A CENSUS OF PLANAR MAPS

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1. Introduction. In the series of "Census" papers, of which this is the fourth, we attempt to lay the groundwork of an enumerative theory of planar maps (**12, 13, 14**). The maps concerned are *rooted* in the sense that some edge is fixed as the *root*, and a positive sense of description and right and left sides are specified for it. This device simplifies the theory by ruling out the possibility of a map being symmetrical.

In this paper formulae are obtained for the number of rooted maps (with n edges), the number of non-separable rooted maps, and the number of 3-connected rooted maps without multiple joins (called c-nets). Some similar enumerations, supplementing the results of earlier papers, are given for triangulations and bicubic maps.

2. Maps. Let G be a topological graph in a 2-sphere or closed plane Π. By this we mean that the edges and vertices of G are disjoint subsets of Π, and that each edge is an open arc in Π whose end-points determine its incident vertices. An edge is a *loop* if its end-points coincide, and a *link* otherwise. The complex $|G|$ of G, a subset of Π, is the union of the edges and vertices of G.

If G is a connected graph having at least one edge, but having only a finite number of edges and vertices, then the dissection of Π which it determines is a *planar map* M, hereinafter called simply a "map." The edges and vertices of G are called *edges* and *vertices* respectively of M. The components of the complement of $|G|$ are the *faces* of M. They are simply connected domains, finite in number. The vertices, edges, and faces of M are its *cells*. Two of them are *incident* if one is contained in the boundary of the other.

A map is *singular* if it has either a loop or a face whose boundary is not a simple closed curve. The following properties of non-singular maps may be taken as axioms in combinatorial discussions:

I. Each edge of a non-singular map is incident with just two faces.

II. Let v be any vertex of a non-singular map M. Then v is incident with at least two edges and at least two faces. Moreover the edges and faces incident with v are the members of a cyclic sequence S_v with the following properties:

(i) The elements of S_v are alternately edges and faces of M, and no edge or face of M appears twice in S_v.

(ii) Two distinct elements of S_v are incident in M if and only if they are consecutive in S_v.

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Evidently S_v is unique, apart from reversal.

A *triangular map* is a non-singular map in which each face is a triangle, that is has just three incident edges.

A *derivable map* is a triangular map M' for which there is given an ordered partition $\{W, W^*, W\dagger\}$ of the vertex-set V having the following properties:

(i) No edge of M' has both ends in the same class W, W^*, or $W\dagger$.

(ii) Each vertex $W\dagger$ is incident with just four edges.

It follows from (i) that each face of M' has one vertex in each of the three sets W, W^*, and $W\dagger$.

Given a derivable map M' we may construct from it a new map M as follows. The vertices of M are the members of W^*. Its edges are in 1–1 correspondence with the members of $W\dagger$, each being the union of a member of $W\dagger$, and the two edges of M' joining it to members of W^*. It is easily seen that these edges and vertices of M make up a connected topological graph. The faces of M are in 1–1 correspondence with the members of W. Each is the union of the corresponding member of W with the incident edges and faces of M'.

We say that M' is a *first derived map* of M.

Consider any two maps M_1 and M_2. A *homeomorphism* of M_1 *onto* M_2 is a homeomorphism of Π onto itself which maps vertices, edges, and faces of M_1 onto vertices, edges, and faces of M_2 respectively. By the theory of simplicial complexes **(7, 8)** two triangular maps are homeomorphic if and only if there is a 1–1 mapping f of the cells of one onto the cells of the other such that f maps vertices, edges, and faces onto vertices, edges, and faces respectively, and such that both f and f^{-1} preserve incidence relations.

For derivable maps M_1 and M_2 we make the further requirement that a homeomorphism of M_1 onto M_2 must preserve each of the sets W, W^*, and $W\dagger$.

In what follows we shall not count two homeomorphic maps as distinct. Accordingly we may say that each derivable map M' is the first derived map of just one map M.

This is not the place to expound a rigorous theory of maps, both singular and non-singular. In this paper we often make statements about maps without proofs. It is believed, however, that the omitted proofs can in every case be constructed without difficulty by standard procedures of graph theory **(1, 6, 10)** or point-set topology **(9)**. In particular we make the following assertion:

Each map M has a unique first derived map M'.

Figure 1 shows a map M (solid lines) and its first derived map M'.

Let v, A, and F be a vertex, edge, and face respectively of a map M. We write $\eta(A, v) = 0, 1$, or 2 according as A is an edge not incident with v, a link incident with v, or a loop incident with v. We write $\eta(F, A)$ for the number of times F is incident with A, that is the number of faces of M' contained in F and incident with any specified edge of M' in A. This number is 0, 1, or 2, by I. We likewise write $\eta(F, v)$ for the number of times F is incident with v, that is one-half the number of faces of M' contained in F and incident with

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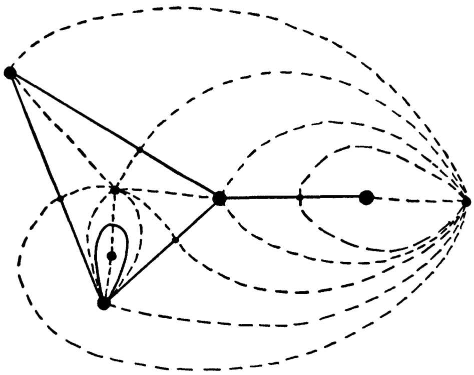

FIGURE 1.

v. By the uniqueness of M' these *incidence numbers* $\eta(X, Y)$ are uniquely determined by the structure of M.

An edge, as we say, has just two "sides," and it has just one face on each side. If the edge is contained in a simple closed curve in $|G|$ it follows from Jordan's Theorem that it is incident with two distinct faces, one in each residual domain of the curve. Hence any edge doubly incident with a face is an isthmus of G.

The *valency* val(v) or val(F) of a vertex v or a face F is the sum of the numbers $\eta(A, v)$ or $\eta(F, A)$ respectively, taken over all edges A of M.

Consider any derivable map M'. Evidently we can repeat the construction for M with W and W^* interchanged. The result is a map M^* called the *dual map* of M. It is easily verified that M is the dual map of M^*. The relation between M and M^* can be partially summarized as follows. There is a 1–1 correspondence θ mapping vertices, edges, and faces of M onto faces, edges, and vertices of M^* respectively, such that both θ and θ^{-1} preserve incidence numbers.

3. Rooted maps.

To *orient* an edge A of a map M is to specify a direction of description. If in addition we specify one side of A as being on the *right* of A, and the other on the *left*, we obtain a *complete orientation* of A.

If A is not an isthmus the left and right sides correspond to distinct faces of M. Otherwise they can be associated with distinct faces of M'.

In a diagram the terms "left" and "right" have an accepted meaning with respect to a directed curve, and we shall draw our diagrams to conform with this convention. Should we wish to interchange right and left sides of A without altering its direction of description we would reflect the diagram of the map in some line of the plane.

A map is *rooted* when some edge A is specified as the *root*, and a complete orientation of A is given.

