PERPLEXING PARADOXES

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UNRAVELING ENIGMAS IN THE WORLD AROUND US

GEORGE G. SZPIRO

Columbia University Press *New York*

Columbia University Press
Publishers Since 1893
New York   Chichester, West Sussex
cup.columbia.edu

Library of Congress Cataloging-in-Publication Data
{to come}

Printed in the United States of America

Add cover/jacket credit information

This book is dedicated to the two of you, Noga and Uriel,
as you continue your journey together. May the paradoxes
in this book inspire and ignite your imaginations,
as your love story unfolds.

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INTRODUCTION

The eminent American philosopher Willard Van Orman Quine defined a paradox as "just any conclusion that at first sounds absurd but that has an argument to sustain it." This captures the spirit of most paradoxes . . . but not all. Some paradoxical statements may sound perfectly innocuous at first and reveal their absurdity only after one reflects more deeply about their implications. Others, seemingly absurd at the outset, may reveal themselves as perfectly innocuous upon further reflection. What most paradoxes have in common is that they elicit surprise, disbelief, bewilderment, and confusion.

Paradoxes are much more than entertaining riddles or intellectual amusements. For millennia, they helped blaze epistemological trails by challenging received wisdom and worldviews. Paradoxes have provided thinkers with food for thought ever since the ancient Greeks discussed philosophical questions. They continue to fascinate thinkers to this day. While philosophy denotes "love of wisdom" (from *philos*, friend, and *sophia*, wisdom), the word *paradox* (from *para*, counter, and *doxa*, opinion) indicates that something seems off and further investigation is advisable. The efforts expended thereon are definitely worthwhile because, as the British philosopher R. M.

Sainsbury remarked, "paradoxes . . . are associated with crises in thought and with revolutionary advances."

Many famous paradoxes date back to questions that the ancient philosophers posed. Theseus, for example, asked whether a ship whose rotten planks have been replaced one by one over the years is the same ship as the original; Zeno inquired whether the champion runner Achilles was able to catch up with a turtle; and Epimenides wanted to know if a Cretan who claimed that all Cretans lie was telling the truth.

In general, one is confronted with a paradox when a statement, seemingly based on valid reasoning and grounded in apparently valid premises, sounds unacceptable. In that case, at least one of the premises is flawed, or the reasoning is false, or—surprise, surprise—the conclusion is, in fact, correct. The latter type are called veridical paradoxes: they appear absurd but are actually true. Falsidical paradoxes, on the other hand, appear false and actually are false because the underlying reasoning is flawed. If, however, the conclusion is absurd, even though the reasoning leading up to it is flawless, then some of the premises on which the argument is grounded must be flawed, or there may exist a contradiction between two apparently equally valid principles, or—horribile dictu—there is as problem with our way of thinking. "Some tacit and trusted pattern of reasoning must be made explicit and henceforward be avoided or revised" (Quine). Such paradoxes are generally called antinomies.

And to quote Quine once more: "A veridical paradox packs a surprise, but the surprise quickly dissipates itself as we ponder the proof. A falsidical paradox packs a surprise, but it is seen as a false alarm when we solve the underlying fallacy. An antinomy, however, packs a surprise that can be accommodated by nothing less than a repudiation of part of our conceptual heritage."

• • •

Some paradoxes can be discussed with bright seven-year-olds. Others are infuriatingly confusing; just when one believes for a fleeting moment that the idea has come into focus, obfuscation sets in, the picture vanishes behind a smoke screen, and one must start all over again.

While paradoxes are generally thought to be the realm of logic, mathematics, and philosophy, they are actually ubiquitous. Type "paradox and . . ." in an internet search box, where ". . ." can stand for nearly any subject, and you will get hundreds of hits. In a few searches I conducted, I found 390 for "paradox and fishing," 362 for "paradox and cheese," and 369 for "paradox and sports."

Though this is a bit tongue in cheek—and when you actually sift through the search results, the list usually ends after a few dozen—in the following chapters, you will find examples of paradoxes from a dozen academic disciplines: logic, mathematics, and philosophy, of course, but also statistics, physics, law, economics, political science, linguistics, literature, theology, and even everyday life. The choice of subjects is rather random; other fields could have been selected, not necessarily "cheese" or "fishing" but evolution, quantum mechanics, medicine, decision-making, relativity, computer science, geography, finance, biology, and many others. If there should be a follow-up volume to this book, I will have occasion to present many, many more paradoxes.

• • •

The format I will follow is to begin each chapter with a question that often sounds trivial. Your initial reaction might be, "So, what's your point?" After delving deeper into the implication of the question, when the absurdity becomes apparent, your next reaction might be, "Wow, I did not think of it that way!" Finally, in the dénouement, when the incorrect or contradictory assumptions are revealed, the faulty reasoning is disclosed, or the paradox is resolved, I expect your final reaction to be, "Aha! Now I get it."

My hope is that this book will make you, dear reader, take statements that sound even just remotely odd with a grain of salt. Don't gloss over incongruences, but prick up your ears. Don't ignore abstruse assertions, but look behind the facade. Spot the paradoxical in apparently innocuous declarations. On the other hand, recognize the innocence of seemingly paradoxical announcements. Though some interpretations in this book are my own and you may not necessarily agree with them, my aim will have been reached if the lighthearted tone that I often use no longer lulls you into a false sense of certitude.

• • •

I began the project of collecting paradoxes in various fields just before the outbreak of the COVID-19 pandemic. During the various lockdowns and quarantine periods that followed, I kept myself busy trying to understand those paradoxes and, once I thought that I did, writing them up. As scary as the first COVID wave was, it allowed my wife, Fortunée, and me to spend months of quality time together, doing gymnastics in the morning, enjoying tête-à-tête lunches and dinners, and, to keep a safe distance from others, talking from our first-floor balcony with our children and grandchildren who were standing on the street below . . . and writing up the paradoxes, some of which you will find in this book.

During that time, I would send various chapters to a list of friends, colleagues, and even to correspondents whom I do not personally know. Many offered suggestions, encouragement, criticism, and comments; I list them here in alphabetical order, with apologies to anybody whom I may have inadvertently omitted: Ron Aharoni, Metin Arditti, Francisco Augspach, Kurt Baumann, Christian Blatter, Jacob

Burak, Eva Burke, Naomi Burke, Barry Cipra, Gary Dreiblatt, Sydney Engelberg, Valery Fabrikant, Marc Gertz, Sally Gertz, Saad Ghazipura, Noga Golan, Nir Grinberg, Thomas Guss, Rüdiger Hillgärtner, Andreas Hirstein, Giora Hon, André Hurni, Eli Jacubovich, Uriel Jaouen-Zrehen, Asaf Karagila, Jonathan Kleid, Noa Labanidze, Edi Landau, George Matsas, Joe Mazur, Norman Megill, Reinhard Meier, Ester Melamed, Jakob Melamed, Ioram Melcer, François Micheloud, René Nordmann, Leigh Pennington, Robert Potts, Alex Radzyner, Daniel Reeves, Y. Rudoy, Celine Schwartz, Michael Schwartz, Saar Shai, Karl Sigmund, Nancy Sinkoff, Charles Smith, Christian Speicher, Daniel Speyer, Bernhard von Stengel, Jim Supplee, Noam Szpiro, Noga Szpiro, Sarit Szpiro, Rudolf Taschner, Hélène Thouvenot, Charlotte Vardy, Charly Wegman, Hans Widmer, Doron Zeilberger, Alicia Zur-Szpiro, and Eliana Zur-Szpiro. I thank them all, but, as I discuss in "The Preface Paradox" (chapter 15), any remaining errors are my own.

• • •

Fortunée and our children, Sarit, Noam, and Noga, listened, most often patiently, whenever I began to expound on yet another paradox. And my mother, in her nineties, diligently read all of them, claiming that she understood nothing but enjoyed reading them anyway. In January 2023, I spent several days with my brother Michael Zur-Szpiro in Switzerland after a skiing accident—his, not mine—and we used the occasion to rearrange the chapters, rework the table of contents, and "compose" the epilogue. (When you get to the end of the book, you will understand why "compose" is set in quotation marks.) Sincere thanks go to my editor at Columbia University Press, Brian C. Smith, who enthusiastically and diligently guided this project along, as he had done with my previous book at CUP, and to Kalie Hyatt and her colleagues from KnowledgeWorks Global Ltd. for conscientious editing.

Above all, I am grateful to Fortunée—not just for the sumptuous meals and the beautiful stone sculptures she creates but above all for her support (and, dare I say, patience) throughout our more than forty years together—and to our children, their spouses, and our grandchildren, who will, I hope, read this book someday.

Tel Aviv, August 2023

PERPLEXING PARADOXES

I

QUOTIDIAN RIDDLES

Making Sense of the Silly and Surprising

Life is complicated. Not everything always makes sense or quite adds up. We begin our venture with everyday amusing bemusements that happen . . . well, every day.

1

MY FRIENDS ARE MORE POPULAR THAN I AM

The Friendship Paradox

Have you noticed that, in general, your friends have more friends on Facebook than you have? That they have more followers on Twitter than you have? That your boyfriend or girlfriend had more romantic partners than you had? In short, that you are less popular, on average, than your friends?

Depressing!

No, don't be miserable. It's simply a fluke of statistics.

Nevertheless, the phenomenon is real. In 1961, in the study *The Adolescent Society*, the sociologist James Coleman showed that most pupils in the twelve high schools that he analyzed had fewer friends than their friends had.

In the digital age, you, dear reader, can easily verify the phenomenon with data from the social networks. Most probably you will be able to confirm that the mean number of friends of your friends, or followers of your followers, is greater than the number of your own friends or followers. In 2012, the average Facebook user had 245 friends. But the average friend on Facebook had 359 friends. Only those with more than 780 friends had friends who, on average, had smaller networks than their own.

If you are a scientist in the publish-or-perish business, you'll discover that your coauthors have more coauthors than you have and, what's even more disheartening, that they also have more publications than you have.

The phenomenon can be observed in many contexts: people disproportionately experience restaurants, beaches, airports, and highways as being more crowded than they actually are on average. Students experience the average class size as being larger than it actually is.

Individuals tend to use the number of friends that their friends have as a basis to determine whether they themselves have an adequate number of friends. Scientists compare the number of their publications to those of their peers. Unfortunately, when using such figures as measures of one's own social or academic competency, most people will feel relatively inadequate. While not all individuals have fewer friends than their friends have, most do.

Scott Feld, also a sociologist, discovered the mathematical explanation for the paradox while investigating the causes and

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consequences of patterns in social networks. He presented his findings at the Sunbelt Social Network Conference in Santa Barbara, California, in 1986, and his paper "Why Your Friends Have More Friends Than You Do" was published in the *American Journal of Sociology* in 1991, three decades after Coleman's study.

The explanation is surprisingly simple.

DÉNOUEMENT

First, people with lots of friends are more likely to be among your circle of friends. Second, when they are, they significantly raise the average number of friends that your friends have.

To illustrate, consider the popular people, that is, those with many friends. They show up in many groups. Wallflowers, on the other hand, with few followers, show up in only a few groups. Hence, the popular few will be encountered by many more individuals than the many introverts.

In Feld's words, "If there are some people with many friendship ties and others with few, those with many ties show up disproportionately in sets of friends. For example, those with forty friends show up in each of forty individual friendship networks and thus can make forty people feel relatively deprived, while those with only one friend show up in only one friendship network and can make only that one person feel relatively advantaged." (That was in predigital times when a group of forty friends was considered very large.)

In the same manner, on average, students experience the average class size as being larger than it is according to college data because, by definition, many students attend the popular classes, whereas few students attend the less popular classes. Thus, on average, students experience a higher average class size than exists for the college because many students are in the large classes and few students are in the small classes.

To give a numerical example, in a class of 150 students, 150 will state that their class has 150 students. In a class of 10 students, 10 will state that it has 10. Therefore, while the true average class size is 80 ([150 + 10] ÷ 2), students will, on average, claim a class size of 141 ([150 × 150] + [10 × 10] ÷ [150 + 10]). The underlying reason is that when computing the weighted average of the class size, the numbers 150 and 10 show up twice in the numerator: once as the numbers to be averaged and once more as their weights.

MORE . . .

By the way, unless you are a fitness freak or bodybuilder, most people you see at the gym are in better shape than you are. Again, no reason to be depressed. These fitter people spend hours each day at the gym, which is why you encounter them in the first place. Whenever you show up, they are likely to be there. The couch potatoes, those who most probably are less fit than you, are absent, and you rarely get to see them. In other words, the people you encounter at the gym are not representative of the general population.

And, yes, we might remark that the same holds for sexual partners. 'Nuff said!

2

WAITING FOR GODOT

The Elevator Paradox

One of the disadvantages of living in a skyscraper is the tedious wait for an elevator. You want to catch the next elevator, either to ascend to the rooftop deck or to descend to street level. Unfortunately, some elevators go up when you want to go down, and some go down when you want to go up. You wait and wait and wait, like Didi and Gogo waiting for Godot.

With elevators shuttling from bottom to top and back again, the average wait for down elevators and up elevators should be about equal.

Correct?

Not if you live on one of the top floors or close to the bottom floor.

In fact, if you are close to the top and want to go down, the time that elapses until a down elevator arrives is much longer, on average, than the time until a useless up elevator arrives. And if you live on one of the lower floors and want to go to the roof, the time until the arrival of an up elevator is much longer, on average, than the time until the arrival of a superfluous down elevator. In general, top-floor residents heading down and bottom-floor residents heading up encounter elevators going in the wrong direction sooner than ones going in the right direction.

The author waiting for the next elevator.

Source: © fsz.

Is this maybe a psychological problem, and the frustrating delays until the arrival of an elevator going in the right direction only *seem* longer?

No, it's real.

The paradox was discovered by the physicists Marvin Stern and George Gamow, who worked on different floors in a seven-story building: Gamow on the second, Stern on the sixth. They often visited each other's offices and had to come to terms with this frustrating phenomenon: the first elevator to arrive was usually headed in the wrong direction.

Several years later, Donald Knuth, the renowned computer scientist, analyzed the conundrum and its dénouement in the *Journal of Recreative Mathematics*.

DÉNOUEMENT

To keep things simple, let's assume that there is just one elevator in the building that continually goes up and then down and then up again and down again, stopping at every floor. Let's also assume that it takes ten seconds for the elevator to move from one floor to the next, including the time it takes to open and close the doors. (We will also assume that the elevator waits zero seconds on every floor.)

From Gamow's position on the second floor, a downward-heading elevator will take ten seconds to reach the bottom and another ten seconds to come up again to the second floor. Hence, the time that elapses between the two observations by a second-floor resident wanting to head up is twenty seconds. After that, the elevator takes one hundred seconds to ascend to the seventh floor and descend again to the second. And so it goes, all day long.

If Gamow is lucky enough to arrive within the twenty-second interval, the first elevator he encounters will go up. If, however, he arrives at the elevator in the hundred-second interval, he must endure a wait of up to one hundred seconds to catch the elevator going in the desired direction. Since his arrival times are purely random, there is only a one-in-six chance (20/120) that the first elevator he encounters will be going up. But there's a five-in-six chance (100/120) that the first elevator will be going down.

Now consider Stern on the sixth floor. An elevator will stop at his floor on its way going up and, twenty seconds later, stop at the sixth floor again, going down. Now Stern could enter the cabin to visit Gamow's office. After that, it will take one hundred seconds for the elevator to descend all the way to the bottom and immediately ascend again to arrive at the sixth floor on its way up. With Stern's arrival times also being purely random, he, too, will have a one-in-six chance that the first elevator he encounters will be going in the right direction, namely down, and a five-in-six chance that the first elevator he encounters will be going the wrong way, up.

That's why the first elevator encountered by higher-floor residents wanting to go down and by lower-floor residents wanting to go up is usually headed in the wrong direction.

MORE . . .

Ancient paternoster elevators, chains of open cabins that loop up and down a building, avoid the paradox entirely because cabins are constantly going in both directions. Fortunately, however, paternosters are no longer allowed in most countries because of the dangers involved in stepping into and out of moving cabins.

• • •

Knuth's analysis is complicated a bit by several factors, though the conclusions remain correct in principle. A high-rise building usually has more than one elevator. There is increased demand at peak times in one direction—going up in the morning, going down after work. Elevators may be programmed to return to the lobby when empty.

And, of course, elevators usually do not stop at every floor unless summoned—and if they do, then not for zero seconds. This may be occasion to appeal to all those dummies who have the annoying habit of pressing both the up and the down buttons, no matter where they want to go, because they don't trust the system.

• • •

One more point: when asking people on the street to count, they will begin, "1, 2, 3." A mathematician, on the other hand, will count, "0, 1, 2, 3." Hence, it would be mathematically correct to consider a building's ground floor, with the entrance and lobby, as floor zero, as is the case all over Europe, instead of calling it the first floor, as is customary in North America. It makes the math so much easier: in Paris and London, you climb three floors to arrive at the third floor; in New York City you walk up only two floors to reach the third.

3

THE PURSUIT OF HAPPINESS

The Paradox of Hedonism

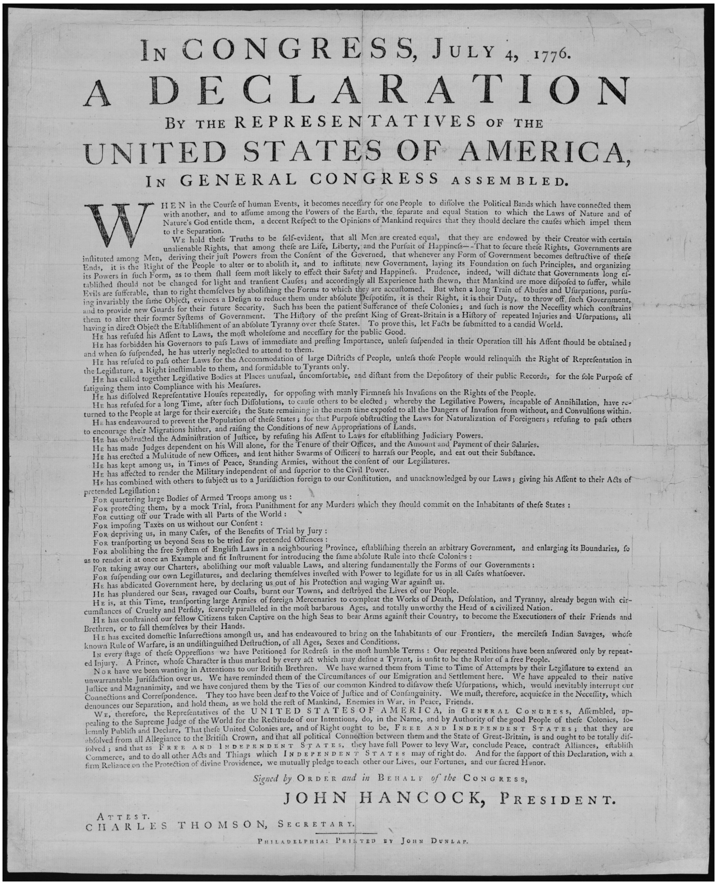
"We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness." So says the United States Declaration of Independence, and any self-respecting citizen should agree with all of these self-evident truths.

