above that any number or amount termed "many" with respect to the circumstances in which it is found, is *murios*. Pseudo-finite model theory, namely the restriction of first-order logic to models with the property that every first-order sentence true in the model is true in a finite model[19], captures one aspect of this idea. With pseudo-finite models particular properties or artifacts of the structure, such as its cardinality, are "divided out", so to speak. This means then that the pseudo-finite structure instantiates a general notion of finiteness, somewhat similar to uses of the word *murios* in Homer as cited above: the unspecified or indefinite (but finite) "many". The notion of pseudo-finite structure is far from what is aimed at here, as pseudo-finite structures are structures which are elementarily equivalent to ultraproducts of finite structures. They are, in a sense, too "sharp" for what is at stake here. However, they do point in the right direction, in that they show that even in classical logic the very notion of finiteness is to some extent blurred, if one suitably restricts the underlying logic. This hint, as we shall show in a future work, is indeed a pivotal one.

We hope that our reflections on the *murios*/*apeiron* pair go some way toward shedding some light on these issues. Our belief is that every convincing

[19]See for example [29].

approach to Ultrafinitism should include the notions of contextual uncountability, of indefiniteness, and of traversing limits. Even better, any such approach should unify these two streams of thought into a single, flexible framework.

This concludes our Very Short History. In our next paper, we shall offer a proposal which strives to take all the mentioned points into account.

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