For rooted maps we adjoin the following requirement to the definition of a homeomorphism. A homeomorphism of M_1 onto M_2 must preserve the root and its specified complete orientation.

Consider a rooted map M, with root A. We can define some corresponding rootings of M' as follows.

One of the two edges A_1' and A_2' of M' in A can be specified as the *first half* of A, the direction of A being from A_1' to A_2'. Let T be the face of M' on the right side of A and incident with A_1'. Let P and Q be distinct members of the class $\{W, W^*, W\dagger\}$. We define the (P, Q)-*rooting* of M' determined by M as follows. The edge of T with one end in P and one in Q is the root. It is directed from P to Q, and T is the face on its right.

On the other hand suppose M' is rooted, the root being directed from P to Q. Evidently there is a unique rooting of M determining the given (P, Q)-rooting of M'. From the foregoing results we deduce:

(3.1) *The number of rooted maps with n edges is equal to the number of* (P, Q)-*rooted derivable maps, P and Q being fixed, with n vertices in* $W\dagger$.

We may use the triangle T to fix a rooting of the map M^*. Its root is the edge corresponding to A. This root has T on its right, and its part in the boundary of T is directed from W to $W\dagger$. The map M^*, thus rooted, is the *dual* of the rooted map M. It is easily verified that M is also the dual of M^*. Duality thus provides us with a 1–1 correspondence of the class of rooted maps onto itself.

Rooting a map destroys its symmetry. More precisely we have:

(3.2) *Let h be a homeomorphism of a rooted map M onto itself. Then h maps each vertex, face, and oriented edge of M onto itself.*

Proof. Let U be the set of all faces F of M such that h maps F onto itself and at least one edge E incident with F onto itself without reversal of direction. Considering the cyclic sequence of edges and faces of M' contained in F we see that each edge and vertex incident with F must be mapped onto itself, without reversal of direction in the case of edges. It follows that if an edge E is incident with two faces F and F_1, F being in U, then F_1 is also in U.

Now U is not null, since it includes the faces incident with the root. Hence, by connection, it includes every face of M.

4. Bicubic maps.

A map is *trivalent* if the valency of each of its vertices is 3. A trivalent map is called *bicubic* if its vertex-set can be partitioned into two disjoint classes U and V so that each edge has one end in U and one in V.

A bicubic map has no loop, and by simple graph-theoretical arguments it can be shown that it has no isthmus either. Since the map is trivalent, it is therefore non-singular.

The faces of a bicubic map can be coloured in three colours so that no two of the same colour have a common edge. A proof of this result can be based on the fact that a solution of Heawood's congruences is obtained by associating the numbers $+1$ and -1 with the members of U and V respectively **(5, 11)**. This 3-colouring is unique, apart from permutations of colours. For when the two faces incident with some edge A are coloured, the colour of the other faces incident with ends of A is determined. Hence, by connection, the entire 3-colouring is determined.

Conversely any 3-colourable trivalent map is bicubic. A vertex can be assigned to U or V according as the cyclic order of colours around it agrees or disagrees with some fixed order, with respect to a fixed direction of rotation in Π.

Given an unrooted bicubic map M we consider the complement R in Π of the union of the faces of a particular colour, red say. In the terminology of **(14)** R is a *band*. The edges of M not incident with red faces define an *even slicing* of R. Conversely by the definitions of **(14)** any even slicing is equivalent to a bicubic map, the "exterior faces" being the members of a specified colour-class.

When a bicubic map is rooted, a distinction is made between the three colours. Those on the left and right of the root are the *left* and *right* colours respectively, and the remaining colour is the *root-colour.*

The main theorem of **(14)** may be restated as follows:

(4.1) *The number of unrooted bicubic maps in which the faces of one colour-class J are distinguished as $J_1, J_2, \ldots, J_k$, and each face J_i has a specified (even) number $2n_i > 0$ of distinguishable vertices is*

$$\frac{(n-1)!}{(n-k+2)!}\prod_{i=1}^{k}\frac{(2n_i)!}{n_i!\,(n_i-1)!},$$

where $n = n_1 + n_2 + \ldots + n_k$.

To make the vertices of J_i distinguishable we have only to specify one as the *representative vertex* of J_i, and to define a positive direction of description of the boundary of J_i. It is then possible to fix any vertex by its position with respect to the representative vertex. But a positive direction of description of J_1 determines a consistent orientation of the map **(8)**, and so fixes a positive direction of description for the boundary of any other face. The labelling of the vertices is thus equivalent to the specification of a representative vertex for each face J_i and the assignment of a positive direction of description to J_1.

When we root a bicubic map M we may consider that J is specified as corresponding to the root-colour. We fix J_1 as the face of J incident with the negative end u of the root A, and take u as the representative vertex of J_1. The positive direction of description of J_1 is fixed as that in which u is followed immediately by the points of the edge incident with J_1 and the face on the left of A. We refer to J_1 as the *root-face.*

The labelling of the remaining $k-1$ faces of J can be carried out in just $(k-1)!$ ways and the remaining representative vertices can be selected in

$2^{k-1}n_2n_3\ldots n_k$ ways. The fact that the $2^{k-1}(k-1)!\,n_2n_3\ldots n_k$ possibilities are topologically distinct follows from (3.2). Using (4.1) we deduce:

(4.2) *The number of rooted bicubic maps in which the root-face has valency* $2t$, *and there are just* q_s *other faces of the root-colour with valency* $2s$ ($s = 1, 2, 3, \ldots$), *is*

$$\frac{(n-1)!}{(k-1)!\,(n-k+2)!}\,\frac{(2t)!}{t!\,(t-1)!}\prod_{s=1}^{\infty}\left\{\frac{(2s-1)!}{s!\,(s-1)!}\right\}^{q_s},$$

where $2n$ *is the total number of vertices and* k *is the number of faces of the root-colour.*

(4.3) *The number of rooted bicubic maps in which there are just* k *faces of the root-colour, each of valency* 4, *is*

$$\frac{2(2k)!\,3^k}{k!\,(k+2)!}.$$

Proof. For rooted bicubic maps of the kind specified we have $4k = 2n$, the number of vertices. Substituting $2k$ for n in (4.2) we obtain (4.3).

5. The number of rooted maps. Duality sets up a 1–1 correspondence between the (W, W^*)-rooted derivable maps on the one hand and the rooted bicubic maps in which each face of the root-colour has valency 4 on the other. Under this correspondence right, left, and root-colours correspond to W, W^*, and $W\dagger$ respectively. Applying this result to (3.1) and (4.3) we obtain:

(5.1) *The number* a_n *of rooted maps with* n *edges is*

$$\frac{2(2n)!\,3^n}{n!\,(n+2)!}.$$

We write

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

Thus $A(x) = 2x + 9x^2 + 54x^3 + 378x^4 + \ldots$. Figure 2 shows the 2 rooted maps with 1 edge, and Figure 3 the 9 rooted maps with 2 edges.

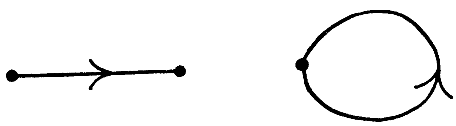

FIGURE 2.