Correct?

No, not all of them. In particular, "the pursuit of Happiness" is open to question. As an aside, note that the Declaration of Independence does not say that there is a right to happiness as such, only to the *pursuit* of happiness.

The school of thought that proposes the pursuit of pleasure and happiness as the most important goal of life was created by the Greek philosopher Aristippus and is called hedonism. But what should hedonists strive for? Alas, it is an often-experienced fact of life that pursuing pleasure or happiness for its own sake often leads to the opposite of the desired goal. One becomes so absorbed with achieving happiness that the hardships on the way or the pain that may result are ignored.

Moreover, unfortunately for the hedonist, the constant pursuit of pleasure interferes with the experience of it. Thus, constant



IN CONGRESS, JULY 4, 1776.

A DECLARATION

BY THE REPRESENTATIVES OF THE

UNITED STATES OF AMERICA,

IN GENERAL CONGRESS ASSEMBLED.

WHEN in the Course of human Events, it becomes necessary for one People to dissolve the Political Bands which have connected them with another, and to assume among the Powers of the Earth, the separate and equal Station to which the Laws of Nature and of Nature's God entitle them, a decent Respect to the Opinions of Mankind requires that they should declare the causes which impel them to the Separation.

WE hold these Truths to be self-evident, that all Men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty, and the Pursuit of Happiness—That to secure these Rights, Governments are instituted among Men, deriving their just Powers from the Consent of the Governed, that whenever any Form of Government becomes destructive of these Ends, it is the Right of the People to alter or to abolish it, and to institute new Government, laying its Foundation on such Principles, and organizing its Powers in such Form, as to them shall seem most likely to effect their Safety and Happiness. Prudence, indeed, will dictate that Governments long established should not be changed for light and transient Causes; and accordingly all Experience hath shewn, that Mankind are more disposed to suffer, while Evils are sufferable, than to right themselves by abolishing the Forms to which they are accustomed. But when a long Train of Abuses and Usurpations, pursuing invariably the same Object, evinces a Design to reduce them under absolute Despotism, it is their Right, it is their Duty, to throw off such Government, and to provide new Guards for their future Security. Such has been the patient Sufferance of these Colonies; and such is now the Necessity which constrains them to alter their former Systems of Government. The History of the present King of Great-Britain is a History of repeated Injuries and Usurpations, all having in direct Object the Establishment of an absolute Tyranny over these States. To prove this, let Facts be submitted to a candid World.

HE has refused his Assent to Laws, the most wholesome and necessary for the public Good.

HE has forbidden his Governors to pass Laws of immediate and pressing Importance, unless suspended in their Operation till his Assent should be obtained; and when so suspended, he has utterly neglected to attend to them.

HE has refused to pass other Laws for the Accommodation of large Districts of People, unless those People would relinquish the Right of Representation in the Legislature, a Right inestimable to them, and formidable to Tyrants only.

HE has called together Legislative Bodies at Places unusual, uncomfortable, and distant from the Depository of their public Records, for the sole Purpose of fatiguing them into Compliance with his Measures.

HE has dissolved Representative Houses repeatedly, for opposing with manly Firmness his Invasions on the Rights of the People.

HE has refused for a long Time, after such Dissolutions, to cause others to be elected; whereby the Legislative Powers, incapable of Annihilation, have returned to the People at large for their exercise; the State remaining in the mean time exposed to all the Dangers of Invasion from without, and Convulsions within.

HE has endeavoured to prevent the Population of these States; for that Purpose obstructing the Laws for Naturalization of Foreigners; refusing to pass others to encourage their Migrations hither, and raising the Conditions of new Appropriations of Lands.

HE has obstructed the Administration of Justice, by refusing his Assent to Laws for establishing Judiciary Powers.

HE has made Judges dependent on his Will alone, for the Tenure of their Offices, and the Amount and Payment of their Salaries.

HE has erected a Multitude of new Offices, and sent hither Swarms of Officers to harrass our People, and eat out their Substance.

HE has kept among us, in Times of Peace, Standing Armies, without the consent of our Legislatures.

HE has affected to render the Military independent of and superior to the Civil Power.

HE has combined with others to subject us to a Jurisdiction foreign to our Constitution, and unacknowledged by our Laws; giving his Assent to their Acts of pretended Legislation:

FOR quartering large Bodies of Armed Troops among us:

FOR protecting them, by a mock Trial, from Punishment for any Murders which they should commit on the Inhabitants of these States:

FOR cutting off our Trade with all Parts of the World:

FOR imposing Taxes on us without our Consent:

FOR depriving us, in many Cases, of the Benefits of Trial by Jury:

FOR transporting us beyond Seas to be tried for pretended Offences:

FOR abolishing the free System of English Laws in a neighbouring Province, establishing therein an arbitrary Government, and enlarging its Boundaries, so as to render it at once an Example and fit Instrument for introducing the same absolute Rule into these Colonies:

FOR taking away our Charters, abolishing our most valuable Laws, and altering fundamentally the Forms of our Governments:

FOR suspending our own Legislatures, and declaring themselves invested with Power to legislate for us in all Cases whatsoever.

HE has abdicated Government here, by declaring us out of his Protection and waging War against us.

HE has plundered our Seas, ravaged our Coasts, burnt our Towns, and destroyed the Lives of our People.

HE is, at this Time, transporting large Armies of foreign Mercenaries to compleat the Works of Death, Desolation, and Tyranny, already begun with circumstances of Cruelty and Perfidy, scarcely paralleled in the most barbarous Ages, and totally unworthy the Head of a civilized Nation.

HE has constrained our fellow Citizens taken Captive on the high Seas to bear Arms against their Country, to become the Executioners of their Friends and Brethren, or to fall themselves by their Hands.

HE has excited domestic Insurrections amongst us, and has endeavoured to bring on the Inhabitants of our Frontiers, the merciless Indian Savages, whose known Rule of Warfare, is an undistinguished Destruction, of all Ages, Sexes and Conditions.

IN every stage of these Oppressions we have Petitioned for Redress in the most humble Terms: Our repeated Petitions have been answered only by repeated Injury. A Prince, whose Character is thus marked by every act which may define a Tyrant, is unfit to be the Ruler of a free People.

NOR have we been wanting in Attentions to our British Brethren. We have warned them from Time to Time of Attempts by their Legislature to extend an unwarrantable Jurisdiction over us. We have reminded them of the Circumstances of our Emigration and Settlement here. We have appealed to their native Justice and Magnanimity, and we have conjured them by the Ties of our common Kindred to disavow these Usurpations, which, would inevitably interrupt our Connections and Correspondence. They too have been deaf to the Voice of Justice and of Consanguinity. We must, therefore, acquiesce in the Necessity, which denounces our Separation, and hold them, as we hold the rest of Mankind, Enemies in War, in Peace, Friends.

WE, therefore, the Representatives of the UNITED STATES OF AMERICA, in GENERAL CONGRESS, Assembled, appealing to the Supreme Judge of the World for the Rectitude of our Intentions, do, in the Name, and by Authority of the good People of these Colonies, solemnly Publish and Declare, That these United Colonies are, and of Right ought to be, FREE AND INDEPENDENT STATES; that they are absolved from all Allegiance to the British Crown, and that all political Connection between them and the State of Great-Britain, is and ought to be totally dissolved; and that as FREE AND INDEPENDENT STATES, they have full Power to levy War, conclude Peace, contract Alliances, establish Commerce, and to do all other Acts and Things which INDEPENDENT STATES may of right do. And for the support of this Declaration, with a firm Reliance on the Protection of divine Providence, we mutually pledge to each other our Lives, our Fortunes, and our sacred Honor.

Signed by ORDER *and in* BEHALF *of the* CONGRESS,

JOHN HANCOCK, PRESIDENT.

ATTEST.
CHARLES THOMSON, SECRETARY.

PHILADELPHIA: PRINTED BY JOHN DUNLAP.

The United States Declaration of Independence.

Source: https://commons.wikimedia.org/wiki/File:Declaration_of_Independence,_Broadside,_printed_by_John_Dunlap_in_Philadelphia.jpg.

pleasure seeking may not yield the most actual pleasure or happiness. It's like trying to be spontaneous—it does not work.

The nineteenth-century moral philosopher and economist Henry Sidgwick, who thought deeply about human nature—he was one of the founders and the first president of the Society for Psychical Research—is credited with coining the phrase "the paradox of hedonism" in his treatise *The Methods of Ethics* (1874). He wrote,

> This brings us to what we may call the fundamental paradox of hedonism, that if the impulse towards pleasure is too predominant it will defeat its own aim. . . . Many middle-aged Englishmen would say business is more agreeable than amusement; but they wouldn't find it so if they transacted their business with a perpetual conscious aim at the pleasure of doing so. The pleasures of thought and study, also, can be enjoyed in the highest degree only by those who have an eagerness of curiosity that temporarily carries the mind away from self and its sensations.

Sidgwick did not operate in a void. A year earlier, in 1873, the utilitarian philosopher John Stuart Mill, had written in his autobiography, "I now thought that [one's happiness] was only to be attained by not making it the direct end. Those only are happy (I thought) who have their minds fixed on some object other than their own happiness. . . . Aiming thus at something else, they find happiness along the way. . . . Ask yourself whether you are happy, and you cease to be so."

DÉNOUEMENT

The singular pursuit of happiness may be self-defeating. A conscious striving for pleasure can actually be destructive, and constant attempts to maximize it tend to frustrate people rather than delight them.

What Sidgwick and Mill proposed is that one gains more pleasure by doing something other than consciously seeking to enjoy oneself. If one wants pleasure, one should concentrate one's attention on the actions that cause pleasure. Happiness will then be attained along the way as a by-product. The neurologist and psychiatrist (and Holocaust survivor) Viktor Frankl put it this way: "Happiness cannot be pursued; it must ensue, and it only does so as the unintended side effect of one's personal dedication to a cause greater than oneself."

For example, if one tends to bask in the honor and respect of one's fellows, one should not strive to gain those accolades. Honor and respect can be won only when one forgets all about winning them and engages instead in *activities* that gain the honor and respect of others. (Politicians: take note!) Thus, happiness is best obtained by pursuing objectives without thinking of the pleasure they will bring.

MORE . . .

How can the pursuit of happiness be counterproductive? Several mechanisms may lead to the paradoxical outcome.

First, the pursuit of sensory pleasures such as drinking alcoholic beverages, smoking tobacco, eating sweets, and having abundant sex may lead to alcoholism, lung cancer, diabetes, and AIDS. Pleasure seeking can lead one into risky experimentation. The high that one experiences from bungee jumping or skydiving may be followed by disaster. Second, habituation inevitably leads to disappointment because experience blunts sensitivity. As pleasure fades with time, the pleasure seeker pursues ever-stronger stimuli, which may lead to increasingly hazardous behavior.

Third, the chase for individual pleasures makes people less sensitive to the needs of others, which leads to moral decay. And fourth, if these are not enough, here's the worst of all: hedonism leads to idleness and boredom.

Therefore, since enjoyment is a by-product of self-actualization, it follows that the pursuit of pleasure yields fewer pleasurable experiences than a life devoted to a cause or to self-development.

4

TIP NOW OR TIP LATER?

The Good Service Paradox

A management consultant is on a trip for work in an out-of-the-way town and has just had dinner at a fancy, expensive restaurant. The maître d' gave her a good table, the waiter was attentive, and the sommelier suggested an excellent wine. It is quite regrettable that she will visit neither the town nor the establishment ever again.

The bill arrives. Happy with the service, she leaves a generous tip. Right?

No, not for a rational person.

A sensible, logical patron would not tip the service staff. Why should she? Dinner is over, and she is very unlikely ever to come again. It would be highly irrational to leave money just lying on the table. Would she do so after purchasing groceries at the supermarket? Or after getting her driver's license at the department of motor vehicles? Or following a doctor's appointment? Would she leave a little something for the cabin attendants on the flight home? She would not do so then, so why do so now?

In contrast to fixed prices paid for products and services, from the perspective of rational economics, tipping is a conundrum. It is voluntary and done, if at all, *after* the interaction. Like philanthropy, altruism, and charity—laudable concepts all—in terms of naked self-interest, gratuities have no raison d'être.

Source: https://commons.wikimedia.org/wiki/File:Jar_for_tips_at_a_restaurant_in_New_Jersey.JPG.

True, hoping for good service and conveying this hope to the restaurant staff with friendly behavior as she enters, the management consultant promises herself that she will leave a good tip. Deep down she may even mean it: good service deserves a reward. But once the meal is over, why?

The problem becomes even more acute because the waiter, a rational professional with instincts honed over many years of experience, realized as soon as the patron entered the establishment that she was a rational businessperson from out of town. He concluded, quite logically, that she would not tip him. Hence, he would be better off expending his efforts on the regular clientele and ignoring her.

So, here's the paradox: the customer would like good service and is prepared to give a gratuity for it, but there's no way she can ensure good service.

Oh, correction! Maybe there is a way to make it happen, even when confronted with a skeptical waiter: tip him beforehand. This would solve the problem, would it not?

No, it would not; prepaying the gratuity simply raises another problem. After pocketing the tip, the rational waiter would no longer have any incentive to provide good service. The consultant is back where she started—minus some cash.

So, what goes? Should one tip before or after the meal?

DÉNOUEMENT

Dangling the promise of a bonus over a service provider as a reward for good performance is a pie-in-the-sky pledge. Only the very naive would fall for such a promise. In fact, in some cultures, offering a tip is considered an insult. And at times, it may even be illegal. (Imagine offering a gratuity to the nice police officer who stopped you for speeding.)

Similar to Parfit's hitchhiker (see chapter 34), the paradox occurs because of a confusion about the relationship between cause and effect. In general, a cause leads to an effect. The effect may be desired or undesired, but what is certain is that the cause comes first and the effect afterward: medication leads to recovery, crime leads to punishment, etc. Similarly, good service should result in a tip at the end, whereas a tip at the start should engender good service.

Hence, to answer the question whether to tip before or after the meal, we must first clarify which is the cause and which is the effect. If a tip is meant to reward good service, then good service is the cause, and the tip is the effect. In this case, the waiter will be obliging, and the tip should come at the end. However, if good service is the result of a tip, then the tip should come first, and good service will follow.

The problem arises when diners and waiters have different points of view. A diner may consider good service a prerequisite for a tip, whereas a waiter may consider a tip a prerequisite for good service. With no tip forthcoming at the start, service will be terrible, and the disappointed diner won't be leaving a tip.

Or the diner may consider good service the result of a tip, whereas the waiter may consider a tip the result of good service. Then the tip is given at the beginning—and service is again terrible because the rational waiter no longer expects any compensation for his good service.

The problem may disappear only if both sides agree on the cause–effect relationship. In this case, the tip can be given either at the beginning or at the end, and both diner and waiter perform their parts of the unwritten understanding.

Take note, however, that there is an additional requirement for the problem to disappear, quite apart from agreement on the cause–effect relationship. The key phrase in the preceding paragraph is "unwritten understanding," which in this case is synonymous with *trust*. Strictly speaking, trustworthiness is incompatible with rational economics, at least in one-time interactions. And that is why the situation of the out-of-town businessperson is so vexing. Even though she and the waiter may agree on which is the cause and which is the effect, the waiter may not trust her to tip him after providing good service, and she may not trust him to give good service after tipping him.

Hence, in the absence of trust, something different is required to make interactions between diners and waiters work: a contract and an enforcement mechanism.

MORE . . .

When purchasing physical goods, paying for and receiving the merchandise generally occur simultaneously. With services, on the other hand, there is a delay between providing and paying for the service. This time gap is the root of the problem. If the parties act strictly rationally, both the prepayment and postpayment periods may lead to problems, as the gratuity-versus-good-service example shows.

Fortunately, in a nation of laws, there exists a solution—of sorts: the parties can commit to a binding contract. For example, many restaurant menus state that a 15 percent gratuity will be added to the final bill if the service was found to be acceptable. Diners thus know that they must pay the gratuity and that the waiter will at the very least provide acceptable, if not perfect, service.

To end the chapter, here's a solution to solve all problems: tip continually throughout the meal.

5

DON'T WORK OUT TO LOSE WEIGHT

The Exercise Paradox

Quentin wants to lose a few pounds. He goes to the gym, runs, swims, and lifts weights. He knows that after a while he will become a slenderer version of his former self. All he has to do is keep it up.

Right?

Most probably not!

Don't get me wrong: this does not mean that Quentin should stop exercising. There are many good reasons to work out. But weight loss isn't one of them. For Quentin to achieve his goal, it's dieting, dieting, and dieting that matter—not exercise.

This is a bit confusing. After all, common sense would tell us that expending energy and burning calories should consume body fat and make Quentin lose weight. So, why is it that in many people's experiences, exercise does not make a person slimmer?

Humans require energy to function. Energy is needed to breathe, pump blood, metabolize, fight off infections, activate the brain, and have sex. The required energy is produced through the intake of food. The consequence: on the one hand, the body accumulates fat reserves when the food intake produces more energy than is needed for the daily routine; on the other hand, when the food intake produces less than what is expended, the body uses up the surplus fat to keep functioning.

The author exercising.

So, the math seems simple: whenever Quentin expends more calories (daily routine plus exercise) than he takes in (food), fat reserves are depleted, and weight loss ensues. Therefore, the secret to losing weight, while keeping up one's daily routine, is to burn more calories than one takes in. The body will make up for the deficit by burning the fat reserves that have accumulated in Quentin's body and he will become slimmer. That's the whole point of exercise.

Hence, simple math would tell us that a combination of more exercise and less food should result in weight loss, provided all else remains constant. But simple math does not tell the entire story. Because the math must be done holistically—over the entire day, week, month . . . —a paradox arises: many people who exercise to slim down experience no weight loss.

DÉNOUEMENT

Three reasons can be given for the paradoxical result, two of which will be obvious after reflection and one of which is somewhat surprising. As we shall see, the phrase "provided all else remains constant" in the previous paragraph is key.

1) Exercise causes hunger, so people tend to eat more after exercising to make up for the calories they burned. They may also feel that they have earned a reward after all that effort. So, in general, "all else" does not stay constant; one eats more after exercise than when doing no exercise. Whatever calories are lost during exercise are made up for later on by consuming more snacks.

2) Exercise is tiring, so people may want to relax for the rest of the day. Hence, after a vigorous workout, they tend to move less than they would without having exercised. They may also think that they've done enough for the day. So, after a five-mile jog in the park, they might head home by car instead of walking and take the elevator instead of climbing the stairs. Again, generally, "all else" does not stay constant; one tends to rest more after working out than without any exercise. In total, people tend to burn fewer calories when they exercise than when they do not.