(5.2) $A(x)$ *is given parametrically by the equations*

$$\xi = 1 + 3x\xi^2,$$
$$A(x) = \tfrac{1}{3}(3-\xi)(\xi-1).$$

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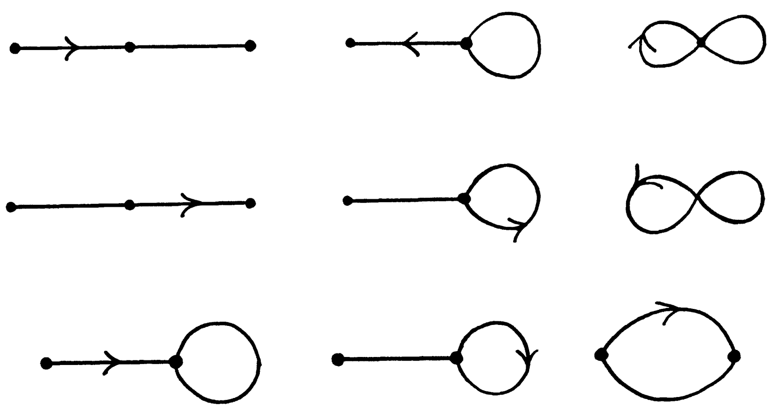

FIGURE 3.

Proof. By Lagrange's Theorem **(15)** the function $A(x)$ defined by these equations satisfies

$$A(x) = \frac{1}{3}\sum_{n=1}^{\infty} \frac{(3x)^n}{n!}\left[\frac{d^{n-1}}{da^{n-1}}\{a^{2n}(4-2a)\}\right]_{a=1}$$
$$= \frac{1}{3}\sum_{n=1}^{\infty} \frac{(3x)^n}{n!}\{4(2n)(2n-1)\ldots(n+2) - 2(2n+1)(2n)\ldots(n+3)\}$$
$$= \frac{1}{3}\sum_{n=1}^{\infty} \frac{(2n)!\,(3x)^n}{n!\,(n+2)!}\{(4n+8)-(4n+2)\}$$
$$= \sum_{n=1}^{\infty} \frac{2(2n)!\,(3x)^n}{n!\,(n+2)!}.$$

6. Non-separable rooted maps. A map M is called *separable* if its edge-set $E(M)$ can be partitioned into two disjoint non-null subsets S and T so that there is just one vertex v incident with both a member of S and a member of T. We then call v a *cut-vertex* of M. For example, any map having a loop and at least one other edge is separable. The maps of Figure 2 are non-separable, but all the maps of Figure 3 except the last are separable.

(6.1) *A vertex v of M is a cut-vertex if and only if there is a face F of M such that $\eta(F, v) \geqslant 2$.*

Proof. Suppose v is a cut-vertex and let $\{S, T\}$ be a corresponding partition of $E(M)$. By connection there is a face F of M incident with members of both S and T. Considering the cyclic sequence of faces of M' in F we find that $\eta(F, v) \geqslant 2$.

Conversely suppose $\eta(F, v) \geqslant 2$ for some face F and vertex v. Then there is a simple closed curve J consisting of v, the vertex w of W in F, and two edges of M' joining v and w. Each residual domain of J contains a vertex of $W\dagger$ and therefore an edge of M. Hence v is a cut-vertex of M.

Corollary I. *v is a cut-vertex of M if and only if it is joined to some $w \in W$ by two distinct edges of M'.*

Corollary II. *M^* is separable if and only if M is separable.*

A non-separable map of two or more edges has no loop. It therefore has no edge doubly incident with any face, by (6.1), Corollary II. So by (6.1) we have:

6.2. *A map of two or more edges is non-separable if and only if it is non-singular.*

Now let M be a rooted map, with root A. Assign the corresponding $(W^*, W\dagger)$-rooting to M'.

An *M'-lune* is a simple closed curve made up of two distinct edges of M' with common ends in W and W^*. Its residual domains are distinguished as its *outside*, containing the root A' of M', and its *inside*. It is a *terminal* M'-lune if its inside is not contained in that of any other M'-lune. It is clear that two distinct terminal M'-lunes have no common edge, and that their interiors are disjoint.

Let $Q(M')$ be the rooted map obtained from M' by rejecting the cells in the interiors of the terminal M'-lunes, and then adopting these interiors as new faces, called *lunes*.

Let N be any non-singular map. Let E be an edge of N with ends u and v and incident faces U and V. We *split* E by replacing it by two new edges E_U and E_V, each joining u and v, which together bound a new face F. U and V are replaced by new faces U_1 and V_1. In the new map, N_1 say, E_U is incident with U_1 and E_V with V_1, while the incidence relations not involving E are all retained.

If N is rooted, and the root is not E, there is an obvious corresponding rooting of N_1. If E is the root of N we adjust the notation so that it is directed from u to v and has U on its right. Then we take the root of N_1 to be E_U, directed from u to v and having U_1 on its right. Clearly N_1 is non-singular. Moreover, if two rooted maps N_1 are derived from N by splitting the same edge E, they are combinatorially equivalent, and therefore homeomorphic.

In practice N_1 can be obtained by joining u and v by an arc L in U. This separates U into two simply connected domains U_1 and F, the boundary of F being $E \cup L$. We write $E = E_V$, $L = E_U$, and $V = V_1$.

Returning to $Q(M')$ we observe that it is obtained from a $(W^*, W\dagger)$-rooted derivable map M_0' by splitting one or more edges joining W and W^*. Moreover, there are no M_0'-lunes and so M_0' is the first derived map of a non-separable rooted map M_0, by (6.1), Corollary I. M_0 is uniquely determined by M. We call it the (non-separable) *core* of M.

If in M' we treat the exterior of any terminal M'-lune as a single new face we obtain a derivable map split along one edge, from W to W^*, which we may regard as the root. We deduce that a_n is the number of ways in which we

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can select a $(W^*, W\dagger)$-rooted derivable map M_0', corresponding to a non-separable rooted map M_0, split it along one or more (W, W^*)-edges, and fill the resulting lunes with (W, W^*)-rooted derivable maps split along their roots, so that the resulting map has just n vertices in W.

If we write b_n for the number of non-separable rooted maps with n edges, and put

$$B(x) = \sum_{n=1}^{\infty} b_n x^n,$$

we can express the above result by the functional equation

$$A(x) = B(x\{1 + A(x)\}^2), \tag{6.3}$$

for the number of splittable edges in M_0' is twice the number of vertices of that map in $W\dagger$.