3) The most surprising explanation is a phenomenon over which one has no control whatsoever. Research has shown that the human body adapts to exercise. It slows down the burning of calories by spending less energy on internal functions, from those of the immune system to those involved in digestion. Researchers believe that the systems in the background become more efficient when one exercises, thus requiring less energy to perform their work. The more calories one burns through exercise, the fewer calories the body requires to keep the internal organs going. So, here, too, "all else" does not stay constant; when exercising, the body uses less energy to operate its internal machinery than when not exercising.

Here's the message to all you exercise buffs: keep doing what you're doing, but don't expect to lose weight by doing it. As the saying goes, weight loss is 80 percent dieting and 20 percent exercise.

MORE . . .

By the way, dieting may also not be the magic bullet it's made out to be. True, when food intake drops, the body burns fat. But—similar to reason 3 given in the dénouement—metabolism may then slow down to conserve resources. So, while one consumes fewer calories, the body also requires fewer calories.

• • •

Another phenomenon also goes under the name "exercise paradox." It says that the importance of exercise is obvious to nearly everybody (not for losing weight, as we saw in this chapter, but for many other health reasons). But though many people profess that they want to exercise, most resist. The phenomenon is probably known to most of us. Psychologists and neurologists have now given it a scientific underpinning and explanation, finding that even if you think that you want to exercise, your brain wants you to be sedentary.

Using electroencephalography, a technique that allows the visualization of brain activity, the scientists' findings suggest that the brain automatically finds sedentary behavior attractive. Individuals who want to work out must activate additional brain resources to counteract the temptation to remain a couch potato.

From an evolutionary point of view, laziness makes absolute sense. Since the conservation of energy is beneficial to humans—who then have more of it available to hunt for food or defend themselves against enemies—it provides an evolutionary advantage.

II

LANGUAGE IS TRICKY

It's Not What You Say, It's What They Hear

One skill that distinguishes modern members of the species *homo sapiens* from other animals is our ability to communicate verbally with each other. But ever since God jumbled our languages at the Tower of Babel, mixups, slipups, confusions, and befuddlements are ubiquitous.

6

CAN'T GET NO SATISFACTION

Morgenbesser's Double Negatives

In common parlance, statements with double negatives like "there's no way you will not attend" and "she is not unlike her brother" are actually positive statements or affirmations. In fact, they translate to "you will attend" and "she is like her brother." Similarly, the Rolling Stones' lyrics "I can't get no satisfaction"—intended by Mick Jagger and Keith Richards to convey frustration with the establishment—express the exact opposite when correctly parsed: "I can get satisfaction."

On the other hand, and this is the point of this chapter, a double positive does not translate into a denial. The statement "there's a street that does lead to Rome" does not indicate a lack of streets leading to Rome. The double positive does not imply a negative; it simply serves to accentuate the positive.

Correct?

Yes—with one famous exception.

It is a story about the philosopher Sidney Morgenbesser of Columbia University. A famous lecturer in linguistics once gave a talk at the university in which he noted that in the English language, double negatives imply a positive, but this does not hold the other way around: double positives do not imply a negative. At that moment, people in the audience sitting near Morgenbesser heard him mutter, "Yeah, yeah!"

Mick Jagger and Keith Richards of the Rolling Stones at the Prudential Center on December 13, 2012.

Source: © SolarScott, CC BY 2.0, https://www.flickr.com/photos/solarscott1955/8275599519/in/set-72157632257706142.

It's a funny story, but is actually no counterexample to the linguist's assertion. Morgenbesser's dismissive interjection, which allegedly turned the double positive into a "no," was a sarcastic comment, not a logical or grammatical observation. This conclusion is borne out by the fact that in written form, "yeah" and "yeah" are separated by a comma. So, in fact, the lecturer, a distinguished Oxford professor of linguistics, was correct in noting that there are no double positives that make a negative.

DÉNOUEMENT

The dénouement is achieved with a bit of straightforward math. Of course, +1 multiplied by +1 equals +1. And −1 multiplied by −1 also equals +1. On the other hand, −1 multiplied by +1 equals −1. One knows this much from elementary school.

Now let's translate this into the language of formal logic. A true statement is denoted by *P*, and the word *not* is usually denoted by a prefixed tilde (the typographical symbol ~). Hence, (~*P)* means that "not *P*" is true. And ~(~*P*) is like multiplying −1 by −1. Hence, it means "not not *P*" is true, which implies that *P* is true.

With this example, we have everything we need to analyze double negatives, double positives, and everything in between. An utterance that combines positives and negatives is like multiplying +1 by −1 or the combination of *P*s with a certain number of tildes.

Let's take the statement "she is not unlike her brother." We denote "she is like her brother" as *P*. Then, the statement "she is unlike her brother" becomes ~*P*, and "she is not unlike her brother" is ~(~*P*), which is equal to *P*. Hence, "she is like her brother."

As an exercise, replace *P* with +1 and ~ with −1 in the previous paragraph and then multiply the terms. You will obtain the same result: (−1) × (−1) × (+1) = +1.

On the other hand, let's analyze "there's a street that does lead to Rome." "A street leads to Rome" would be denoted by +1. "That does" is also a positive statement, and it, too, is denoted by +1. So, we have (+1) × (+1), which equals +1. Hence, "a street leads to Rome."

Back to Morgenbesser's interjection. His "yeah" and "yeah" were not "multiplied" or logically combined. They were only juxtaposed, which is indicated, as mentioned earlier, by the comma that separates them once the utterance is transcribed. Morgenbesser simply expressed the same opinion twice: (+1) *and* (+1), or "*P* is true, *P* is true," which means exactly that: *P* is true. In spite of his dismissive tone, Morgenbesser agreed with the linguist.

MORE . . .

Note that—like Morgenbesser's faux double positive, which did not express a negative—a double negative does not always imply a positive. Vernacular English, or street language as spoken by many people, often employs double negatives not to convey a positive meaning but to double down on the negative. To such speakers, the statement "I don't know nothin' " does not imply the possession of some knowledge but rather emphasizes the speaker's ignorance.

How about triple negatives? What does "I ain't got no time for no bullshit!" convey? Well, $(-1) \times (-1) \times (-1) = -1$. Or, in logical notation, with "no time for bullshit" represented by $\sim BS$, we get, after some rearrangement of the terms, $\sim(\sim(\sim BS)) = \sim BS$. The speaker obviously wants to express politely, but in her own style, that she has no time for small talk.

And finally, dear reader, try your understanding on "I ain't got no time for no bullshit no more!"

• • •

It is not true in all languages that double negatives imply a positive. Take French, for example, in which double negatives are prescribed by the rules of grammar. To negate a verb, it must be embedded into "*ne . . . pas*" ("no . . . not"), "*ne . . . rien*" ("no . . . nothing"), or "*ne . . . aucun*" ("no . . . none"). These are prefixes and suffixes that, each on its own but also in conjunction, indicate negation.

7

DON'T TRUST FRIENDS

False Friends

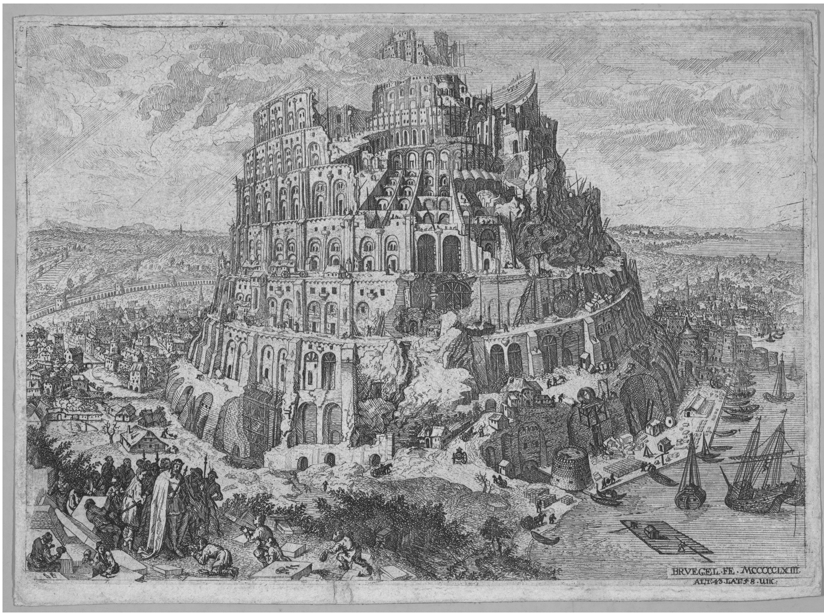
Translating from one language into another is a tricky business. One must not simply convert word for word but convey the meaning of the text and the emotions that the author wants to evoke. In the Book of Genesis, people purportedly found this out the hard way when they were about to build the Tower of Babel and God confused their language so that they could no longer understand one another.

Sometimes, however, one catches a lucky break, and the meaning of a word or phrase is so obvious that the translation imposes itself. For example, the Italian *acqua calda* is obviously "cold water."

Correct?

Not at all; in fact, it is the exact opposite: *acqua calda* is "*hot* water." "Cold water" is *acqua fredda*.

Examples abound in all pairs of languages. The German *sympathisch* translates into English as "likable," whereas "sympathetic" must be translated as *mitfühlend*. The English "six" sounds like the Turkish *sekiz*, but the latter actually denotes the number eight. The French *monnaie* does not mean "money" but "coins," whereas *les coins* is not "small change" but "corners." A gift is a present in English, but *Gift* is "poison" in German. The English "physician" is a *docteur* in



The Tower of Babel, painting by Anton Joseph von Prenner, after Pieter Bruegel the Elder.

Source: Courtesy of the Metropolitan Museum of Art, https://www.metmuseum.org/art/collection/search/415714.

French, whereas the French *physicien* is a "physicist" in English. And "a person of weak physique" should be translated as *une personne peu robuste*, according to the French linguists Maxime Kœssler and Jules Derocquigny.

The German *diskret* can be "discreet" in English but also "discrete," a mathematical term denoting "separated" or "discontinuous." And while we're looking at math, here's a favorite of mine: the English "billion" is *Milliarde* in German, whereas the German *Billion* is "trillion" in English. And the German *Trillion* translates into "quintillion." It's nothing if not confusing.

The term "false friends" was coined for such mistranslations, albeit in French, by Kœssler and Derocquigny in their 1928 book *Les faux amis, ou les trahisons du vocabulaire anglais (conseils aux traducteurs)* (*False Friends, or the Treachery of the English Vocabulary: Advice to Translators*).

In praise of the book, a colleague of the authors sent a letter from Berkeley, which they reproduced before the preface: "Reasoning by analogy is the most precious and the most disastrous of resources. Your book reminds us of the dangers," the letter said. "Familiarity [is] both tempting and treacherous: the pitfalls of a phrase are never to be dreaded more than when it presents itself as a cousin of our own phrase."

DÉNOUEMENT

Caldo and "cold" are false cognates, words that sound alike but are not related. The etymological root of the Italian *caldo* is the Latin *calidus* ("warm" or "hot"). The English "cold," on the other hand, derives from an old Norse word that has become today's *kalt* in German. And *Caldo* has become "scalding" in English.

The English "much" and the Spanish *mucho* are words that mean the same thing but came about via different origins. The English word derives from the Proto-Indo-European *meghs* ("big" or "great"), whereas the Spanish word derives from the Latin *multus* ("much" or "many").

False cognates are purely coincidental. The English "sheriff" and the Arabic *sharif* evolved independently in each language. "Kayak," derived from the Inuit language, and the Turkish *kayak* both refer to a fishing vessel but also appeared independently. And the German *haben* ("to have") means the same as, but is not related to, the Latin *habere*.

Then there are cognates that are related but have gained different meanings over the centuries. The Spanish and Italian word *firma* and the German word *Firma* both derive from the Latin *firmare* ("to confirm"), but the meaning in Spanish and Italian is "signature," whereas the German word refers to an incorporated firm.

Another example is "gymnasium," which denotes an exercise hall in English but, with an upper-case "G," a high school in German. Both meanings derive from the Greek *gymnasion*, a place where men and boys exercised in the nude.

The German *Meer* is *zee* in Dutch and "sea" in English, whereas the Dutch *meer* is *See* in German and "lake" in English.

And a German *Unternehmer* ("businessman") had better not introduce himself as an "undertaker" but as an "entrepreneur." On the other hand, "undertaking" refers not to a funeral but to an enterprise or an initiative, which is not quite the same as an *entreprise* ("corporation" or "organization") in French.

MORE . . .

In the preface of *Les faux amis*, Kœssler and Derocquigny singled out as an example of a "*détestable anglicisme*" the word "investment" as employed by economists and journalists to denote the placement of money. Properly, they maintain, the verb "to invest" refers to clothing, meaning "to clothe in the official robes of an office" and derives from the Latin *investire* ("to clothe in, cover, or surround"). This "*barbarisme*" must have resulted from the incompetence of a translator, the authors write.

Unfortunately, they thereby unwittingly revealed their own incompetence in matters of finance by translating "He has invested his money in the stocks" as "*Il a placé son argent en rente*." But *rente* refers to fixed-income financial instruments, like savings accounts and bonds, rather than stocks, which are risky . . . investments.

8

JANUS WORDS

The Antagonym Paradox

1) "From then on, it was all downhill!"
2) "Albert rents the apartment."
3) "Berta is holding me up."
4) "Cecil consulted with the king."
5) "The reception was cool."
6) "Censors screened the movie."
7) "David fought with Excalibur."

We know what these statements mean, don't we?
No, we don't.

1) Did it become easy going, or was it getting worse?
2) Is Albert a tenant or a landlord?
3) Is Berta supporting me or impeding my progress?
4) Did Cecil give or get advice?
5) Was the reception glacial, or was it a "hot" party?
6) For the public or from minors?
7) Is Excalibur a comrade in arms, an enemy, or a weapon?

The Roman god Janus is usually depicted as a two-faced statue, which symbolizes duality, as in, for example, creation and destruction, beginning and end, light and darkness, past and future. So-called

Statue of Janus in Vienna, Austria.

Source: © lienyuan lee, https://commons.wikimedia.org/wiki/File:Statue_of_Janus_%E5%82%91%E7%BA%B3%E5%A3%AB%E5%83%8F_-_panoramio.jpg.

Janus words have two contradictory meanings. They are often also referred to as antonyms or contronyms.

Janus words are a subset of homonyms, words that sound the same but have different meanings; for example, "The price of entry to the country fair was fair," or "We have a reservation, but I have my reservations about that restaurant."

What distinguishes Janus words from other homonyms is that they have not only different but opposite meanings.

DÉNOUEMENT

Janus words may give rise to confusion, and often it is only the context that makes clear what is meant. But sometimes, even the context may not suffice. For example, "oversight" may refer to attentive and responsible care or to an inadvertent omission or error. "To overlook" may mean to watch and control an operation, or it may mean not to notice something. An action that is sanctioned may mean that it is approved or that it is punished.

This is what makes machine translations so tricky: without the context, it is impossible to determine what is meant by a Janus word, and it takes advanced artificial intelligence to determine the sense. There are other classes of words and pairs of words that may also lead artificial intelligence or human intelligence astray.

Homophones: words that sound the same but have different meanings and different spellings:

I have a pair of pears.
I can see the sea from afar.
I led the people to the store of lead.
While I am overseas, my partner will oversee the operations.

Homographs: words that are spelled the same but have different meanings:

You don't need to lie down to tell a lie.
I object to keeping this object.
She is content with the content.
The bridegroom gives his bride a present and presents her to his parents.

Heteronyms: words that have the same spelling but different meanings when they are pronounced differently:

I have a tear in my eye as I tear up this piece of paper.
The wind blows while I wind the clock.
I lead the people to the store of lead.

Sometimes the difference in pronunciation is very subtle. Take Andrzej, who tells his tutor, "I'm trying to polish up my English." The tutor's answer? "No need, your English is Polish enough."

Some heteronyms are pronounced the same, but the stress is on a different vowel:

The garden was used to prod*U*ce pr*O*duce.
The insurance was inv*A*lid for the *I*nvalid.
The soldier decided to des*E*rt his post in the d*E*sert.

Some words can belong to several categories: "fair" is both a homonym and a homograph. "Tear" is both a homograph and a heteronym. "Sea" and "see" are both homonyms and homophones. And, as pointed out earlier, Janus words are homonyms with not only different but opposite meanings.

Some Janus words are meant to be ironic or are used to add emphasis. "Pretty ugly" is not a contradiction in terms; rather, "pretty" is meant to emphasize "ugly." The same holds for "incredibly trustworthy." "Cool" and "in" are used as ironic Janus words, as, for example, in "this fireplace is cool" and "outdoor activities are in."

Michael Jackson's hit "Bad" is a case in point. As the singer described in an interview, it is a song about a kid from a bad neighborhood who gets to go away to a private school: "He comes back to the old neighborhood when he's on a break from school, and the kids from the neighborhood start giving him trouble. He sings, 'I'm bad, you're bad, who's bad, who's the best?' He's saying when you're strong and good, then you're bad."

MORE . . .

One of my pet peeves is about a category of words that belong to the class of Bushisms, words that have a prefix for emphasis but also a pre-prefix that cancels or revokes the emphasis or emphasizes the emphasis.

For example, why disentangle ropes when you could simply "distangle" them, why disembark from a ship when you could simply "disbark" the *barque* (French for "small boat")? And why does some information remain undisclosed when it could simply remain closed, why are some people who like to work on their own considered uncooperative while they are simply operative? Finally, couldn't judges be "judiced" instead of unprejudiced and unbending officials "disclined" (from the Old English prefix "dys-" used to indicate negation, and the Latin *clinare*, meaning "to bend") instead of disinclined?

Of course, even George W. had a problem with Bushisms because "they misunderestimated" him.

9

PENTASYLLABIC HAS FIVE SYLLABLES

The Grelling–Nelson Paradox

Words are classified as nouns, verbs, prepositions, conjunctions, etc. Adjectives are the class of words that describe objects, like *tall* building, *cold* climate, and *red* flower. Adjectives can be further classified into grammatical types, for example, descriptive, quantitative, demonstrative, and interrogative. Examples are *seven* days, *sufficient* money, *these* flowers, *each* request, and *whose* shoes.

A different, more quirky method of classifying adjectives is one that divides them into two groups according to whether they describe themselves or not. The former are called autological adjectives and the latter heterological adjectives. To illustrate, the word *black* as printed on this page is autological since it describes itself. Conversely, the word *white* as printed here is heterological. The word *pentasyllabic* (i.e., consisting of five syllables) itself consists of five syllables and is therefore autological; *monosyllabic* (i.e., consisting of one syllable) is not.

Since adjectives are either autological or heterological, they can all be classified unambiguously into one of the two categories. Right?

Wrong!

Take the word *heterological*.