To solve this equation we write

$$u = x\{1 + A(x)\}^2.$$

Letting ξ be as in (5.2), and writing $\eta = 1 - \xi$, we find

$$27u = -(1 - \xi)(4 - \xi)^2,$$

$$A(x) = B(u) = -\tfrac{1}{3}\eta(2 + \eta),$$

$$\eta = \frac{-27u}{(3 + \eta)^2}.$$

Hence, by Lagrange's Theorem,

$$B(u) = \sum_{n=1}^{\infty} \frac{(-27u)^n}{n!} \left[\frac{d^{n-1}}{da^{n-1}}\left\{\frac{1}{(3 + a)^{2n}}\left(\frac{-2(1 + a)}{3}\right)\right\}\right]_{a=0},$$

$$B(x) = -\frac{2}{3}\sum_{n=1}^{\infty} \frac{(-27x)^n}{n!} \left[\frac{d^{n-1}}{da^{n-1}}\left\{\frac{1}{(3 + a)^{2n-1}} - \frac{2}{(3 + a)^{2n}}\right\}\right]_{a=0}$$

$$= \frac{2}{3}\sum_{n=1}^{\infty} \frac{(27x)^n}{n!}\left\{\frac{(2n - 1)(2n)\ldots(3n - 3)}{3^{3n-2}} - \frac{2(2n)(2n + 1)\ldots(3n - 2)}{3^{3n-1}}\right\}$$

$$= 2\sum_{n=1}^{\infty} \frac{x^n}{n!}\frac{(3n - 3)!}{(2n - 1)!}\{3(2n - 1) - 2(3n - 2)\},$$

$$B(x) = 2\sum_{n=1}^{\infty} \frac{(3n - 3)!}{n!\,(2n - 1)!}x^n. \tag{6.4}$$

Thus

$$B(x) = 2x + x^2 + 2x^3 + 6x^4 + 22x^5 + 91x^6 + 408x^7 + 1938x^8 + \ldots.$$

Figure 4 shows the unrooted non-separable maps of from 3 to 5 edges. A number attached to an edge indicates in how many distinct ways the edge can be rooted. If a map is symmetrical the number is attached to only one member of each equivalence class of edges. Beneath each map we give (in brackets) the total number of its rootings.

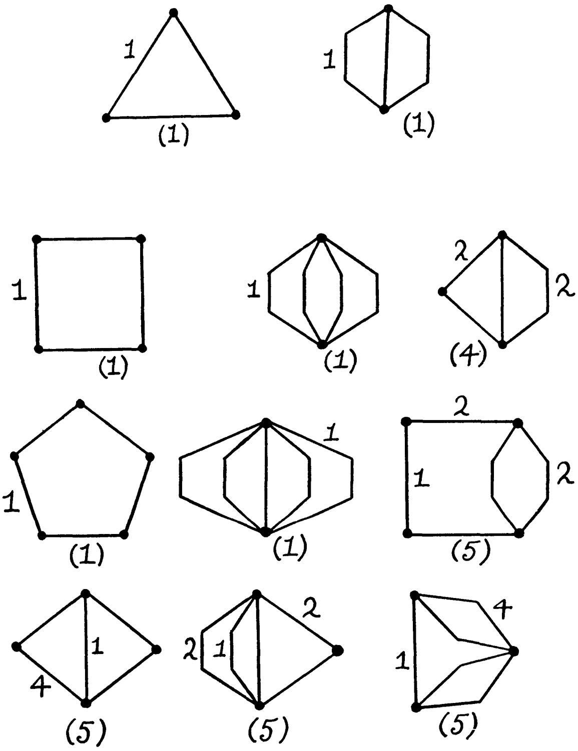

FIGURE 4.

7. Extensions. Let M_1 be a non-singular rooted map, with root A. Let B_1 be another edge of M_1 with an arbitrarily fixed complete orientation. Let it be directed from end a_1 to end b_1 and let the faces on its right and left be U_1 and W_1 respectively. Let M_2 be a second non-singular rooted map with root B_2. Let B_2 be directed from a_2 to b_2, and let the faces on the right and left be U_2 and W_2 respectively.

Let M_1 and M_2 be split along B_1 and B_2 respectively, so that new lunes F_1 and F_2 are introduced into M_1 and M_2. Let F_1 be filled with the split map M_2, by the following topological identification. The boundary of F_1 is identified with that of F_2, a_1 with a_2, and b_1 with b_2. The edge incident with U_1 is identified with that incident with U_2, and the edge incident with W_1 is identified with the edge incident with W_2. The exterior of F_2 is identified with the interior of F_1. Finally let U_1 and U_2 be united with their common incident edge to form a single new face U, and similarly for W_1 and W_2. This operation is

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illustrated in Figure 5. The resulting map M is an *extension of* M_1 *at* B_1 *by* M_2. It is readily seen to be non-singular. Its combinatorial structure, and therefore its homeomorphism class, is uniquely determined.

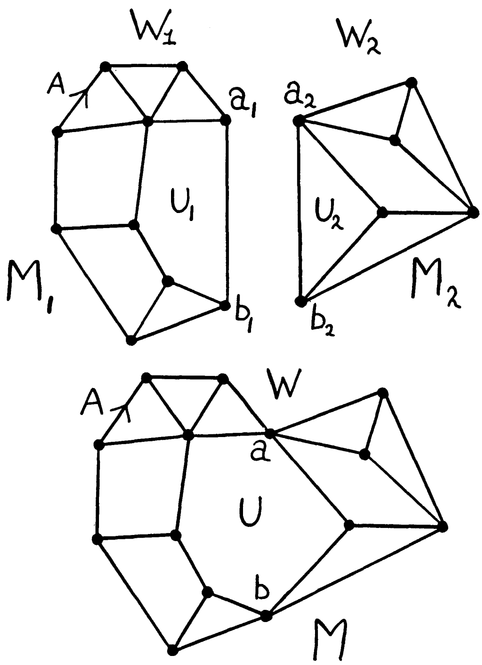

FIGURE 5.

If M_1 and B_1 are fixed, but M_2 is required to have only n edges, the number of extensions of M_1 at B_1 is b_n. Each extension increases the number of edges of M_1 by $n - 2$.

We may repeat the operation of extension, operating each time at a different edge of the original map M_1. The result is a *multiple extension* of M_1. It is convenient to regard a single extension of M_1 as a special case of a multiple extension, and even to count M_1 as a multiple extension of itself.

A 2-*separator* of a non-singular map M is defined by two distinct vertices a and b and two distinct faces U and W such that

$$\eta(U, a) = \eta(U, b) = \eta(W, a) = \eta(W, b) = 1.$$

We write such a 2-separator as $[a, b; U, W]$. Joining a and b by arcs in U and W we obtain a simple closed curve J separating $E(M)$ into two non-null subsets X and Y. We call the pair $\{X, Y\}$, which is uniquely determined by the 2-separator, the corresponding 2-*separation*. The 2-separation is ***proper*** if both X and Y have more than one edge. A non-singular map with no proper

2-separation is called 3-*connected*. A 3-connected map with more than 3 edges is a *c-net*.

It is clear that any 2-separator of M determines one of M^*. Hence the dual of a c-net is a c-net. (See (6.2) and (6.1), Corollary II.)

The map M constructed earlier in this section as an extension of M_1 at B_1 by M_2 has a 2-separator $[a, b; U, W]$, where a is formed by identifying a_1 and a_2, and b by identifying b_1 and b_2. (See Figure 5.) The corresponding 2-separation is $\{E(M_1) - \{B_1\}, E(M_2) - \{B_2\}\}$. Conversely any rooted map M with a 2-separator $[a, b; U, W]$ and a corresponding 2-separation $\{X, Y\}$, with the root in X, can be represented as such an extension. M_1 and M_2 are obtained from M by replacing the edges in the appropriate residual domain of J by a single edge joining a and b.