WHITE BLACK

It either describes itself, or it does not. If it describes itself, it is autological. Following the definition of *autological*, *heterological* is autological. On the other hand, if that word does not describe itself, then it is heterological. But if *heterological* is heterological, then, following the definition of *autological*, it is again autological.

Get it? A paradox!

It is an amusing pastime to come up with autological and heterological words, not only adjectives. *Unhyphenated* is autological, but *un-hyphenated* is not. *English* is autological, but when a German speaker writes "*Englisch*," it is heterological. *Printed*, when printed on a page, is autological; *written*, when printed, is heterological. Both *tiny* and *elongated* could qualify as autological, as could nouns like *buzzword*, *lingo*, and indeed *noun*, as well as technical terms like *terminus technicus* and *lingua franca*. *Cliché* is, of course, a cliché. *Erudite*? I would classify *erudite* as an erudite word; hence, it is also autological. *Magniloquent* is similarly autological. *Neologism* used to be a new word, hence autological, but is no more and is thus now heterological. When shouted, the word *loud* is autological; so is *quiet* when whispered. But *quiet*, when screamed, and *loud*, when muttered under your breath, are heterological. As a final example, try to utter *unutterableness*. If you can, it is heterological; if you cannot, it is autological.

The paradox was devised by the German mathematician and logician Kurt Grelling (1886–1942), a student of the eminent mathematician David Hilbert (1862–1943) and his colleague, the philosopher

Leonard Nelson (1882–1927). (Grelling was deported and killed by the Nazis in Auschwitz because of his Jewish descent.) The motivation for Grelling and Nelson's paper describing this problem was an attempt to further analyze Russell's paradox (see below).

DÉNOUEMENT

Let's call the set of all words that describe themselves *A* and the set of all words that do not describe themselves *H*. Then the question is, does the word *heterological* belong to *A*?

If it belongs to *A*, then the word describes itself; thus, *heterological* is autological. We have a contradiction.

If it does not belong to *A*, then it does not describe itself; hence, it must belong in *H*. But *heterological* does describe itself, so it must belong in *A*. Again, we have a contradiction.

After some reflection, one can see that the Grelling–Nelson paradox is identical to the conundrum that Bertrand Russell posed (see chapter 31): Figaro is a barber in Seville. He must shave all of Seville's men who do not shave themselves—and only those. Does Figaro shave himself?

• • •

In order to resolve the paradox, Russell distinguished between sets and members of sets. Statements that refer to a set are of a higher logical type than statements that refer to an element of a set. When it is not clear whether a statement refers to a set or to an element of a set, confusion arises.

In the Grelling–Nelson paradox, the set of heterological words, *H*, is of a higher-order logic than is the word *heterological*. According to the theory of logical types, the question whether *heterological* belongs to *H* is meaningless. It is somewhat akin to asking whether a fruit basket tastes nice when what is actually being questioned is whether the apples within the basket are tasty.

MORE . . .

I once visited San Diego on a journalistic assignment. The city has a color-coded trolley-based transportation system consisting of the Green, Blue, and Orange Lines (with Purple and Yellow Lines planned for the middle of the century). I was to take the Green Line from Twelfth and Imperial to Old Town. A blazing red trolley arrived with the displays on the front and sides of the car indicating that this was indeed the Green Line. So, the Green Line is red! Further research revealed that the trolleys of the Blue and Orange Lines are also painted red. The designations of the San Diego trolley lines are heterological! And, to make things really confusing, there is no autological Red Line in San Diego.

• • •

A famous phenomenon in experimental psychology is called the Stroop effect. The experiment goes like this: the words *red*, *green*, *blue*, and *purple* are printed on a sheet of paper but in different colors: *green* is printed in red, *blue* in green, *purple* in yellow, and so on. Subjects are asked to name the colors in which the words are printed. Invariably, they are thrown off; when words and colors are mismatched, subjects take longer to name the color, and they make errors. The incongruity of the word and the color, that is, heterological printing, leads to the Stroop effect.

10

A ROSE IS A ROSE IS A ROSE

The Langford–Moore Paradox

To explain concepts like *father*, *sister*, *bachelor*, and *pentagon*, one may state that a father is a male parent, a sister is a female sibling, a bachelor is an unmarried man, and a pentagon is a five-sided polygon. Such statements affirm that *father* is identical to *male parent*, *sister* to *female sibling*, *bachelor* to *unmarried man*, and *pentagon* to *five-sided polygon*. We have successfully analyzed these concepts.

But are these statements informative?

No!

Huh? What could be clearer than "a father is a male parent"? Well, here's the deal: whenever concepts are identical, we can legitimately substitute one for the other. And when we do so, what do we get? "A father is a father," "a sister is a sister," "a bachelor is a bachelor," and "a pentagon is a pentagon." These statements, though true, are not informative at all. They are trivial; they are tautologies. On the other hand, "time is money" is an informative statement (time is worth money, and wasted time entails costs) but it is not a correct analysis. And father could refer to papa-bear, sister to a comrade-in-arms, bachelor to an academic degree, and Pentagon (with an upper-case P) to the headquarters of the Department of Defense.

Source: "The Pentagon" (https://flic.kr/p/4m7mNY) by David B. Gleason is licensed under CC BY-SA 2.0 (https://creativecommons.org/licenses/by-sa/2.0/).

Thus, if the analysis is correct, it is not informative; if it is informative, it is not correct. Hence, an analysis cannot simultaneously be both correct and informative.

Trust philosophers to muddy the waters.

In the *Concise Oxford Dictionary*, *analysis* is defined as a "resolution into simpler elements." In the *Oxford Dictionary of Philosophy*, it is defined as "the process of breaking a concept down into more simple parts, so that its logical structure is displayed." This is also the thesis proposed by the English philosopher George Edward Moore (1873–1958), one of the founders of analytic philosophy: "A thing becomes intelligible first when it is analyzed into its constituent concepts." But as a thing is broken down into its parts, one eventually reaches a dead end: terms cannot be further broken down; they become unanalyzable and undefinable.

In Moore's words, "You can give a definition of a horse, because a horse has many different properties and qualities, all of which you can enumerate. But when you have reduced a horse to his simplest terms, then you can no longer define those terms. They are simply something which you think of or perceive, and to any one who cannot think of or perceive them, you can never, by any definition, make their nature known."

It would be like trying to define a pentagon to a *Homo neanderthalensis* or the color yellow to a blind person. Or like trying to explain to a three-year-old that the right hand is the one with the thumb pointing to the left.

DÉNOUEMENT

In philosopher speak, the subject of a statement to be analyzed—father, sister, bachelor, or pentagon—is called the *analysandum* (i.e., the item to be analyzed). The object of the statement—male parent, female sibling, unmarried man, or five-sided polygon—is the *analysans* (i.e., the expression proposed as an explanation).

If the analysandum and the analysans have the same meaning, then the analysis is a trivial tautology. If they do not mean the same thing, then the analysis is false.

Try that on the following statements: "an idiot is a fool," "a friend is a buddy," "garbage is trash," "a rug is a carpet," and "money is cash." The first three are trivial since the analysandum and the analysans mean the same thing: idiot = fool, friend = buddy, garbage = trash. The other two statements are false since the analysandum and the analysans have different meanings: money and cash are both means of payment but are not the same thing, and a rug can be a carpet but also a blanket.

Hence, an analysis is either a trivial tautology and therefore uninformative, or it is false.

• • •

To begin an analysis, even a good definition can be informative only if you already have some prior knowledge of the term. It can be a starting point to a superficial understanding of a concept, but more context is often needed for deeper knowledge as to what it really is.

The ancient Greek definition of a point is a case in point: "A point is that which has no part" leaves a lot to be desired if you've never heard of a point before. Only by going through the further definitions and postulates in Euclid's *Elements* does it gradually become clear what a point is. Analyses of many concepts follow the same pattern: one begins with an incomplete definition, but as one gains sufficient experience with the term one eventually gets it.

MORE . . .

For more than a century, poetry buffs have been puzzling over the famous line "A rose is a rose is a rose is a rose" from the 1913 poem "Sacred Emily" by the American poet Gertrude Stein. Grammatically incorrect since it improperly juxtaposes three main clauses, the line evokes the law of identity, one of the three fundamental laws of thought: "*A* is identical to *A*." (The other two are the law of noncontradiction: "*A* and *not A* cannot both be true," and the law of the excluded middle: "*A* must either be or not be.")

Thus, the poem's line is true but not at all informative; it is a trivial tautology, thrice over. But Stein did not have tautology in mind; in fact, she did not want to lead the concentration of the reader away from the flower by defining the rose in terms other than that of the rose itself. Rather, she had the intensification of meaning in mind. Possibly inspired by Shakespeare's "A rose by any other name would smell as sweet," Stein expressed that what matters is what something *is*, not what it is *called*. It is what it is! What you see is what you get.

• • •

The Langford–Moore paradox (so called because the American logician Cooper Harold Langford [1895–1964] made it known more widely in a paper published in 1942) is also known as the paradox of analysis and may be compared to the paradox of inquiry (a.k.a. Meno's paradox; see chapter 38). If you know what you're looking for, inquiry is unnecessary; if you don't know what you're looking for, inquiry is impossible. Therefore, inquiry is either unnecessary or impossible.

III

UNBELIEVABLE BUT TRUE

There's More Than Happy Endings

There's nothing like books, movies, and plays to take one's mind off the daily grind and allow readers and viewers to relax. Relax? Who wants to *relax* when the suspense grips and you're on the edge of your seat? Bring on the next episode . . .

11

WHODUNIT? HE DONE IT!

The Paradox of Suspense

Suspenseful movies—for example, Frank Reicher's *Suspense* (ha!) and Alfred Hitchcock's *Psycho*—are often well worth watching again. In fact, a movie's appeal can be gauged by, among other variables, the number of repeat viewings by fans, and its financial success depends in part on purchases of DVDs and downloads by devotees who have already seen it. Streaming services show their subscribers a list of movies that they have already viewed, suggesting they "watch again."

Upon the first viewing of a whodunit, viewers are kept in suspense until the final scenes, in which the culprit is revealed. But why would people want to rewatch a movie when they already know how it will end? It should be quite boring to sit through a film whose conclusion is no longer uncertain. Nevertheless, experience shows that even upon repeat viewings, audiences wait with bated breath for the climactic ending.

Why?

• • •

Cognitive psychologists have determined that people feel suspense when they fear a bad outcome, hope for a good outcome, or are uncertain about which will come to pass. Thus, according to the *Stanford Encyclopedia of Philosophy*,

1) Suspense requires uncertainty.
2) Knowledge of a story's outcome precludes uncertainty.
3) People feel suspense in response to some stories when they have knowledge of the outcome.

One may accept that statements 1 and 2 are common knowledge. But if they are true, then statement 3 must be wrong: the knowledge of a story's outcome at a repeat viewing should prevent the emergence of any feeling of suspense. So, how is it that some films can be thrilling even on recurrent viewings or, as the author of the encyclopedia entry asked more generally, "that some narrative artworks can still seem suspenseful on repeated encounters"? A lively discussion of this question has been ongoing for several decades in psychology, philosophy, art criticism, and film theory.

The word *suspense* comes from the Latin *suspendere*, meaning "to hang up," in the sense of suspending belief. The phenomenon that occurs at repeat viewings was named the paradox of suspense by the philosopher Robert Yanal: "To raise suspense, a narrative not only withholds information . . . it implies several possible alternative outcomes. . . . This uncertainty as to the narrative's outcome would seem to be a necessary condition for suspense, for it would seem that a person cannot be in suspense regarding an outcome he already knows."

Nevertheless, repeat viewings of a film may continue to invoke suspenseful feelings. The tension associated with anticipation and uncertainty persists even though the spectator knows what will happen. This seems paradoxical.

In fact, a movie can sometimes be more suspenseful if uncertainty is reduced. When the bomb's fuse is lit and the countdown begins, we already know that the scene will end with an explosion; nevertheless, we are on the edge of our seats.

DÉNOUEMENT

The paradox arises because of the incompatibility of the three statements listed earlier. Since they cannot be satisfied simultaneously, there are three ways to unravel the paradox, one for each statement.

In contrast to statement 1, the philosopher Aaron Smuts claims that suspense does not require uncertainty. According to his "desire–frustration theory of suspense," all one needs is a strong desire to change things combined with an ongoing inability to do so. The theory holds that suspense results when one's desire to affect the outcome of an imminent event is frustrated. For example, a movie informs a viewer of impending danger, but the viewer is unable to warn the film's hero. She wants to shout, "Watch out! He's right behind you!" but cannot. The inability to use one's knowledge to affect an outcome creates suspense.

A competing theory called "entertained uncertainty" does require uncertainty, but it need not be genuine uncertainty; it can be imagined. Even if we know that a film will end a certain way, we can—while watching it—still imagine that it will end differently. Merely to entertain the idea of being ignorant of an event's outcome suffices to create suspense. Like good fiction, a good film uses the power of imagination to produce emotional reactions. (It's like getting all worked up about an imagined slight by someone that never actually happened.)

On the other hand, one may reject statement 2 and claim that mere knowledge of a story's outcome need not preclude uncertainty. In the "momentary forgetting theory," suspense does require uncertainty, but viewers are believed to become so absorbed in a movie that they briefly forget that they already know the outcome. They engage in an imaginative exercise in which they pretend that they are uncertain.

Finally, one may reject statement 3. In the "emotional misidentification theory," Yanal argues that repeat viewers do not feel suspense. They simply feel the anticipation of what they know will occur and confuse it with suspense. The audience's excitement relies on the ability to perfectly anticipate what is to come. In this theory, not only does anticipation not require uncertainty, but it actually depends on the knowledge of the outcome. Uncertainty evokes curiosity, not suspense.

MORE . . .

Why do music lovers relisten to symphony concerts and theatergoers reattend performances over and over again? Suspense because of an unknown ending is not what keeps them enthralled. There are other reasons to consume works of music and theater repeatedly. Often the anticipation of the musicians' expressiveness and the performers' dramatic acting makes the public come back for more.

And why do young children want to reread fairy tales—or have them reread—many times? According to the psychologist Bruno Bettelheim, fairy tales help children work through problems such as separation anxiety, oedipal conflict, and sibling rivalry. Though they already know how the tale ends, tension remains because they may need multiple readings in order to come to terms with these problems, at which point they lose interest in the fable.

Finally, why do literature buffs reread novels? Because they change their perspective each time. On the first reading, they may assess the psychology of the characters, on the second sociological associations, and on the third historical or political implications. Tension remains because demanding texts are hermeneutically inexhaustible; they call for constant reinterpretation depending on the reader's perspective, location, and historical context.

12

TO WALLOW IN SORROW

The Paradox of Tragedy

It's the play's third and final act. The hero stabs himself and gasps his last breath, the heroine drinks the poisoned cocktail and sinks to the ground, the spurned lover shrieks in anguish. The audience gasps, and tears stream down many cheeks. The curtains fall, and thunderous applause fills the hall.

A little later, in the bar around the corner, the theatergoers can't get enough of it. They loved, simply *loved* the piece.

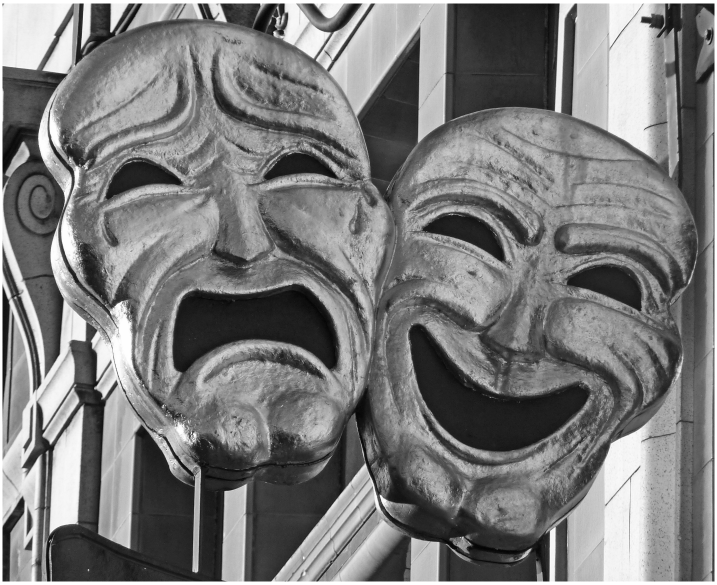
They what? Loved a play that filled them with grief?

Yes!

Clearly, tragedies provoke negative emotions in spectators: sadness, pity, heartache, loneliness, disappointment, guilt, shame, regret, fear, disgust, shock, horror, distress, anger, and indignation.

But here's the paradox: though it *feels* bad to experience these sensations, theatergoers seek out performances that evoke them. Spectators seem to appreciate tragic events in art that they would abhor in real life. They apparently love to wallow in sorrow. The muses of tragedy as well as comedy are common symbols of theater.

The puzzle can be traced back to Aristotle, who in chapter 13 of his *Poetics* asks, "What is the poet to aim for, and what is he to avoid, in constructing his plots?" His answer: "For the finest form



"Tragedy and Comedy," Scarbrough Hotel, Leeds.
Source: © Tim Green, https://www.flickr.com/photos/atoach/8094737104.

of tragedy, the plot . . . must imitate actions arousing pity and fear." The point is driven home in chapter 14: "The tragic fear and pity may be aroused by the spectacle. . . . The tragic pleasure is that of pity and fear, and the poet has to produce it by a work of imitation."

Two millennia later, the Scottish philosopher David Hume (1711–1776) discussed the "unaccountable pleasure" evoked by "a well-written tragedy." Incomprehensibly, the tragedy is pleasing, though that pleasure depends on "sorrow, terror, anxiety," and other naturally disagreeable emotions: "The more [the spectators] are touched and affected, the more are they delighted with the spectacle. . . . They are pleased in proportion as they are afflicted, and never are so happy as when they employ tears, sobs, and cries to give vent to their sorrow, and relieve their heart, swollen with the tenderest sympathy and compassion."

DÉNOUEMENT

Several hypotheses have been offered to explain the allure of tragedies. As pointed out, the first goes back to Aristotle. Feeling pity and fear by attending a tragic play or viewing a sad movie (though, to be precise, Aristotle did not write about movies) might prove to be cathartic: the experience purifies and expels these painful emotions: "Painful art helps drive out painful emotions in a flood of tears."

The second hypothesis, put forth by Hume, suggests that the discomfort of attending the performance of a tragedy is converted into something pleasant and positive through the author's skills, the actors' talents, and the play's intricacy, plausibility, and moral tone. The pleasurable sentiments evoked during the performance compensate spectators for the painful emotions they must endure at the same time.

Another hypothesis, which involves what is called a "meta-response," states that spectators are happy to congratulate themselves after attending a tragic play or movie because they are so kindhearted. They experience positive emotions because only compassionate people like themselves have the ability to feel pain for others.

The "rich experience hypothesis" maintains that people attend tragic plays, watch depressing films, and read sad books because doing so allows them to endure situations of distress without any risk to themselves. In general, one would have to suffer through agonizing circumstances to experience fear, anger, horror, disgust, or misery in real life. By only seeing such emotions played out on a screen or stage, or by reading about them in a book, these emotions can be experienced at a safe distance.

Then there's the "control hypothesis." It claims that one can enjoy negative emotions brought about in a play, film, or book because one retains control. One can always walk out of a theater or put down a book.

Finally, though it does not seem to have been mentioned in the literature, people may also enjoy the feeling of *schadenfreude* (gloating): "I'm glad it's he and not me."

MORE . . .

With the exception of the control hypothesis, all the explanations presented in this chapter may be subsumed under utilitarianism, a moral philosophy proposed in the eighteenth century by Hume's contemporary, the English philosopher Jeremy Bentham (1747–1832). Utilitarianism proposes that human beings strive (or should strive) to maximize their overall happiness and wellness. In *An Introduction to the Principles of Morals and Legislation* (1789), Bentham even suggested an algorithm—he called it the "felicific calculus"—to compute the amount of pleasure that a specific action is likely to cause by balancing "hedons" (fictitious units of pleasure) against "dolors" (fictitious units of pain). Apparently, the hedons elicited by watching a tragedy or reading a sad book outweigh the dolors.

Food for thought: hearing a joke again does not elicit another fit of uproarious laughter.

13

MOVED TO TEARS

The Paradox of Fiction

Tension has been building up for the novel's first two hundred pages. Heart throbbing and pulse racing, you continue to turn the pages. Anxiously, you wonder what will happen. Then, on page 205, the climax: the spurned lover falls into the arms of his beloved, the sick friend is on the way to recovery, the wayward son returns home, the ranch is saved. Breathing a sigh of relief, you put down the book, and your pulse returns to normal.

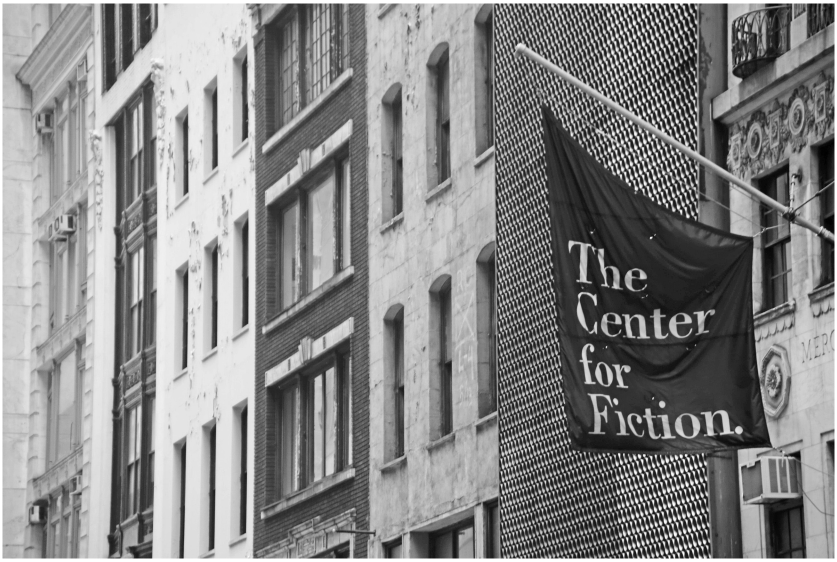
But, wait a minute: it's only a novel! Nobody was about to get hurt, either physically or romantically; it's all fiction. Are you irrational, incoherent, and inconsistent if you are moved by an invented story?

Well, maybe.

In general, one would agree with each of the following three statements:

1) We are genuinely moved (e.g., to tears, to anger, to horror) by fiction.
2) We know that what is portrayed in fiction is not real.
3) We are genuinely moved only by what we believe is real.

But if statements 2 and 3 are true, then statement 1 cannot be true. Or, expressed differently, whoever is moved by fiction is not rational. But as we know from experience, statement 1 *is* true.



The Center for Fiction, 17 East Forty-Seventh Street, New York City.

Source: © Paul Sableman, https://commons.wikimedia.org/wiki/File:The_Center_for_Fiction_(12702798163).jpg.

People like you and me, that is, totally rational people (we hope), *are* moved by what they know does not exist. Novels and stories, even though they simply sprang from authors' minds, have the ability to move us (see also chapter 12 on the paradox of tragedy). That does seem irrational. A paradox!

In 1975, the philosophers Colin Radford and Michael Weston wrote an article entitled "How Can We Be Moved by the Fate of Anna Karenina?" in which they argued that emotional responses to works of fiction are irrational. They called this the "paradox of fiction."

"If you are at all humane," they wrote, "you are unlikely to be unmoved by what you read. The account is likely to awaken or reawaken feelings of anger, horror, dismay or outrage."

So far, so good. But then comes the twist: "But now suppose you discover that the account is false. If the account had caused you to grieve, you could not continue to grieve. . . . If you learned later that the account was false, you would feel that in being moved to tears you had been fooled, duped."

The article spawned a lively back-and-forth in the academic press during the following decades; half a century later, the debate is still ongoing.

DÉNOUEMENT

Several explanations have been proposed in attempts to solve the paradox. One is that readers are quite aware that they are consuming fiction; nevertheless, they pretend to be horrified, amused, or saddened. Fully conscious that they are simulating emotions, readers enter a game of make-believe. The emotions that they experience are not genuine; they are quasi-emotions. The fiction that one reads provides the basis for games of make-believe. This "pretend hypothesis" may also explain why readers experience suspense even on repeat readings of the same book (see also chapter 11 on the paradox of suspense). The pretend hypothesis maintains that statement 1 is false.

In contrast to the pretend hypothesis, the "illusion hypothesis" maintains that we believe that what we read is true. The author's skill in recounting events and in describing scenery and characters so overwhelms the reader that one is deceived into believing that what one is reading is factual. While temporarily suspending disbelief, one deems the romance described in a novel's pages to be a true story. There's a problem, however. In the case of science fiction or fairy tales, the illusion hypothesis renders readers, if only for the course of the reading, superstitious, if not irrational; it turns them into believers in aliens, witches, and vampires. Hence, according to the pretend hypothesis, statement 2 is false.

Finally, a "thought hypothesis" has been proposed to explain the paradox of fiction. It claims that belief in the truthfulness of events, as they are described in a novel, is not necessary in order to be moved. Rather than believing in the actual existence of the portrayed characters or the occurrence of the depicted events, all we need to do is envision them. For example, a novel may provoke thoughts about real people in similar situations or indeed about oneself in an unconscious redirection of feelings from a character to oneself. In effect, the thought hypothesis contradicts statement 3.

MORE . . .

The pretend hypothesis is similar to what I would call "Pavlovian amusement." To wit: a well-known comedian comes on stage and, without uttering a word, makes a grimace. The grimace itself is not funny at all. Nevertheless, the audience bursts into roaring laughter. By convention, spectators pretend that they are hilariously amused whenever this comedian does his signature grimace. The display of any other sentiment (for example, bored indifference) would be considered inappropriate.

• • •

Another paradox is related to the paradox of fiction. In contrast to the intricate plot of *Anna Karenina*, so-called junk fiction—for example, the romance novels sold at supermarket checkout counters and airport bookstores—is extremely formulaic: generic storylines follow a standard and extremely limited repertoire of scripts.

With only minor variations, narratives like "boy sees girl, boy wants girl, boy gets girl" and "girl sees boy, girl wants boy, girl gets boy" are regurgitated over and over again. Nevertheless, certain readers can't get enough of them; as soon as one is finished, another one is picked up. It's a paradox—the paradox of junk fiction.

14

HIDDEN BY QUOTATION MARKS

The Quinification Paradox

Play is a verb.
Table is a noun.
Is a fragment of a sentence is a fragment of a sentence.

Now consider this:

The sentence " 'Is a noun' is a noun" is false.
The sentence "The sentence ' "Is a noun" is a noun' is false" is true.
The sentence "The sentence 'The sentence " 'Is a noun' is a noun" is false' is true" is true.

All clear?

• • •

First of all, why is the sentence " 'Is a noun' is a noun" false? It is false because "is a noun" is a fragment of a sentence, not a noun. But apart from of a bit of parsing, there is actually no problem here, and with some imaginative punctuation or, better yet, insertion of parentheses, the meaning of the last statement becomes clearer:

The sentence {The sentence [The sentence ("Is a noun" is a noun) is false] is true} is true.

So, the statements presented earlier are correct. What about the following statement (I'll call it the *P* statement): " 'Yields falsehood when preceded by its quotation' yields falsehood when preceded by its quotation." Is the *P* statement true or false?

Short answer: if it's true, it's false; if it's false, it's true.

Long answer: let's say that a statement composed of a string of words that is preceded by the same string of words in quotation marks is *q*-shaped. We see that the *P* statement is *q*-shaped.

Now, the example " 'Is a fragment of a sentence' is a fragment of a sentence" shows that *q*-shaped statements can be true. Hence, the *P* statement is false, since—being *q*-shaped—it can be true—even though it states that it is a falsehood.

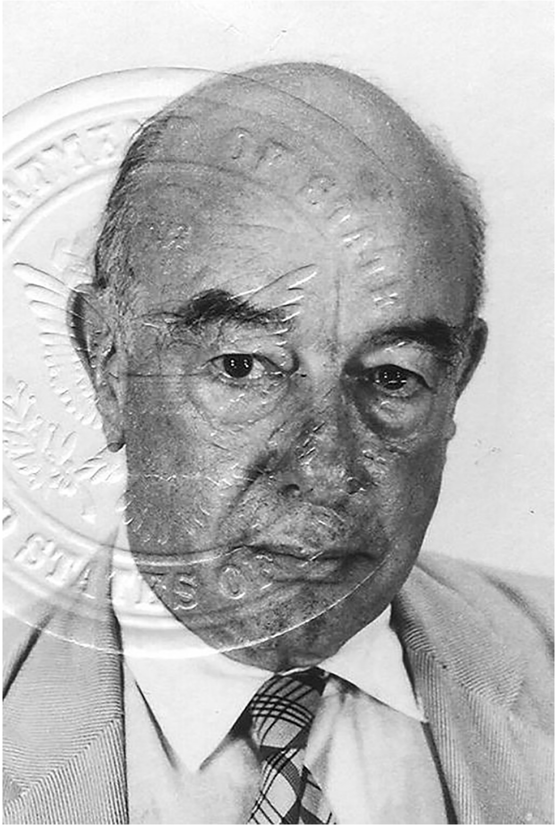
On the other hand, the example " 'Is a noun' is a noun" shows that *q*-shaped statements can be false. Hence, the *P* statement—which says that *q*-shaped statements are false—is true, even though—being *q*-shaped itself—it should be false.

To summarize: if the *P* statement is false, then it says of itself that it must be true. But if the *P* statement is true, then it says of itself that it must be false.

Quine's paradox, named after the American philosopher Willard Van Orman Quine (1908–2000), is very similar to the liar's paradox—with an important difference.

Let me first review the liar's paradox. It tells the story of a Cretan named Epimenides who asserts that all Cretans lie. So, since all Cretans lie, he is telling the truth. But if he, a Cretan, tells the truth, not all Cretans lie.

The liar's paradox, as well as Russell's paradox (see chapter 31) and several others in the present collection, are based on the vexing problem of self-reference. Many philosophers believed that self-reference was a pathological situation, the elimination of which



Passport photo of Willard Van Orman Quine (1908–2000).

Source: https://commons.wikimedia.org/wiki/File:Willard_Van_Orman_Quine_passport_cropped.jpg.

would remove the paradox. Quine did not agree; his objective was to show that self-reference is not the only reason for such paradoxes. To prove his point, he sought a paradox that was not based on self-reference. The statement he devised—a string of words preceded by the same string in quotation marks (which I call "*q*-shaped")—is grammatically correct and does not refer directly to itself.

Nevertheless, it creates a paradox. Hence, a lack of self-reference—which is what differentiates Quine's statement from the liar's paradox—does not resolve the paradox.

In honor of Quine, the concatenation of a sentence with its quotation has been called "quinification," and *q*-shaped statements are referred to as "quined."

DÉNOUEMENT

Self-reference paradoxes may arise when elements of a set—a set of things or people—refer to themselves. Take the fairy tale *Rumpelstiltskin*. If someone asserts, "*Rumpelstiltskin* is a lie," there is no self-reference since the subject ("*Rumpelstiltskin*") and the complement ("a lie"), which are linked by the verb ("is"), belong to different sets: the former to the set of fairy tales, the latter to the set of utterances. In contrast, in the statement "This sentence is a lie," the subject ("This sentence"), the complement ("a lie"), and the entire statement ("This sentence is a lie") belong to the set of utterances. That alone need not present a problem, in the same manner as "The sentence is a lie" presents no problem. But by asserting something about "*This* sentence . . ." the verb ("is") links not a subject ("The sentence") but the entire statement ("This sentence is a lie") to the complement. In effect, the statement asserts, "The sentence 'This sentence is a lie' is a lie." Such a self-reference is what creates the paradox in a manner that "*Rumpelstiltskin* is a lie" does not.

To show that self-reference is not the only reason for such paradoxes, Quine sought a statement that referred to itself without referring to itself. (If that sounds paradoxical, then you begin to understand why he needed to come up with the convoluted quinification.)

To illustrate: in a statement like "The king's speech yields a falsehood when preceded by its quotation," the verb ("yields") links the subject ("The king's speech") to the object ("a falsehood"). Though both subject and object belong to the set of utterances, there is no paradox.

In a *q*-shaped sentence, the string of words enclosed in quotation marks assumes the role of the sentence's subject. This is where confusion sets in: usually the subject is just a noun. But as *q*-shaped sentences show, it does not have to be.

So, at first, no paradox is apparent because the quotation marks hide the actual subject behind a veil of ignorance. Only when the veil is lifted and one is able to inspect what is inside the quotation marks is the subject revealed to be identical to the object. But neither object nor subject link to the entire sentence; hence, there is no self-reference as such. Whatever reference there is is only indirect, and Quine's objective to show that self-reference is not required for this type of paradox has been achieved.

MORE . . .

The paradox is inherent in any sentence that discusses truth and falsity in any language (like English) that contains words or sentences that can be used to describe themselves (see also chapters 9, 31, and 36). One usually expects words like *I* or *this* or something similar in sentences that refer to themselves. In quined sentences, however, the subject names itself through the use of quotation marks, and the verb links the subject both directly to the object and indirectly to the entire sentence.

• • •

Food for thought: has Quine's objective really been achieved? It could be argued that the word *its* in the phrase "preceded by its quotation" does indicate self-reference. Hmmm . . .

15

ALL REMAINING ERRORS ARE MY OWN

The Preface Paradox

The author of a PhD dissertation in the sciences has checked and rechecked each statement in her treatise. Moreover, her doctoral advisers, several referees, and the editor of an academic publishing house have also verified all statements. Nevertheless, she adds a disclaimer to the preface: "My dissertation is based entirely on facts, and the content has been carefully checked by me; I thank the numerous experts who have also verified the accuracy of all statements. Any remaining errors, however, are my own."

In nonfiction books, such everyday expressions of gratitude are common practice. But the disclaimer, though routine in academic texts, may surprise. Is it just feigned humbleness? Or does the author—after all that fact-checking and cross-referencing—truly believe that errors remain?

It is quite rational for the author to believe that each statement is true. Unfortunately, the author knows from previous experience that in spite of her best efforts, errors are inevitable. After all, *errare humanum est*, to err is human. Hence, it is also rational for her to believe that the disclaimer in the preface is true.

Nevertheless, the disclaimer seems paradoxical. On the one hand, the author claims that all facts in the book are true. On the other hand, she simultaneously admits that this very claim may be false. We have a claim immediately followed by a disclaim (or, rather, *disclaimer* in proper English).

Clearly, this is inconsistent. Or contradictory. Or both. Or neither. But one thing is for sure: we are faced with a paradox.

As prosaic as the disclaimer sounds, it has led to numerous debates and a sizable literature in the philosophical community.

It all started with a competition in *Analysis*, an esteemed journal for short papers in philosophy published by Oxford University Press. In June 1955, A. M. MacIver from Southampton University posed a prize question: "How can I think it possible that I might be mistaken?" A year later, MacIver mourned the fact that only two competitors had submitted answers, one of whom did not even see the point of it, while the other thought the question could be dodged. "I find this not only disappointing but odd (though not quite unexpected)," MacIver wrote. "Can it be that there is really no problem? If so, I wish someone would explain to me what confusion has led me to think that there is."

A decade later, the cudgel was taken up by D. C. Makinson in an article entitled "The Paradox of the Preface," in which he concluded that "such statements express a state of mind which, one may argue, it is impossible to have." Over the next decades, several other learned papers followed. We'll concentrate on a 1987 paper by the philosopher John Williams from Singapore Management University.

DÉNOUEMENT

Let's analyze the example that gave the preface paradox its name. We denote the statements in the book as S_1, S_2, . . . , S_n and the disclaimer in the preface as D. Though each S_i is believed to be true, the disclaimer says that their combination is false: $D = \sim(S_1 + S_2 + \ldots + S_n)$.

Hence, the author's set of beliefs is expressed as follows: S_1, S_2, . . . , S_n, D. First, she is perfectly rational in believing that each assertion S_i by itself is true. But, second, she also believes that D is true, which says that not all S_i are true. Is there a contradiction?

No, explained Williams. The author's set of beliefs is obviously inconsistent—but not contradictory. Had the author believed $S_1 + S_2 + . . . + S_n + D$, that would have been contradictory according to Williams since it corresponds to the following:

$$(S_1 + S_2 + . . . + S_n) + \sim(S_1 + S_2 + . . . + S_n)$$

The crux of the paradox lies in the acceptance or rejection of the "conjunction principle." This principle asserts, erroneously as Williams and other philosophers maintain, that a belief that several propositions are true entails the belief that their conjunction is also true; that is, if $S_1, S_2, . . . , S_n$ are believed to be true, then $S_1 + S_2 + . . . + S_n$ is also believed to be true. Once the conjunction principle is recognized as misguided and abandoned, the paradox disappears.

The distinction is important: all believers who hold contradictory beliefs hold inconsistent ones, but not all believers who hold inconsistent beliefs hold contradictory ones. Dropping the conjunction principle permits the possibility of rational inconsistent belief. And since the relevant beliefs expressed in the preface are inconsistent rather than contradictory, they are not paradoxical but entirely rational.

MORE . . .

With some basic probability theory, it is quite rational for the author of a book to believe that each statement in her work is true and, simultaneously, to believe that there are errors. Let's stipulate that whenever a statement has a probability of being true of at least 99.9 percent, one is entitled to believe that it is true.