8. The number of rooted c-nets.

We may use the notion of an extension to count the non-separable rooted maps of 3 or more edges in another way. We separate these maps into three "types" and deal with each type separately. In each case we denote the root of the map M by A.

Type I. *The map M_A obtained from M by fusing A with its two incident faces, to form a single new face, is separable.*

Suppose the number of cut-vertices of M_A is k. Simple graph-theoretical arguments show that the defining graph G_A of M_A is the union of $k + 1$ non-separable graphs $G_1, G_2, \ldots, G_{k+1}$ with the following properties:

(i) If $1 \leqslant i < j \leqslant k + 1$, then G_i and G_j have no common edge.

(ii) If $|i - j| \geqslant 2$, then G_i and G_j have no common vertex.

(iii) If $i = j - 1$, then G_i and G_j have just one common vertex v_i.

(iv) The ends of A can be denoted by v_0 and v_{k+1}, so that $v_0 \in V(G_1) - \{v_1\}$ and $v_{k+1} \in V(G_{k+1}) - \{v_k\}$.

The vertices $v_1, v_2, \ldots, v_k$ are the cut-vertices of G_A. (See Figure 6.)

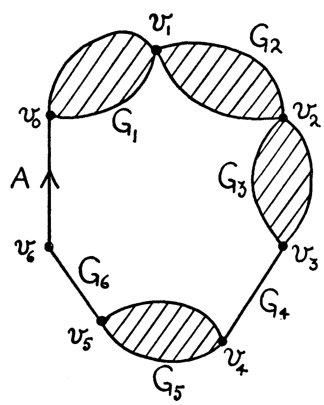

FIGURE 6.

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If we replace each G_i by a single arc A_i joining v_{i-1} and v_i we obtain a polygon with $k+2$ edges $A, A_1, A_2, \ldots, A_{k+1}$. It follows that the maps of Type I are the multiple extensions of the rooted maps defined by polygons of 3 or more edges, by rooted non-singular maps of three or more edges which are not of Type I.

Let q_n, $n \geqslant 3$, be the number of rooted non-singular maps of n edges which are not of Type I. Write

$$Q(x) = \sum_{n=3}^{\infty} q_n x^n.$$

We can now write the foregoing result as follows:

$$B(x) = 2x + x^2 + Q(x) + x\sum_{s=2}^{\infty} \{x + x^{-1}Q(x)\}^s \tag{8.1}$$

$$= x + \frac{x}{1 - (x + x^{-1}Q(x))},$$

$$Q(x) = \frac{x(B(x) - 2x)}{B(x) - x} - x^2, \tag{8.2}$$

$$= x^3 + 3x^4 + 11x^5 + 40x^6 + \ldots . \tag{8.3}$$

The rooted maps of Type I are therefore enumerated by the generating function

$$B(x) - 2x - x^2 - Q(x) = \frac{(B(x) - 2x)^2}{B(x) - x}.$$

Type II. *The defining graph G of M is the union of three or more connected graphs such that the intersection of any two of them consists solely of the two ends of A.*

Let the ends of A be a and b. We define an (a, b)-*arm* of G as a minimal connected subgraph H which includes a and b and has no vertices, other than a and b, incident with edges of G not in H. The (a, b)-arms of G have G as their union, and the intersection of any two of them consists solely of a and b.

We observe that M is of Type II if and only if the number of its (a, b)-arms is at least three. One (a, b)-arm consists solely of the root A and its two ends. By II the (a, b)-arms occur in a cyclic sequence, any two consecutive ones being separated by a face which is incident with both a and b and which has incident edges only in these two (a, b)-arms.

If we replace each (a, b)-arm by a single edge joining a and b we obtain a map defined by two vertices and $k \geqslant 3$ edges joining them. We conclude that the rooted maps of Type II are the multiple extensions of the simple rooted maps of this kind. We may now repeat the argument for Type I, with q_n redefined as the number of rooted non-singular maps with n edges which are not of Type II. All the steps in the proof remain valid. Hence the rooted maps of Type II are enumerated by the same function,

$$\frac{(B(x) - 2x)^2}{B(x) - x},$$

as those of Type I. We can explain this coincidence by proving that the maps of Type II are the duals of those of Type I.

It is clear from the definitions that no rooted map belongs both to Type I and to Type II.

Type III. *M belongs neither to Type* I *nor to Type* II.

Referring to Figure 4 we observe that a rooted map of Type III must have at least 6 edges.

We write any 2-separation of a rooted map M of Type III as an ordered pair $\{X, Y\}$, with $A \in X$. A *terminal* 2-separation, defined by a *terminal* 2-separator, is a proper 2-separation $\{X, Y\}$ such that no other proper 2-separation $\{X', Y'\}$ satisfies $Y \subset Y'$.

For rooted maps of Type III we have the following theorems:

(8.4) *If* $[a, b; U, W]$ *is a terminal* 2-*separator of* M, *then* a *and* b *are not both ends of* A.

Proof. Otherwise there would be at least three (a, b)-arms, one having the single edge A, and so M would be of Type II.

(8.5) *Let* $[a, b; U, W]$ *and* $[a_1, b_1; U_1, W_1]$ *be distinct terminal* 2-*separators of* M. *Then the pairs* $\{a, b\}$ *and* $\{a_1, b_1\}$ *are distinct.*

Proof. Suppose $a = a_1$ and $b = b_1$. Let the given 2-separators determine 2-separations $\{X, Y\}$ and $\{X_1, Y_1\}$ respectively. Let H be the (a, b)-arm containing A. Then H includes no edge of Y or Y_1. Using (8.4) we deduce that $\{E(H), E(M) - E(H)\}$ is a proper 2-separation of M, and that $Y \cup Y_1 \subseteq E(M) - E(H)$. This contradicts the hypothesis that $[a, b; U, V]$ and $[a_1, b_1; U_1, W_1]$ are terminal.

(8.6) *Let* $[a, b; U, W]$ *and* $[a_1, b_1; U_1, W_1]$ *be distinct terminal* 2-*separators of* M, *determining* 2-*separations* $\{X, Y\}$ *and* $\{X_1, Y_1\}$ *respectively. Then* $Y \cap Y_1 = \emptyset$.

Proof. Suppose $Y \cap Y_1$ is non-null. If a_1 and b_1 are both incident with members of X, or both with members of Y, then $Y = Y_1$ by (8.5) and the terminal condition. Otherwise the 2-separation $\{X \cap X_1, Y \cap Y_1\}$ is improper and M is of Type I.

In view of these results we may assert that M is a multiple extension of a rooted map M_0 obtained from M by replacing the set Y in each terminal 2-separation $\{X, Y\}$ by a single edge joining the vertices of the corresponding terminal 2-separator. The map M_0 is 3-connected since any cut-vertex or proper 2-separation of M_0 would be converted by the operation of multiple extension into a cut-vertex or proper 2-separation of M. Moreover M_0 has at least four edges since otherwise the extension would be of Type I or Type II. Thus M_0 is a c-net. We call it the (3-connected) *core* of M.

Conversely it is easily verified that any multiple extension of a rooted c-net M_0 has M_0 as its 3-connected core.

Let c_n, $n \geqslant 4$, be the number of rooted c-nets of n edges. Write

$$C(x) = \sum_{n=4}^{\infty} c_n x^n.$$

Since any non-singular rooted map of 3 or more edges is of just one of the types I, II, and III we can combine the foregoing results into the following functional equation, in which $U(x)$ denotes $B(x) - 2x$:

$$U(x) = x^2 + \frac{2U^2(x)}{x + U(x)} + \frac{xC(x^{-1}U(x))}{x^{-1}U(x)}. \tag{8.7}$$

Writing $z = x^{-1}U(x)$ we obtain

$$C(z) = z^2 - \frac{2z^3}{1+z} - xz. \tag{8.8}$$

To complete the determination of the function $C(z)$ we must obtain x in terms of z.

In the notation of Section 6 we have, putting $y = u^{-1}U(u)$,

$$y = u^{-1}B(u) - 2 = u^{-1}A(x) - 2$$
$$= \frac{9\eta(2+\eta)}{\eta(3+\eta)^2} - 2 = \frac{-\eta(3+2\eta)}{(3+\eta)^2},$$
$$\eta = -y\frac{(3+\eta)^2}{3+2\eta}, \qquad 27u = -\eta(3+\eta)^2,$$
$$9\frac{du}{d\eta} = -(\eta+1)(\eta+3).$$

Hence, by Lagrange's Theorem,

$$u = \sum_{n=1}^{\infty} \frac{y^n}{n!}\left[\frac{d^{n-1}}{d\eta^{n-1}}\left\{(-1)^n \frac{(3+\eta)^{2n}}{(3+2\eta)^n}\left(\frac{-1}{9}\right)(\eta+1)(\eta+3)\right\}\right]_{\eta=0}.$$

But x is the same function of z as u is of y. Hence

$$x = -\frac{1}{9}\sum_{n=1}^{\infty}\frac{z^n}{n!}(-1)^n\left[\frac{d^{n-1}}{d\eta^{n-1}}\left\{\frac{(3+\eta)^{2n+2}}{(3+2\eta)^n} - \frac{2(3+\eta)^{2n+1}}{(3+2\eta)^n}\right\}\right]_{\eta=0}$$
$$= -\frac{1}{9}\sum_{n=1}^{\infty}\frac{z^n}{n!}(-1)^n\left[\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{d^r}{d\eta^r}\left\{\frac{1}{3+2\eta}\right\}^n\cdot\frac{d^{n-r-1}}{d\eta^{n-r-1}}\{3+\eta\}^{2n+2}\right.$$
$$\left. -2\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{d^r}{d\eta^r}\left\{\frac{1}{3+2\eta}\right\}^n\frac{d^{n-r-1}}{d\eta^{n-r-1}}\{3+\eta\}^{2n+1}\right]_{\eta=0}$$

$$= -\frac{1}{9}\sum_{n=1}^{\infty}\frac{z^n}{n!}(-1)^n$$

$$\times\left\{\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(-2)^r(n+r-1)!}{3^{n+r}(n-1)!}3^{n+r+3}\frac{(2n+2)!}{(n+r+3)!}\right.$$

$$\left.-2\sum_{r=0}^{n-1}\binom{n-1}{r}\frac{(-2)^r(n+r-1)!}{3^{n+r}(n-1)!}3^{n+r+2}\frac{(2n+1)!}{(n+r+2)!}\right\}$$

$$= -\frac{1}{9}\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)!\,z^n}{n!\,(n-1)!}$$

$$\times\left\{27\sum_{r=0}^{n-1}\binom{n-1}{r}(-2)^r\frac{2n+2}{(n+r)(n+r+1)(n+r+2)(n+r+3)}\right.$$

$$\left.-18\sum_{r=0}^{n-1}\binom{n-1}{r}(-2)^r\frac{1}{(n+r)(n+r+1)(n+r+2)}\right\}$$

$$= -\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)!\,z^n}{n!\,(n-1)!}$$

$$\times\left\{\sum_{r=0}^{n-1}\binom{n-1}{r}(-2)^r\frac{2(2n-r)}{(n+r)(n+r+1)(n+r+2)(n+r+3)}\right\}$$

$$= -\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)!\,z^n}{n!\,(n-1)!}\left[\sum_{r=0}^{n-1}\binom{n-1}{r}(-2)^r\left\{\frac{n}{n+r}\right.\right.$$

$$\left.\left.-\frac{3n+1}{n+r+1}+\frac{3n+2}{n+r+2}-\frac{n+1}{n+r+3}\right\}\right],$$

(8.9) $$x = \sum_{n=1}\frac{(2n+1)!\,z^n}{n!\,(n-1)!}\left\{n\int_0^1(1-t)^3t^{n-1}(2t-1)^{n-1}dt\right.$$

$$\left.-\int_0^1(1-t)^2t^n(2t-1)^{n-1}dt\right\},$$

$$= -\sum_{n=1}^{\infty}R_nz^n, \text{ say.}$$

We denote the integral

$$\int_0^1 t^{n+s}(2t-1)^n dt, \qquad n \geqslant 0,$$

in the cases $s = 0, 1, 2,$ and 3, by J_n, K_n, L_n, M_n respectively. Using integration by parts we obtain the following identities, valid for $n \geqslant 1$:

(8.10) $$J_{n-1} = \frac{3}{n} - \frac{4(2n+1)}{n}J_n,$$

(8.11) $$K_{n-1} = \frac{1}{4n} + \frac{1}{4}J_{n-1},$$

(8.12) $$L_{n-1} = \frac{1}{2n} - \frac{n+1}{2n} J_n,$$

(8.13) $$M_{n-1} = \frac{3n+2}{8n(n+1)} - \frac{n+2}{8n} J_n.$$

Using these formulae we can express R_n in terms of J_n, as follows:

(8.14) $$R_n = \frac{(2n+1)!}{8(n!)^2} \{(27n^2 + 9n - 2)J_n - (9n-2)\}.$$

We use this relation to define R_0. Applying (8.10) we obtain a recursion formula for R_n. The result is

(8.15) $$S_n R_{n-1} + 2S_{n-1} R_n = \frac{2(2n)!}{(n!)^2}, \qquad n \geqslant 1,$$

where $S_n = 27n^2 + 9n - 2$. By (8.8) we have also

(8.16) $$c_{n+1} = 2(-1)^{n+1} + R_n, \qquad n \geqslant 3.$$

It is easy to verify that $J_0 = 1$ and $R_0 = 0$. Formula (8.15) enables us to compute R_1, R_2, R_3, and so on. In this way a table of values of c_n has been constructed (see Table I).