Now, consider this: the present book contains about 3,000 sentences. (I haven't really counted.) For each sentence, $S_1, S_2, \ldots, S_{3,000}$, I have made 99.9 percent sure that it is correct. (I am being far too optimistic.) Hence, I am entitled to believe that each sentence is true (I hope). What is the probability that all sentences are true? According to basic probability theory, the chance of all sentences being true is $0.999^{3,000}$, which equals 0.0497. . . . In other words, there is a less than 5 percent chance that all sentences are correct and a probability of more than 95 percent that at least one sentence is false. Hence, the disclaimer in the introduction to this book is certainly in order.

• • •

The two following examples are similar to, but not the same as, the preface paradox. An airport manager who announces, "The aircraft will take off at 11:30, but I could be wrong," apparently believes that the flight will take off at 11:30 but also believes that it might not. Or an interior decorator who declares, "I want to paint the wall green, but I may not," is stating that he wants to paint a wall green but simultaneously believes that he may not.

Such declarations express a state of mind that is arguably impossible to have. But the conflicting proclamations hinge on the use of modal verbs like *can*, *could*, *may*, *might*, and *must*. Modal verbs indicate likelihood, permission, ability, or obligation. Their use in the second part of a statement relativizes the first part of the statement and hence indicates neither inconsistency nor contradiction.

• • •

The preface paradox is the opposite of Moore's paradox (see chapter 32), which deals with statements like "It's raining, but I don't believe that it's raining." Moore's paradox expresses a disbelief in truth, whereas the preface paradox expresses a belief in falseness.

IV

YOU DO THE MATH

Numbers Don't Lie—Go Figure!

Mathematics is nothing if not exact, clear cut, and unambiguous, right? Well, think again! Even the queen of sciences hosts unusual ambiguities, obscure puzzles, and perplexing paradoxes.

16

CHOCOLATES FROM THE TRAYS

The Axiom of Choice

There is a famous shop in Zurich, Switzerland, for all things chocolaty called Sprüngli. Behind the counters, well protected by glass covers, are trays of pralines and truffles, dark chocolates with pistachios on top, chocolate-covered orange slices, marzipan-filled cubes, and many more delicacies.

Customers point to the various trays, and neatly uniformed salespeople wielding tongs pick one item from each of the trays to which the customers point. Then the chosen items are packed into little cardboard boxes, and the happy customers leave with neat little packages filled with the goodies.

Right?

According to some mathematicians, no, this is not right. In fact, according to their deeply held conviction, this is impossible.

Huh? I have gone through this procedure at Sprüngli's many, many times, so how can anybody claim that I am unable to get the chocolates? Well, fortunately some mathematicians say that it *is* possible.

Aha! So what exactly is possible according to some mathematicians but impossible according to others? It's the salespeople wielding the tongs, that's what.

Sprüngli.

Source: https://commons.wikimedia.org/wiki/File:Sprüngli.jpg.

The problem is that while the confections in various trays differ, within each tray the delicacies are identical. The *amandines*, the *nougatines*, the *visitandines*—each quite unlike the others—are indistinguishable among themselves. So how can the salespeople choose which almond cookie to put into the box? Unsure which item to pick, they must be at a loss at every customer request, just as Buridan's ass was immobilized between two bales of hay. Only if the customer specifies "the leftmost *florentin*" or "the second *Luxemburgerli* from the bottom" can a pick be made. It simply won't do to say, "Please just give me one of those champagne truffles."

The theoretical debate over whether the salespeople are able to pick items from the various trays or whether they will hesitate at every request has stopped philosophers and mathematicians in their tracks for decades—though you would not believe it when observing the hustle and bustle at Sprüngli's in Zurich.

An example by the philosopher Bertrand Russell renders this counterintuitive assertion—that it is impossible to make selections—more comprehensible. Lord Chichester has a big collection of shoes and must decide which pair to wear to the queen's garden party. To make up his mind, he asks his butler to bring him one shoe from each pair. The butler has no problem with this request: he simply brings the left shoe of each pair for the lord's appraisal.

Now for suitable socks. Lord Chichester asks the butler to bring one sock from each pair of his sock collection. This time, the butler does have a problem. Which sock should he choose? Within each pair, the two socks are identical. As I shall describe, to be able to follow the lord's instruction, the butler requires the so-called axiom of choice. Only if he accepts it can he bring one sock from each pair.

• • •

The deep mathematical question is whether choices can be made automatically. As pointed out earlier, some mathematicians believe they can; others believe they cannot. To wit: a computer algorithm cannot be instructed to "just choose one" from a collection of items; it must be instructed specifically which one. It is an indisputable fact, however, that the salespeople at Sprüngli's are able to make choices among identical items. Hence, the group of nonbelievers must somehow come to terms with this phenomenon.

It was the mathematician Ernst Zermelo (1871–1953) who provided the answer. Zermelo investigated a conjecture of Georg Cantor, the founder of set theory, which stated that every set of objects can be well ordered. In simple terms, the "well-ordering principle" means that for any set, and for any subset of that set, there exists an ordering such that the smallest element within the set can be identified. In a bit of roundabout reasoning, this provides the loophole out of the dilemma for salespeople and butlers but mainly for doubtful mathematicians.

DÉNOUEMENT

So what's the way out for the conflicted mathematicians? On the one hand, they think that choices cannot be made; on the other hand, the salespeople at Sprüngli's obviously do make choices. The answer is typical for mathematicians: simply assume that choices can always be made. Even better, stipulate it as an axiom: for any set *X* of nonempty sets (e.g., for all trays filled with goodies at Sprüngli's), there exists a choice function *f* defined on *X*. That is, some sort of intuition tells the salespeople which item to choose. The axiom states that a choice can always be made but without specifying how.

This is what Russell illustrated with the example of the shoes and socks: whenever a selection rule can be stated—for example, "always select the left shoe"—the axiom is not needed. But when no such rule can be specified because the items have no distinguishing features—for example, "bring me one sock from each pair"—the axiom that such a choice *can* be made is required before the butler, the Sprüngli salespeople, or the mathematician can proceed.

The axiom of choice is one of the most discussed axioms of mathematics. Many theorems use it in their proofs, some of them nearly without taking notice. Its importance is comparable to the importance in geometry of Euclid's parallel postulate. With it, we have geometry as we know it, without it, we get elliptic and hyperbolic geometries. Similarly, with the axiom of choice many theorems can be proved, without it, the theorems may be wrong or the proofs incomplete. By and large, the mathematical community accepts the axiom of choice. Hence, even the theorems that require it in their proofs are in general considered correct.

MORE . . .

Zermelo proved that the axiom of choice is true if one assumes that the well-ordering principle is true. And vice versa. Now, recall that the well-ordering principle simply says that there exists some kind of ordering and that according to this ordering, there exists a smallest element. How the elements of the set are ordered—for example, alphabetically, numerically, by temperature, or by color—and what *smallest* means in terms of that ordering is left unspecified.

But can every conceivable set be well ordered? Does, say, the set of positive integers have a smallest element? Yes, it is the number one. On the other hand, the set of real numbers does not have a smallest element because for every teeny-weeny number that you can think of, there exists an even teeny-weenier one.

All this may be confusing, but human intuition does not always follow what is mathematically correct. The axiom of choice agrees with the intuition of most people; the well-ordering principle is contrary to the intuition of most mathematicians.

However, we may get an inkling of how the two are equivalent. The well-ordering principle says—without further specification—that there is some order that allows the smallest element to be picked. The axiom of choice says—without specifying how—that a choice can be made. If one can do one, one can do the other.

One problem with the axiom of choice is that it sometimes leads to counterintuitive results as with the Tarski–Banach paradox (not discussed in this book).

• • •

By the way, I reiterate: while the staff at Sprüngli's *can* make choices, computers cannot. For more on this, see chapter 19.

17

ROUNDING CROOKED NUMBERS

0.999 . . .

It is often easier to deal with nice round figures like 6 or 18 rather than "crooked" numbers like 6.01356837 or 17.986757321. This is why we often round numbers up or down so that 6.01356837 becomes 6, and 17.986757321 becomes 18. We realize that a small error arises when rounding, but very often the resulting ease of computation is worth that error. But can some crooked numbers be rounded up without incurring any error at all?

Yes.

The never-ending number 0.999999 . . . , for example, with an infinite number of 9s after the decimal point, turns out to be not only *nearly equal* to 1.0 but *exactly equal* to 1.0.

How can that be? Surely, even with an infinite number of 9s after the decimal point, the resulting number must be an itsy-bitsy, teeny-weeny bit less than 1.0. Well, no, it's not, as we shall see.

And what about numbers like 3.19999 . . . and 7.63529999 . . . ?

Decimal fractions were invented by the Arab mathematician Abu'l-Hasan al-Uqlidisi in the ninth century and reinvented by the Persian scholar Jamshid al-Kashi in the fifteenth century. In the sixteenth century, the Flemish physicist, mathematician, and engineer Simon Stevin represented numbers with unending decimals. And in 1758, the Swiss mathematician, astronomer, and philosopher Johann

Heinrich Lambert proved that the decimal representation of π never ends; that is, an infinite number of digits follow the decimal point.

Leonhard Euler (1707–1783), also Swiss, was one of the foremost mathematicians of the eighteenth century. Together with the Bernoulli brothers, cousins, and uncles—his close friends and mentors (all Swiss as well)—he was largely responsible for developing the infinitesimal calculus on which modern mathematics and engineering is based. But Euler also dealt with algebra as it is nowadays taught in high schools. In 1770, he published a two-volume textbook on the subject. The first volume contained 562 consecutively numbered paragraphs; the second volume added another 250.

The book was not directed at mathematicians but was meant to be read by anybody. In fact, Euler—who had lost his eyesight four years earlier and by the time of publication was nearly totally blind—dictated the text to a tailor who wrote everything down as Euler spoke. It is said that the great teacher's explanations were so clear that this young man, being of only mediocre intellect, not only understood everything that Euler dictated but was soon able to solve algebraic problems himself.

It is in the chapter on infinite decimal fractions, in paragraph 524 to be exact, that the number 0.9999 . . . made its first appearance.

• • •

In *Mathematics: A Very Short Introduction*, Timothy Gowers claims that the statement "0.9999 . . . equals 1.0" is a convention, albeit an indispensable one in conventional mathematics. Because if 0.9999 . . . were *unequal* to 1.0, then what would the difference between the two numbers be? If it were unequal to 0, then it would have to be something infinitesimally small but nevertheless larger than 0. And what could that be? A 0 with an infinite number of 0s after the decimal point, followed by a 1? (0.0000 . . . 1) For such a number to

Leonhard Euler.

Source: https://commons.wikimedia.org/wiki/File:Leonhard_Euler_2.jpg.

exist, a completely new, unconventional mathematics would have to be invented.

On the other hand, if it is 0 (which, in fact, it is), then 0.9999 . . . cannot be unequal to 1.0 since, whenever the difference between two numbers is 0, the two numbers are equal to each other.

DÉNOUEMENT

Let's denote the number 0.9999 . . . by X:

$$X = 0.9999\ldots$$

If we multiply both sides by 10, we get

$$10 \times X = 9.9999\ldots$$

If we deduct X from both sides of the equation, we get

$$9 \times X = 9$$

And now it's obvious: $X = 1$

QED (*quod erat demonstrandum*, Latin for "this is what had to be proved").

Another way to see that 0.9999 . . . is equal to 1.0 is as follows: the fraction one-third (1/3) is written in decimal notation as 0.3333 . . . If we multiply this by 3, we get 0.9999 . . . On the other hand, 3 times 1/3 is equal to 1.0. Voilà: 0.9999 . . . is equal to 1.0.

• • •

0.9999 . . . is not the only number that is equal to another number. For example, the number $Z = 3.19999\ldots$ is equal to 3.2. Why? If you multiply both sides by 10 and then deduct Z from both sides, you get $9 \times Z = 28.8$. Now divide 28.8 by 9, and there you have it: $Z = 3.2$.

As an exercise, do the same with, say, 7.63529999 . . .

The argument also works the other way around. Hence, the number 4.57 is identical to the number 4.569999 . . .

MORE . . .

Leonhard Euler arrived at this conclusion in a more sophisticated way. He was not really interested in numbers ending in 9999 . . . Rather, his interest was kindled by so-called progressions. Progressions are mathematical constructs of the following form:

$$1 + k + k^2 + k^3 + k^4 + \ldots$$

(Nowadays, we call Euler's progression a geometric series.) Can such a never-ending progression be summed? Euler showed that it can, as long as k is less than 1. In fact, he showed that in this case, the sum of such an infinite series is equal to $^1/_{(1 - 1/k)}$.

Guess what Euler used to illustrate his result. Yes, it was the number 0.9999 . . . since it can be written as an infinite progression:

$$0.9999\ldots = 0.9\ (1 + {}^1/_{10} + {}^1/_{10}{}^2 + {}^1/_{10}{}^3 + {}^1/_{10}{}^4 + \ldots)$$

With $k = {}^1/_{10}$, the sum of the progression inside the parentheses becomes $^1/_{(1 - 1/10)} = {}^{10}/_9$. Hence, $0.9999\ldots = 0.9 \times ({}^{10}/_9) = 1$. QED.

18

ON OR OFF?

Thomson's Lamp

It is 23:58, two minutes before midnight (11:58 p.m. for you Americans). You're having a sleepless night and switch on the lamp on your bedstand. After one minute, you switch it off again. After half a minute, you switch it on. After another quarter of a minute, you switch it off. And so on: you switch it on again and off again, and on again and off again, each time halving the elapsed time.

At exactly midnight, is the lamp on or off?

Unclear.

But first things first: the time periods—1, ½, ¼, ⅛ . . .—add up to exactly two minutes. So, as the number of flips approaches infinity, the time approaches midnight.

Next: at all times, the lamp is either on or off. It cannot be both, and it cannot be neither. As we get close to midnight, whenever the lamp is on, a split-second later it is off again. And whenever it is off, it is switched on a split-second later. So, what's the situation when the clock strikes twelve?

The conundrum was devised in 1954 by the British philosopher James F. Thomson. He wrote that the lamp "cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never

Source: https://commons.wikimedia.org/wiki/File:MarsPerSundstedtAteljeLyktan.png.

turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction."

Thomson was investigating what he called "supertasks," a series of acts that are performed infinitely often in a finite amount of time. Supertasks come in many forms; one example is Zeno's well-known paradox of Achilles and the turtle: if the turtle has a head start, Achilles can never catch up with it because he must always reach the point where the turtle was, but the turtle keeps moving.

But the question is whether such tasks are possible, in theory or in the real world. Some thinkers argue that the mere notion of performing "an infinite number of acts" is self-contradictory because it actually means performing "one more act than any finite number." And one physicist claims that any attempt to carry out a supertask in the real world would produce a divergence of the curvature of spacetime, resulting in the formation of a black hole. The formation of the black hole would then kill the operator, thus ending the supertask.

If supertasks were possible, then as-yet unproven conjectures in number theory (for example, the Goldbach conjecture) could be proven, and the truth or falsehood of certain undecidable propositions could be determined in a finite amount of time, through a brute-force verification for all natural numbers.

DÉNOUEMENT

To resolve the paradox, note that Thomson's experiment does not contain enough information to determine the state of the lamp at exactly midnight. The switch is flipped starting at 23:58 and then after 1, 1½, 1¾, $1^7/_8$, . . . , $1^{255}/_{256}$, . . . minutes, but in Thomson's description, the hands of the clock never complete the rotation of the full two minutes—just as Achilles never reaches the turtle. Hence, the state of the lamp is specified at every instance until just before midnight but not at midnight itself. So, Thomson's claim that the lamp could not be on because it was never on without being turned off again, and vice versa, applies only to instants of time *strictly before midnight*. Hence, the question is poorly posed and cannot be answered.

To illustrate this, let's number the flips of the switch: $n = 1, 2, 3, \ldots$ Whenever n is odd, the light is on; whenever n is even, the light is off. Hence, another way to express this problem is to ask whether the "final" number (after all of 1, 2, 3, . . . have been accounted for) is even or odd. And when you put it that way, the question is seen to be nonsensical.

• • •

I will present three more dénouements: one mathematical, one probabilistic, and one physical.

A mathematical solution runs as follows. We denote the state of the lamp by 0 if it is off and by +1 if it is on. And we denote an on flip as +1 and an off flip as −1. Starting with 0 (the lamp is off), we determine the state of the lamp at midnight by adding up the flips:

$$S = 1 - 1 + 1 - 1 + 1 \ldots$$

We can rewrite this as

$$S = 1 - (1 - 1 + 1 - 1 + 1 \ldots),$$

and recognize that

$$S = 1 - S$$

Therefore,

$$S = \tfrac{1}{2}$$

Oh, so, the lamp is half on and half off? Nice try, but no cigar (see also Grandi's paradox in chapter 20).

Another attempt to answer the question uses probability. What are the chances that the lamp will be on at midnight? Well, it is on for one minute and off for half a minute, then on for a quarter of a minute and off for an eighth of a minute, and so on. Obviously, it is on twice as long as it is off. Therefore, the probability of the light being on at midnight must be ⅔. Ha ha!

Finally, a physical explanation. For the lamp to be on or off, there must be a time interval during which the lamp finds itself in that state. But at the exact switching times, no such interval exists since the state of the lamp differs immediately to the left and immediately to the right. So, the state of the lamp is undefined at every switching time. In particular, around midnight, there is no interval during which the lamp is in a constant state.

The consensus among philosophers seems to be that Thomson's lamp is not a matter of paradox but of an incomplete description.

MORE . . .

Why do we only consider two minutes?

According to first-year calculus, the sum of an infinite series beginning with A and always adding the previous value multiplied by h ($h < 1$) is equal to $A/_{(1-h)}$. Thus,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots = {}^{1}/_{(1-\frac{1}{2})} = 2$$

19

RANDOMNESS IS NOT RANDOM

The Random Numbers Paradox

Is the following sequence of digits random? 7, 3, 3, 9, 2, 8, 7, 8, 2, 0

Is the following sequence of bits random? 1, 1, 1, 1, 1, 1, 1, 1, 1, 1

Yes and yes.

Yes and no.

No and yes.

No and no.

Dunno!

The last answer is correct: we simply do not know.

• • •

When coins are tossed to kick off a football game, they fall half the times heads, half the times tails (or, as mathematicians like to say, zero or one). In games with dice, the numbers between one and six appear, each with a probability of one-sixth. And in casinos, the roulette ball falls on a number between zero and thirty-six about 2.7 percent of the time. What these tosses have in common is that the numbers thrown are purely random; the subsequent tosses are independent of the previous ones. Gamblers who think that after several throws on tails a toss of heads must necessarily follow will sooner or later go bankrupt.

Sequences of numbers that meet three criteria—all numbers are equally probable, independent of the previous ones, and unpredictable—are called random, and they are very important in many areas of daily life. Gambling is just one of the activities for which you need random numbers. Random numbers are required in economics, medical research, science, and computer science. For opinion polls, participants are selected by means of random numbers; in medical research, test subjects are randomly assigned to different groups, and the names of recruits are drawn by lot.