TABLE I

n	c_n	n	c_n
4	0	15	7,296
5	0	16	24,460
6	1	17	82,926
7	0	18	284,068
8	4	19	981,882
9	6	20	3,421,318
10	24	21	12,007,554
11	66	22	42,416,488
12	214	23	150,718,770
13	676	24	538,421,590
14	2,209	25	1,932,856,590

The c-nets of up to 11 edges are shown in Figure 7. The numbers are to be interpreted as for Figure 4.

From (8.15) we obtain, for $n \geqslant 2$,

$$\frac{R_n}{S_n} + \frac{1}{2}\frac{R_{n-1}}{S_{n-1}} = \frac{(2n)!}{(n!)^2}\frac{1}{S_{n-1}S_n},$$

$$\frac{1}{2}\frac{R_{n-1}}{S_{n-1}} + \frac{1}{4}\frac{R_{n-2}}{S_{n-2}} = \frac{1}{2}\frac{(2n-2)!}{((n-1)!)^2}\frac{1}{S_{n-2}S_{n-1}},$$

$$\frac{R_n}{S_n} - \frac{1}{4}\frac{R_{n-2}}{S_{n-2}} = \frac{(2n-2)!\,(2S_{n-2}.2n(2n-1) - n^2 S_n)}{2(n!)^2 S_{n-2}S_{n-1}S_n}$$

$$= \frac{n(2n-2)!\,(189n^3 - 891n^2 + 1098n - 352)}{2(n!)^2 S_{n-2}S_{n-1}S_n}.$$

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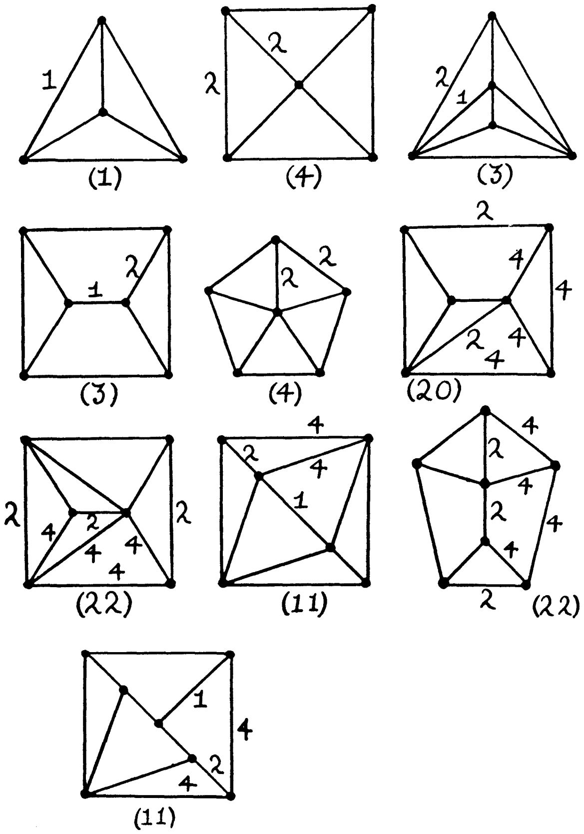

FIGURE 7.

Applying Stirling's Theorem we find that for large n the expression on the right is asymptotically

$$\frac{7n^{-9/2}4^n}{8.729\sqrt{\pi}}.$$

From these results we can deduce first that $R_n/S_n \to \infty$ as $n \to \infty$ and then that

$$\frac{R_n}{S_n} \sim \frac{7n^{-9/2}4^n}{8.729\sqrt{\pi}}\left\{1 + \frac{1}{4^3} + \frac{1}{4^6} + \frac{1}{4^9} + \ldots\right\} = \frac{7n^{-9/2}4^n}{8.729\sqrt{\pi}} \cdot \frac{64}{63},$$

whence

$$c_n \sim R_{n-1} \sim \frac{2n^{-5/2}4^n}{243\sqrt{\pi}}. \tag{8.17}$$

9. Unrooted c-nets. The plausible, but unproved, assumption that almost all c-nets are unsymmetrical implies that the number of unrooted c-nets of n edges is asymptotically

$$\frac{c_n}{4n} \sim \frac{n^{-7/2}4^n}{486\sqrt{\pi}}.$$

c-nets have been studied in connection with the theory of simple perfect rectangles **(2, 3, 4)**. Experience suggests that an unsymmetrical c-net nearly always gives rise to n simple perfect rectangles of order $n - 1$. As such a rectangle can be obtained from two (dual) c-nets, we can now conjecture with some confidence that the number of simple perfect rectangles of order n is asymptotically

$$\frac{n^{-5/2}4^n}{243\sqrt{\pi}}.$$

10. Simple triangulations. The triangulations studied in **(12)** are the rooted 3-connected triangular maps. Such a map is called *simple* if each simple closed curve made up of 3 edges is the boundary of a face. In **(12)** the number of simple triangulations with $n + 3$ vertices is denoted by $\phi_{n,0}$, and it is shown **(12, § 7)** that, for $n > 0$,

$$\phi_{n,0} = \frac{1}{2^{2n+3}}\left\{-(-1)^n(16n + 10) + (3n + 3)!\sum_{j=0}^{2n+1}\frac{(3n + 1 - j)!\,(3j - 3n - 4)}{j!\,(3n - j + 4)!\,(2n + 1 - j)!}\right\}. \tag{10.1}$$

Here we discuss a simplification of this formula which was overlooked in **(12)**.

From (10.1) we have

$$\begin{aligned}\phi_{n,0} &= \frac{1}{2^{2n+3}}\left\{-(-1)^n(16n + 10) + \frac{(3n + 3)!}{(2n + 1)!}\sum_{j=0}^{2n+1}\binom{2n + 1}{j}\frac{3j - 3n - 4}{(3n - j + 4)(3n - j + 3)(3n - j + 2)}\right\}\\ &= \frac{1}{2^{2n+3}}\left[-(-1)^n(16n + 10) + \frac{(3n + 3)!}{(2n + 1)!}\sum_{j=0}^{2n+1}\binom{2n + 1}{j}\left\{\frac{3n + 1}{3n - j + 2} - \frac{6n + 5}{3n - j + 3} + \frac{3n + 4}{3n - j + 4}\right\}\right]\\ &= \frac{1}{2^{2n+3}}\left[-(-1)^n(16n + 10) + \frac{(3n + 3)!}{(2n + 1)!}\left\{(3n + 4)\int_0^1 (1 - t)^2 t^n (1 + t)^{2n+1}dt - 3\int_0^1 (1 - t)t^n(1 + t)^{2n+1}dt\right\}\right].\end{aligned}$$

Using integration by parts a recursion formula reminiscent of (8.15) can be obtained from this expression. This time we find, for $n > 1$,

$$\phi_{n-1,0}S_n + 4\phi_{n,0}S_{n-1} = \frac{7(3n)!}{n!\,(2n + 1)!}, \tag{10.2}$$

where S_n is now $8n + 5$. With the help of (10.2), $\phi_{n,0}$ has been computed up to $n = 21$ as shown in Table II.

TABLE II

n	$\phi_{n,0}$	n	$\phi_{n,0}$
0	1	11	164,796
1	1	12	897,380
2	0	13	4,970,296
3	1	14	27,930,828
4	3	15	158,935,761
5	12	16	914,325,657
6	52	17	5,310,702,819
7	241	18	31,110,146,416
8	1,173	19	183,634,501,753
9	5,929	20	1,091,371,140,915
10	30,880	21	6,526,333,259,312

11. A further discussion of bicubic maps. Let f_n denote the number of rooted bicubic maps of $2n$ vertices. Write

$$F(x) = \sum_{n=1}^{\infty} f_n x^n.$$

The value of f_n can be obtained by summation from (4.2). We note that

$$\sum_{n=1}^{\infty} \frac{(2n)!\, x^{n-1}}{n!\,(n-1)!} = 2(1-4x)^{-3/2},$$

$$\sum_{n=1}^{\infty} \frac{(2n-1)!\, x^{n}}{n!\,(n-1)!} = \tfrac{1}{2}\{(1-4x)^{-1/2} - 1\}.$$

Applying these results to (4.2) we obtain

$$F(x) = x\sum_{k=1}^{\infty} \frac{1}{(k-1)!}\, x^{k-3}\left(\frac{d}{dx}\right)^{k-3}\left[\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}^{k-1} 2(1-4x)^{-3/2}\right],$$

$$xF(x) = 2\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}\left(\frac{d}{dx}\right)^{k-3}\left[\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}^{k-1} \times \frac{d}{dx}\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}\right],$$

$$\frac{d}{dx}(xF(x)) = 2\sum_{k=2}^{\infty} \frac{x^{k-2}}{(k-2)!}\left(\frac{d}{dx}\right)^{k-3}\left[\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}^{k-1} \times \frac{d}{dx}\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}\right]$$
$$+ 2\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}\left(\frac{d}{dx}\right)^{k-2}\left[\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}^{k-1} \times \frac{d}{dx}\left\{\frac{(1-4x)^{-1/2}-1}{2}\right\}\right].$$

Here $\left(\frac{d}{dx}\right)^{-1} f$ is interpreted as

$$\int_0^x f\,dx.$$

Let $\lambda(x)$ be defined implicitly by

$$\lambda(x) = x + x\left\{\frac{(1 - 4\lambda(x))^{-1/2} - 1}{2}\right\} = \frac{x}{2}\{1 + (1 - 4\lambda(x))^{-1/2}\}. \tag{11.1}$$

Then by Lagrange's Theorem we have

$$\frac{d}{dx}(xF(x)) = \left\{\frac{\lambda(x) - x}{x}\right\}^2 + 2\left\{\frac{\lambda(x) - x}{x}\right\} = \frac{\lambda^2(x)}{x^2} - 1. \tag{11.2}$$

Now

$$\begin{aligned}\lambda^2(x) &= \frac{x^2}{4}\left\{1 + 2(1 - 4\lambda(x))^{-1/2} + \frac{1}{1 - 4\lambda(x)}\right\}\\ &= \frac{x^2}{4}\{2 + 2(1 - 4\lambda(x))^{-1/2}\} + \frac{x^2}{4}\left\{\frac{1}{1 - 4\lambda(x)} - 1\right\}\\ &= x\lambda(x) + \frac{x^2\lambda(x)}{1 - 4\lambda(x)}.\end{aligned}$$

Since $\lambda(x)$ is not identically zero this implies

$$\begin{aligned}\lambda(x) &= x + \frac{x^2}{1 - 4\lambda(x)},\\ 4\lambda^2(x) - (1 + 4x)\lambda(x) + x(1 + x) &= 0,\\ \lambda(x) &= \frac{1 + 4x - \sqrt{1 - 8x}}{8},\end{aligned}$$

since $\lambda(x)$ has a zero constant term. Hence

$$\begin{aligned}32\lambda^2(x) &= (1 + 4x)^2 - (1 + 4x)(1 - 8x)^{1/2} - 8x - 8x^2\\ &= 1 + 8x^2 - (1 + 4x)(1 - 8x)^{1/2}.\end{aligned}$$

Applying these results to (11.2) we obtain

$$\frac{d}{dx}(xF(x)) = \frac{1}{32}\left\{\frac{1}{x^2} - 24 - \frac{(1 - 8x)^{3/2}}{x^2} - \frac{12(1 - 8x)^{1/2}}{x}\right\}.$$

On integration we find

$$\begin{aligned}F(x) &= \frac{1}{32x^2}\{-1 + 12x - 24x^2 + (1 - 8x)^{3/2}\}\\ &= 6\sum_{n=0}^{\infty}\frac{(2n + 1)!}{n!\,(n + 3)!}2^n x^{n+1}\\ &= x + 3x^2 + 12x^3 + 56x^4 + 288x^5 + 1584x^6 + 9152x^7\\ &\qquad + 54912x^8 + 339456x^9 + \dots .\end{aligned}$$

The bicubic maps of not more than 8 vertices are shown in Figure 8.

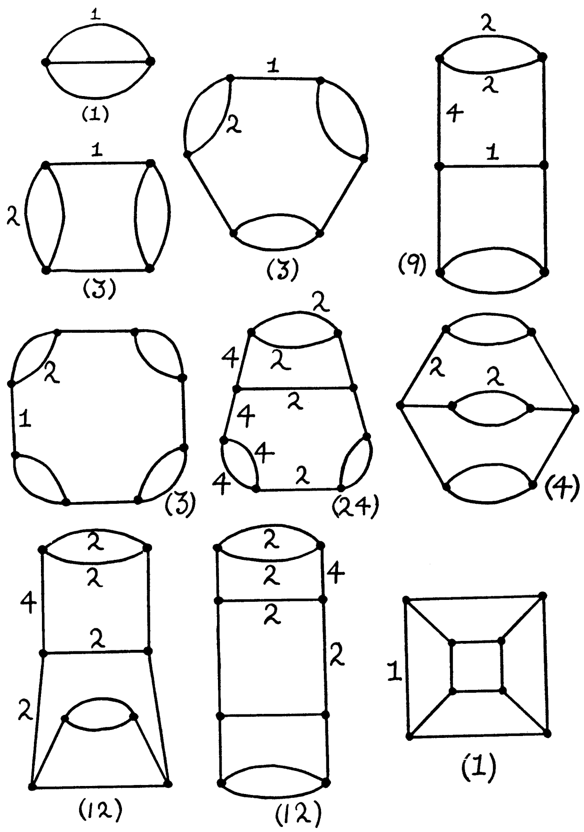

FIGURE 8.

Each rooted bicubic map can be represented as a multiple extension of a 3-connected bicubic core. If

$$G(x) = \sum_{n=1}^{\infty} g_n x^n$$

enumerates the rooted 3-connected bicubic maps of $2n$ vertices we can accordingly deduce the functional equation

$$F(x) = G(x(1 + F(x))^3).$$

From this we find

$$G(x) = x + x^4 + 3x^6 + 7x^7 + 15x^8 + \ldots .$$

But so far the coefficients are more easily obtained by actual counting.

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