One of the most important applications of random numbers is in simulation studies. Random numbers indicate which of several alternatives should occur in a scenario, and many scenarios are run and inspected. Such efforts began with the Manhattan Project in World War II when the Americans were developing the first atomic bombs. Since nuclear explosions cannot be tested experimentally, they had to make do with simulations.

It is not only dangerous phenomena that are routinely simulated. To calculate the interaction of external and internal influences in complex systems using the laws of probability is often far too difficult. Instead, scientists, managers, and engineers take recourse to simulations. Aircraft manufacturers, for example, simulate the behavior of aircraft under various weather conditions and pilot reactions. Companies use simulations to run through scenarios to find out how procurement costs, collective bargaining, and strikes affect profits. Economists simulate how economic decisions influence each other and affect inflation and unemployment.

In computer science, algorithms can be massively accelerated when random numbers are used. For example, in the 1950s, the so-called bubble-sort algorithm was used to order a list of millions of names alphabetically. When Quicksort, an algorithm that makes use of random numbers, was launched in 1961, the running time could be accelerated by several orders of magnitude. In cryptography, random numbers are used in addition to prime numbers, which must be chosen at random. And using simulations, even the irrational number π can be approximated.

An indispensable condition of a simulation is that the numbers used to mimic various scenarios are truly random. But how can we be certain that a number sequence is truly random? Here's the paradox: we can't! If a sequence of numbers were indeed purely random, we could not recognize it as such—for if we did, it would not be random.

DÉNOUEMENT

For many applications, millions of random numbers are required. One might think that they can simply be generated by computers. However, this is not the case. Computers are deterministic systems; a computer must be told exactly what to do. An instruction such as "pick heads or tails" or "select a number between zero and nine" cannot be followed by a computer. The best that computers can do is to produce so-called *pseudo-random numbers*, which look as if they were randomly generated but in fact were produced by deterministic, albeit complex, algorithms. Since the days of the Manhattan Project, mathematicians and computer scientists have been striving to find ever-better algorithms for the production of numbers that get close to satisfying the three randomness-criteria mentioned earlier.

Common to most such algorithms is that the numbers are in some way based on the preceding numbers in the sequence. This means that one of the criteria, namely independence, is always violated. To mitigate the effect of this violation, computer scientists help themselves by choosing the very first number of the sequence, the "seed," as randomly as possible—such as the millisecond at which one clicks "Enter"—and then making the relationships of further numbers with their predecessors as obscure as possible. This is done mostly with the help of so-called *one-way functions*, which are very easy to compute (such as by multiplying prime numbers) but hard to reverse (such as factorizing a number into its primes). It is then impossible to determine how the members of a pseudo-random sequence follow each other.

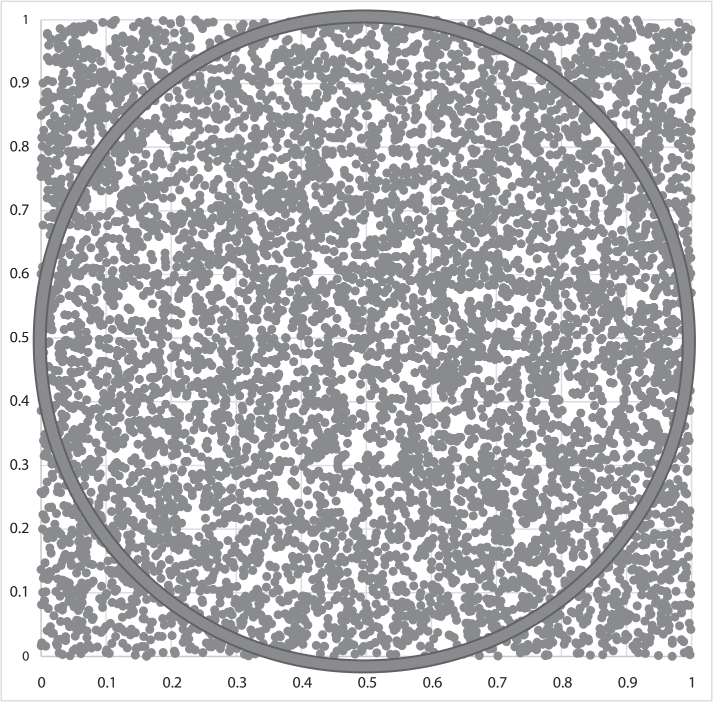
Purists, however, believe that even pseudo-random numbers are inadequate. Simulations have been known to occur that yielded incorrect results because generators were used that exhibited a minimal, albeit hidden, systematic bias.

That's why researchers often make do with numbers based on natural phenomena, such as atmospheric noise, lava lamps, and weather conditions, or human activity, such as the movements of a computer mouse, stock market prices, or time delays in typing. Computer scientists have also shown how several time sequences—for example weather conditions, stock market movements, etc.—can be mixed to extract as much randomness as possible.

But even these methods are inadequate. After all, meteorologists and investment advisers often have some success in forecasting weather conditions and stock prices. And even tosses of coins and dice are not entirely random. Since they obey Newton's laws, they are predictable, at least in principle: if the initial conditions and the forces acting on them are known precisely, the outcome of tosses could theoretically be predicted.

This leaves only one method that most scientists believe generates truly unbiased random numbers: quantum mechanics. The Swiss company ID Quantique, for example, has developed a generator that makes use of quantum effects. Photons are fired at a semitransparent mirror; half the photons fly through the mirror; the other half are deflected. Photon counters register the bit as zero in one case and as one in the other. Since, according to the laws of quantum mechanics, it cannot be predicted which case will occur, the sequence of zeros and ones is completely random. But even that is inadequate because the setup is never completely exact. For example, for the numbers to be purely random, the mirror would have to be placed at an angle of precisely 45.00000 . . . degrees.

So, back to our question: How can we ascertain if a sequence of numbers or bits is random? Since inspection of the sequence itself



Simulating the number π.

Source: © George Szpiro.

does not provide an answer, the best one can do is inspect the process that generated it. If the process is close to random, we may be confident that the sequence is close to random.

MORE . . .

How can the irrational number π be simulated with random numbers?

In the coordinate system, a square with a side length of 2.0 is defined around the origin. The area of the square is thus 4.0. A circle with radius $r = 1.0$ is drawn into this square; its area is πr^2, which is equal to 3.141 . . . Now, if eight thousand random numbers between −1 and +1 correspond to the x and y coordinates of four thousand random points within the square, 78.539 . . . % of the points fall within the circle. Thus, the proportion of points falling within the circle (78.539 . . . % of 4.0) approaches π.

For even more on random numbers, see my forthcoming book *Random Numbers: All the Best Bits*, 2024.

20

ZERO OR ONE? THAT IS THE QUESTION

Grandi's Paradox

What is the sum of the series 1 − 1 + 1 − 1 + 1 − 1 + . . . if you continue it to infinity? The partial sums—the sums obtained when you continue the series just for certain number of steps—are 1, 0, 1, 0, . . . and so on. The series appears not to have a sum. In mathematical lingo, it is said to diverge since it does not *converge* to a definite sum.[1]

Is that really so? Maybe we can explore the series further by putting parentheses around the sum's expressions like so:

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots = (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + \ldots = 0$$

Aha! The sum of the series is zero. Correct?

Not really. Let's try again, but this time let's put parentheses around the expressions of the series a little differently, like so:

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots = 1 + 0 + 0 + 0 + \ldots = 1$$

1. Infinite series that sum to a definite number are said to converge. Series that do not converge are said to diverge. Most often, divergent sums continue to grow toward infinity. Series such as the present one neither diverge toward infinity nor converge. They would better be called "*non*vergent."

Oops! Now the sum of the series is one.

So, zero or one, that is the question. A paradox!

Several mathematicians in the seventeenth and early eighteenth centuries came across the series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ The first serious treatment of this series was performed in 1703 by the Italian monk Luigi Guido Grandi.

Grandi concluded that the series sums neither to one nor to zero but to one-half! How did he justify this? With a story. Two brothers, Titius and Mavius, inherit an expensive stone from their father whose will specifies that they are forbidden to sell it. The brothers must decide how to divide their inheritance equally and agree that they will henceforth exhibit the stone in their respective museums on alternate days. Each museum's cabinet will be occupied by the stone one day and empty the next. The arrangement will also bind their heirs—for eternity. Thus, Grandi summarized, each family would own the stone half the time.

The monk's cute tale did not find universal acceptance, however. In particular, G. W. Leibniz saw no relation to the problem at hand, though he, too, was of the opinion that the series sums to one-half.

DÉNOUEMENT

Let's call the sum of Grandi's series, if it exists, S:

$$S = 1 - 1 + 1 - 1 + \ldots$$

And let's perform some mathematical operations. First, we deduct S from 1, to get

$$1 - S = 1 - (1 - 1 + 1 - 1 + \ldots) = 1 - 1 + 1 - 1 + \ldots$$

But the series on the right-hand side is exactly Grandi's series whose sum is equal to S. So we have $1 - S = S$, or, in other words, $1 = 2S$. Lo and behold, this gives $S = \frac{1}{2}$, just as Grandi and Leibniz believed.

The operations we performed seem perfectly legal. But there is a caveat: mathematically they are not legal for series that do not converge! So we need another justification.

Enter Ernesto Cesàro, a nineteenth-century Italian mathematician.[2] He suggested a more acceptable summation method for divergent series that consists of taking the average of the partial sums, which are 1, 0, 1, 0, 1, 0, . . . To illustrate, after n steps (for even n), the sum of the partial sums is $n/2$, and dividing this by n gives the Cesàro sum: ½.

MORE . . .

The series 1 − 2 + 3 − 4 + 5 − 6 + 7 − 8 + . . . poses a similar but more difficult problem. Its partial sums are 1, − 1, 2, − 2, 3, − 3, . . . As we proceed along this series, the partial sums grow alternately toward plus and minus infinity. Clearly, the series diverges, and this time even Cesàro cannot save it because the Cesàro sum also diverges.

Nevertheless, in 1749, much before Cesàro's time, the Swiss mathematician Leonhard Euler believed that this series summed to ¼, though he admitted, "*Cela doit paroître bien paradoxe*" ("This must seem quite paradoxical").

We can again show, albeit without rigor, how one arrives at that result. Let's call the sum, if it exists, T. Then,

2. Cesàro died tragically at the age of forty-seven while trying to save his son from drowning. Sadly, his son did not survive either.

$$\begin{aligned} 2T &= 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots + 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots \\ &= 1 + (-2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots) + 1 - 2 + (3 - 4 + 5 - 6 + 7 - 8 + \ldots) \\ &= 1 + 1 - 2 + (-2 + 3 - 4 + 5 - 6 + 7 - 8 + \ldots) + (3 - 4 + 5 - 6 + 7 - 8 + \ldots) \\ &= 0 + (-2 + 3) + (3 - 4) + (-4 + 5) + (+5 - 6) + (-6 + 7) + (7 - 8) + \ldots \\ &= 0 + 1 - 1 + 1 - 1 + \ldots \end{aligned}$$

But this is just the Grandi series, which, as we know from what we found earlier, sums to ½. Hence, $2T = \frac{1}{2}$ or $T = \frac{1}{4}$, just as Euler surmised.

V

LET'S GET PHYSICAL

Nothing happens until something moves.

—ALBERT EINSTEIN

Paradoxes abound, not only in relativity and quantum theory but also in classical mechanics. Ever since Isaac Newton discovered gravity—a force that's invisible—people have been puzzled by what magic rules our existence. Weird science rules the world . . .

21

WHY IS IT DARK AT NIGHT?

Olbers's Paradox

During the day, the sun shines brightly to light up the earth. And during the night?

We may stipulate that the universe is infinite in all directions. In this infinite universe, an infinite number of stars glitter in the sky. Of course, the light of a star diminishes the farther away it is from the earth. But since there is an infinite number of stars, their combined light should be sufficiently strong on its way to the earth to make the sky seem bright, even at night.

Correct?

Obviously not, as we observe each night.

First of all, what is the argument that the night sky should be bright?

The farther away a light source, the larger the lit-up area but the weaker its brightness. One can confirm this by holding a flashlight to a wall. As the flashlight moves away from the wall, the area of the lit-up circle increases with the square of the distance. And the brightness of the lit-up circle becomes weaker.

Weaker by how much? Since the light source emits a finite amount of energy, the same amount of light must be spread over a larger area. And since the lit-up area increases by the square of the distance, the brightness of the lit-up circle decreases by the square root of the distance.

The sky at night.

Source: https://commons.wikimedia.org/wiki/File:Night_Sky_Stars_Trees_02.jpg.

Now let's consider the number of stars. We may assume that stars are homogeneously distributed throughout the universe, that is, that their density is similar everywhere. We now split the universe into thin, concentric layers around the earth, like the layers of an onion. In each layer are a certain number of stars. Archimedes already knew, based on his calculations of the surfaces of spheres, that the number of stars in each layer increases with the square of the layer's distance from the earth. For example, in a layer that is twice as far from the earth as another one, there are four times as many stars.

Taken together, this would mean the following: (a) the brightness of each star *decreases* with the square root of the distance; (b) the number of stars *increases* with the square of the distance of the layer from the earth; (c) hence, each layer should produce the same amount of light; and (d) with an infinite number of layers, the combined light that hits the earth should be . . . infinite.

So, why is the night sky dark?

The paradox was formulated in 1823 by the German astronomer and medical doctor Heinrich Olbers in a paper entitled "On the Transparency of the Universe." An explanation was needed to settle the questions whether the universe is, in fact, infinite and whether an infinite number of stars are distributed homogeneously within that infinite space. The fact that the night sky is dark would speak against the existence of an infinite universe.

But Olbers did not believe that. He thought that nights are dark because intergalactic dust absorbs the light. It was not a good explanation, however; the absorbed energy would have heated the dust until it emitted as much light as it absorbed. To be fair, in Olbers's time it was not yet known that heat and light were different forms of energy and that one could be transformed into the other.

In 1884, Lord Kelvin gave a lecture in Baltimore on the issue and suggested a correct solution. The lecture was published in 1901 under the title "On Ether and Gravitational Matter Through Infinite Space."

DÉNOUEMENT

We stipulated that the universe is infinite, but we made no assertion about its age. Indeed, it all started with the big bang, fourteen billion years ago.

Lord Kelvin explained that since light travels at a finite speed, albeit a very fast speed, it takes time for the stars' light to reach the earth. Hence, the light of only a finite part of the universe can reach the earth. Light emanating from sources more than fourteen billion light-years distant from the earth has not reached us yet.

Additionally, Albert Einstein predicted, and cosmologists have since confirmed, that the universe expands in all directions: everything moves away from everything else. When stars travel away from the earth, the so-called Doppler effect kicks in: the wavelengths of sound and light are stretched, like the pitch of an ambulance's siren that increases as the ambulance moves closer to you but then diminishes as it moves away. In the same manner, the wavelengths of light emanating from stars moving away from the earth increase, shifting them into the infrared part of the spectrum, which is invisible to the human eye.

To summarize, the sky is dark at night because (1) the light of many stars has not yet reached the earth, and (2) an infinite number of stars are traveling away from the earth. The light emanating from them, though directed toward the earth, has shifted to wavelengths that are invisible.

These are the reasons it is dark whenever the sun does not light up the sky.

MORE . . .

Surprisingly, one of the first people to suggest this explanation for the dark night sky was not a scientist but a writer, the American author Edgar Allan Poe (1809–1849). In his prose poem "Eureka" (actually a 150-page treatise), he suggested that light from faraway stars may not have reached the earth yet. The pertinent part of the essay reads as follows:

> Were the succession of stars endless, then the background of the sky would present us a uniform luminosity, like that displayed by the Galaxy—since there could be absolutely no point, in all that background, at which would not exist a star. The only mode,

> therefore, in which, under such a state of affairs, we could comprehend the voids which our telescopes find in innumerable directions, would be by supposing the distance of the invisible background so immense that no ray from it has yet been able to reach us at all.

But it is unclear whether Poe actually believed that the universe is infinite. He may have given his explanation simply as a sort of tongue-in-cheek rationalization, even slightly ridiculing it. This is borne out by reading on: "No astronomical fallacy is more untenable, and none has been more pertinaciously adhered to, than that of the absolute illimitation of the Universe of Stars."

Deep down, Poe apparently believed that the infinite universe is a fallacy.

22

GATHERING IN THE MIDDLE

The Tea Leaves Paradox

Place a marble on a carousel, and let the carousel spin. Drive a car at high speed around a bend on a racetrack. Centrifugal forces pull the marble and the car outward. It's a basic law of physics.

Likewise, when stirring a cup of tea that contains tea leaves floating in the water, the leaves will flee toward the outside, that is, toward the rim of the cup, because of the centrifugal force.

Correct?

Incorrect! The tea leaves do not flee outward; rather, they gather in the middle of the cup, at the bottom. If you don't believe it, go make yourself a cup of tea now and observe.

• • •

Before we embark on the discussion about the paradox, let me mention that the centrifugal force acting on the marble or on the car is not a real force since real forces act among several bodies. In the case of a marble rotating on a carousel or a car careening around a bend, there is no second body. Strictly speaking, what pulls the marble and car outward is an *acceleration*.

• • •

It was no less a luminary than Albert Einstein who explained the seeming paradox of the tea leaves in a talk to the Prussian Academy of Sciences in 1926, ten years after presenting his somewhat less mundane general theory of relativity at the same venue. In the lecture, published later that year in a German scholarly journal, Einstein set himself the task of describing why, when looking in the direction of flow, meandering rivers in the Northern Hemisphere tend to erode more on the right bank whereas those in the Southern Hemisphere tend to erode more on the left bank.

To make the problem more vivid, Einstein began with an everyday observation: "Imagine a flat-bottomed cup full of tea. At the bottom there are some tea leaves, which stay there because they are a little heavier than the liquid they have displaced. When setting the brew in motion by stirring it with a spoon, the leaves will soon collect in the center of the bottom of the cup."

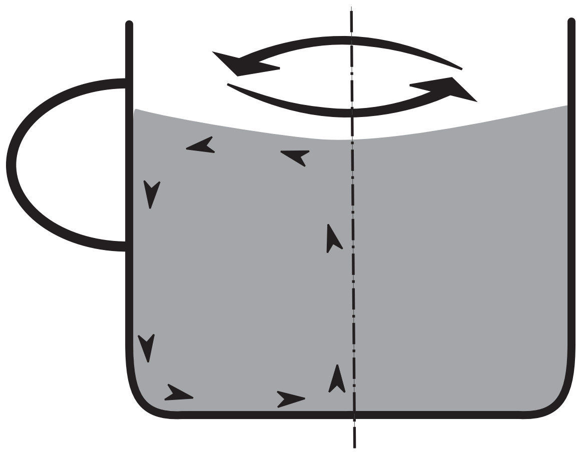
Why do they not flee in the opposite direction, toward the rim, as would be expected from the centrifugal force?

DÉNOUEMENT

The water is, in fact, set into a horizontal, circular motion by the spoon, but near the rim and along the bottom of the cup there is friction. This slows the water's motion. Hence, the water at the center of the cup spins faster than the water at the perimeter and bottom. Thus, the centrifugal force in the outer and bottom regions is weaker than in the center.

In Einstein's words, "The result of this will be a circular movement of the liquid . . . which goes on increasing until, under the influence of ground friction, it becomes stationary. The tea leaves are swept into the center by the circular movement."

The figure for this chapter provides an illustration of how Einstein explains the tea leaves paradox. We see that in addition to



The tea leaves paradox.

Source: https://commons.wikimedia.org/wiki/File:Tea_leaf_Paradox_Illustration.svg.

the primary horizontal movement generated by the spoon, secondary circular flows descend vertically from the surface of the liquid, along the side of the cup to the bottom, and back up again. The tea leaves are pushed by the flow to the side of the cup and then to the bottom, toward the center, where they remain since they are heavier than water.

To see how the vertical flow occurs, let's follow an individual water molecule. Pushed by the centrifugal force, molecule A moves outward toward the rim. There it can go no farther, and another molecule, molecule B, must move out of the way to make room for it. Where can B go? Only downward because of gravity. So, at the rim, molecule A forces molecule B down, and molecule B forces all the molecules below it to move farther downward until they hit the bottom of the cup. There they have no place to go, except toward the center, along the bottom. In the center of the cup, they crash into the molecules coming from the cup's opposite direction and must move upward again, leaving the heavy tea leaves behind.

Paradox resolved!

MORE . . .

Einstein's primary goal in writing the paper was not to explain the perplexing but unexciting phenomenon of tea leaves gathering at the bottom of a cup but the erosion of rivers. It had long been known that in the earth's Northern Hemisphere, the banks on the right sides of rivers tend to be steeper than on the left sides. In the Southern Hemisphere, the situation is reversed: left river banks are steeper than the right ones. This is true no matter whether the rivers flow north, south, west, east, or in any direction in between. (However, along the equator, nothing of the like occurs.)

The phenomenon arises, Einstein explained, because of the interplay between the friction along river beds and banks on the one hand and the earth's rotation on the other. Since the earth rotates eastward (counterclockwise as seen from above the North Pole, and clockwise as seen from below the South Pole), this creates an acceleration known as the Coriolis force. In the Northern Hemisphere, it deflects moving water—and hurricanes—to the right and in the Southern Hemisphere to the left of its initial direction. (Like the centrifugal force acting on a marble on a carousel or on a car careening around a bend, the Coriolis force is an acceleration, not an actual force.)

How does this create erosion? Because of friction, a river's water moves faster the farther away it is from the banks and from the bed; hence, it moves fastest at the top in the middle. And since, in the Northern Hemisphere, water is being displaced to the right because of the earth's rotation, a vertical circulation is created within the river from the top to the right and down again, as in the right part of the teacup shown in the figure. This is what creates the erosion on the right sides of rivers. In the Southern Hemisphere, where the rotation is clockwise (as seen from below the South Pole), water is displaced toward the left, and lefthand river banks become more eroded than those on the right, as in the left part of the teacup shown in the figure.

• • •

How would that translate to race cars going around a bend at high speed? If—heaven forbid—the cars hit the racetrack's outer boundary, they will be tossed back inside. After a while, a collection of wrecked cars will have gathered in the middle of the racetrack—just like the tea leaves in the teacup.

23

SHAKEN, NOT STIRRED!

The Brazil Nut Effect

Shake a jar of nuts of mixed sizes vigorously up and down to get a good mix. Open the jar, and what do you see? A good mix? Unlikely. More likely: the heavier nuts, owing to their weight, will have sunk to the bottom, and the lighter ones will have crawled to the top.

Right?

Wrong again! The big, heavy nuts are on top, and the light ones are at the bottom.

Repeat the experiment by placing an almond at the bottom of a jar. Then fill the jar halfway with rice. Cover the jar so nothing spills, and start shaking. Lo and behold, after a few energetic wobbles, the almond magically rises to the top.

The effect seems paradoxical at first because we are used to a different scenario: when letting a mixture of two liquids, say oil and water, stand still for a while, the heavier liquid will sink because of gravity, thus displacing the lighter liquid toward the top.

But the situation here is totally upside down!

Source: https://commons.wikimedia.org/wiki/File:Brazil_nuts1.jpg.

DÉNOUEMENT

The explanation can be given in two words: *percolation* and *convection.*

Whenever a shake elevates a big nut ever so slightly, a small void opens up below it. Immediately, smaller nuts from the sides will tumble into the empty space below the biggie, thereby preventing the biggie from falling back into the void, thus keeping it at the higher level. Upon the next shake, the big nut is again lifted a bit, and smaller nuts tumble into the void, preventing the biggie from settling back. This continues until the biggie hits the surface, where it usually remains, just bouncing up and down. Sometimes it does go under, only to reappear again a few shakes later.

Percolation—which occurs when water drips through coffee grounds—is the process that describes the small nuts' migration through the voids between the larger nuts down to the bottom of the jar. Convection describes the push of the larger nuts upward toward the top of the pile.

MORE . . .

But wait a minute, a physical system, like the jar of nuts or the bowl of rice with the almond, always tries to attain a state of minimum energy. This means that the center of gravity should be as low as possible. How can the center of gravity be low if the heavy nuts are on top and the light ones are on the bottom? Well, we must take a holistic view: though the large Brazil nuts are heavier, there is a lot of air around them, which is just empty space. The small nuts, gathered at the bottom, are much more densely packed, with very little weightless air between them. Since the density of the collection of small nuts is higher than the density of the collection of large nuts, the center of gravity does, in fact, move downward when the jar or bowl is shaken, and the Brazil nuts rise.

• • •

As reasonable as the dénouement sounds, the issue is not yet fully resolved, and research on the Brazil nut effect is ongoing.

Scientists believe that several more factors are at play. And that makes the Brazil nut effect even more interesting.

Some suspect, for example, that friction, the amplitude and frequency of the vibrations, the geometry of the container, and the air pressure in the space between the nuts may contribute to the effect. This is because when the big nuts and the small nuts have the same densities, they remain mixed even after thorough shaking. Indeed, there may even be a reverse Brazil nut effect that occurs under certain circumstances. The ratio of the densities of the big nuts versus the small nuts is one reason for this effect.

Another possible explanation is inertia. In the case of the jar of rice and one almond, the grains of rice, which have less mass and face less friction than the almond, may be more easily accelerated and may get to the interstitial openings faster than the larger, heavier nut. Though the biggies might have more momentum, they are faced with more friction and resistance from the gang of small grains of rice.

And I just mentioned that the geometry of the container may play a role. Apparently, convection currents have been observed in a martini glass–shaped container that sent biggies to the bottom, keeping little grains on top.

Speaking of which, is this why James Bond preferred his martinis "shaken, not stirred"? Probably not. However, a generation ago, a team of six researchers published a paper in the *British Medical Journal* showing that the antioxidant activity of shaken martinis is superior to that of stirred martinis. Now, couple that with the antioxidants present in the Brazil nuts floating at the top of the jar, and we know why 007 managed to get out of any spat.

24

COLD AND COLDER

The Mpemba Paradox

Put two containers filled with water into a freezer, one that is already cold and another that is still warm. Which will freeze first? The one that's already cold, of course. After all, it has already proceeded part-way to the freezing point.

Right?

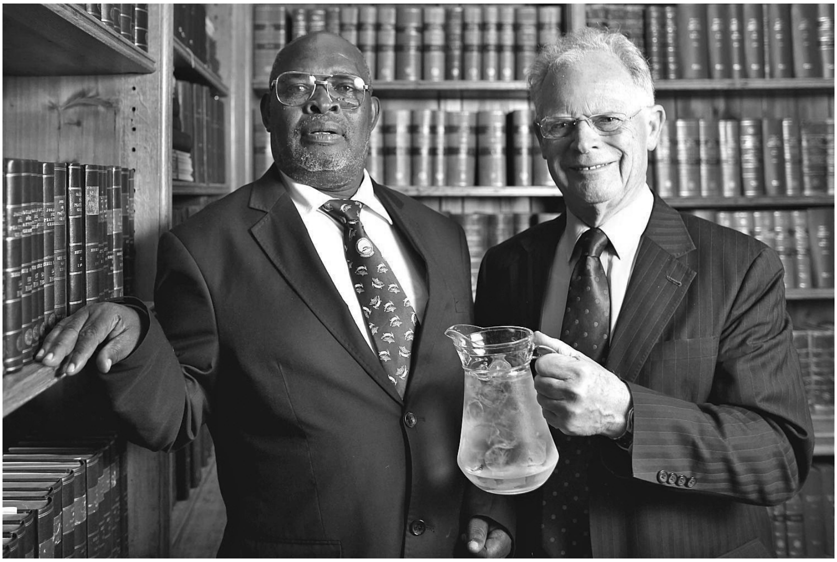
Wrong!

As it has turned out in many experiments, the container with the warmer water may freeze faster. The one that's already cold takes longer.

• • •

In 1963, Erasto Mpemba, a Tanzanian high school student was taking a cookery class at Magamba Secondary School in Tanganyika. The day's project was to prepare ice cream. The students were to bring milk to a boil, mix it with sugar, let it cool down, and put a jar of the mixture into the freezer.

Afraid of losing the last available space in the freezer, Erasto did not wait long enough and put his not-yet-cold jar into the freezer anyway. Imagine his surprise an hour and a half later when he noted that his ice-cream mixture had frozen before those of his classmates.



Erasto Mpemba and Denis Osborne.

Source: © Ben Gurr, *The Times*, January 11, 2013.

Some time later, the school's principal invited a physics lecturer from University College in Dar es Salaam, the later diplomat Denis Osborne, to give a lecture at the high school. During the question-and-answer period, Erasto asked about the phenomenon that he had observed to the snickering of colleagues and teachers: "If you take two similar containers with equal volumes of water, one at thirty-five degrees Celsius and the other at one hundred degrees Celsius, and put them into a freezer, the one that started at one hundred degrees Celsius freezes first. Why?"

The professor did not know the answer. Indeed, he did not even know whether what Mpemba described could be true. But he was intrigued. Back at his lab, he performed experiments and discovered that the high school student had been correct. In 1969, they published a paper together describing the effect and thus immortalizing the Mpemba name. Mpemba went on to work in the Wildlife Division of Tanzania's Ministry of Natural Resources and Tourism.

• • •

Actually, Mpemba was not the first to discover the phenomenon. Aristotle had already observed that people, "when they want to cool water quickly, begin by putting it in the sun," concluding that "the fact that the water has previously been warmed contributes to its freezing quickly." The English philosopher Francis Bacon (1561–1621) noted that "lukewarm water freezes more easily than that which is quite cold," and the French thinker René Descartes (1596–1650) wrote that "one can see by experience that water that has been kept on a fire for a long time freezes faster than other water."

DÉNOUEMENT

To perform the appropriate experiment is not easy because the question being asked is not very precise. What does "to freeze" mean? Zero degrees Celsius? Or when water begins to turn into ice? Or when the entire contents of a container has turned into ice? In their paper, Mpemba and Osborne defined it as the time it took for the first ice crystals to form. But that is not easy to observe, especially in a closed refrigerator.

And even after the question has been clarified, many parameters can be varied in the experimental setup, including the amount of water in the containers, the type of water, the size and shape of the containers, the size and shape of the freezer, and the temperature of the freezer. It is no wonder that the Mpemba paradox is an ongoing project and that experts still do not agree on an explanation for the phenomenon. Several have been proposed.

One is evaporation: if the containers are left open, the hot water will evaporate more quickly than the cool water, and its volume will decline. With a smaller volume, the water has less heat to cool and thus cools faster. The hot water freezes first because there's less of it to freeze.

Another explanation is that hot water generally holds less dissolved gas than cold water. The reduced amount of dissolved gas may change the ability of the water to conduct heat, or it may change the freezing point of the water by a significant amount.

Convection is a third explanation. In general, heat rises, so the contents at the top of the container will be hotter than the average temperature in the container. Since heat is mostly lost at the surface, and since the surface is hotter than elsewhere in the container, heat is lost faster than one might think based on the average temperature of the container. Thus, even when the average temperature of the "hotter" container has descended to the initial temperature of the "cooler" container, it is still warmer at the top and the hotter container therefore experiences a faster rate of cooling. Thus, its rate of freezing may overtake the cooler container.

A fourth explanation may be that the hot container somehow changes the environment around it, thus affecting the cooling process in some complex fashion.

Finally, water may remain liquid at zero degrees Celsius and freeze only at a somewhat lower temperature. This is called "supercooling." It could be that water that was initially warmer supercools less than cold water, thus freezing first.

MORE . . .

The Mpemba paradox has some practical applications. One is obvious: ice-cream makers should freeze their mixtures while they are still hot because they will freeze faster. But there are more: in the winter, a car should be washed with cold water because hot water will freeze on it more quickly. A skating rink, on the other hand, should be flooded with hot water because it will freeze faster. And do put the doggie bag with your restaurant leftovers into the fridge immediately; don't wait for it to cool down.

25

SUCK OR SPOUT?

The Sprinkler Paradox

Sprinklers are S-shaped gardening devices, mounted horizontally on a pipe, through which water is guided and spouted out. The nice thing is that as soon as the water is turned on, the device starts turning counterclockwise and keeps turning, thus watering the entire flower bed.

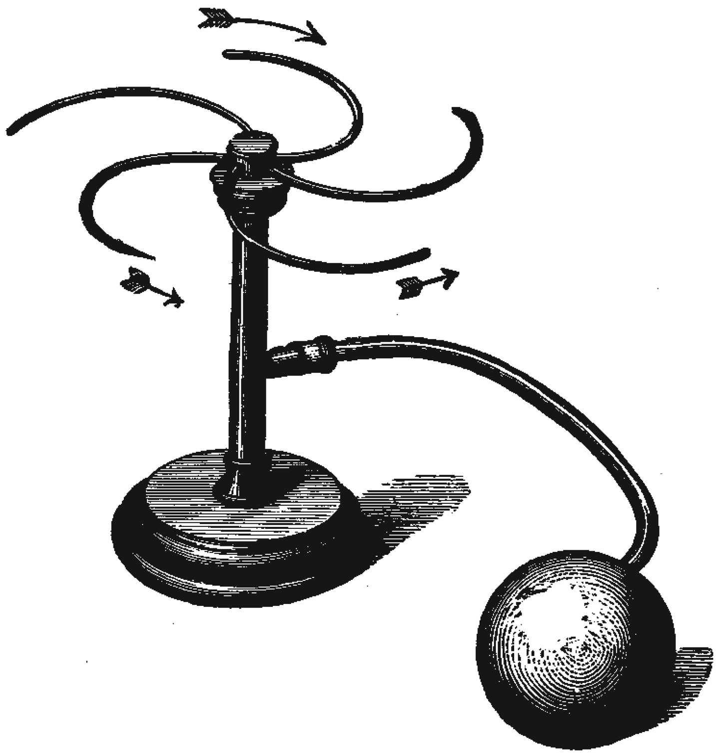
The motion derives from the recoil of the water: as the water is pushed out the nozzle, conservation of momentum requires that there be a counteracting backward momentum. That is what pushes the device into a counterclockwise rotation. It is the same recoil that you feel when holding a shower hose in your hand and turning on the faucet. If you don't hold it tightly, it will jump backward out of your hand.

Now, if such an S-shaped device is suspended in a tub filled with water, and it sucks in water, rather than spouting it out, will the device rotate in the other (clockwise) direction?

No. . . . Yes. . . .

It depends on whom you ask.

Some physicists argue on theoretical grounds that the sucking device should not move at all. Others have demonstrated in experiments that it does rotate and indeed clockwise. On YouTube, one can find video clips showing that such S-shaped devices both do and do not rotate clockwise when they suck water in.



Ernst Mach's illustration.

Source: https://commons.wikimedia.org/wiki/File:Reaction_wheel.pdf.

So, what gives?

The device that sucks water in instead of spouting it out is often called a Feynman sprinkler, even though the Nobel Prize winner Richard Feynman did not invent it. In fact, he objected to its being called that even though he made it notorious in one of his books. I will honor the famous physicist by *not* naming it after him. It was the Austrian physicist and philosopher Ernst Mach who wrote about the phenomenon in 1883 in "*Die Mechanik in ihrer Entwickelung*" ("The Development of Mechanics"). One would think, he wrote, that when the contraption sucks in water, the opposite rotation should arise than when spouting: "But this does not happen, in general."

Sixty years later, Feynman, then still a student, devised an experiment with his colleagues to test the phenomenon experimentally. They submerged an S-shaped device into a glass container filled with water and let it suck in the water. As Mach had predicted, apart from an initial tremble, the sucking device did not budge. But the brilliant Princeton students apparently still did not believe that nothing should happen and increased the flow of water by raising the pressure—until the glass container exploded. And that was the end of it. Feynman never explained why nothing happened except for the explosion, or what he had expected would happen.

DÉNOUEMENT

The confusion arises because one would think that sucking and spouting are symmetric phenomena. However, sucking is not spouting played backward.

When the device spouts, Mach explained, a narrow jet of water is directed into the air in front of it. The recoil of this jet is what pushes the S-shaped device in a counterclockwise direction. It's all because of the law of the conservation of momentum. The law indicates that the spouting water goes one way and, to conserve momentum, the sprinkler goes the other. This forces the S-shaped spouting sprinkler to rotate counterclockwise.

But when the device sucks water in, it is not a thin column of water that enters the spout. The water comes from all directions. Hence, there is no jet to produce a recoil and no clockwise rotation when the S-shaped device sucks the water in.

To illustrate this, one can perform an experiment at home. Stand in front of a fan: your hair is blown backward in the jet of air.

Now go and stand behind the fan. Your hair does not blow at all. In fact, you feel practically nothing. The air that is sucked from behind the ventilator to the front comes from all directions behind the fan and is only then channeled into a column in front.

Does the law of the conservation of momentum not hold in this case? Does the incoming water not go one way and the sprinkler the other? The explanation is that the incoming water, now a focused column, hits the bend of the S-shaped sprinkler and pushes against the sprinkler's forward push, thus counteracting the tendency to rotate; the sprinkler remains at a standstill.

MORE . . .

Clear enough. So, why is there any disagreement? Even Ernst Mach seems to have been aware that some might not agree, as evidenced by his careful wording. The reverse rotation that one might expect does not happen "in general," he wrote, adding that "as a rule," there is no noticeable rotation. Was he aware of a problem?

In fact, some experiments do show movement, indeed a clockwise rotation. But that is caused by other factors. The incoming water creates vortices inside the sprinkler and thus dissipates energy. By the time the column of water hits the bend, it has lost some of its momentum. Thus, the counteracting force is a tiny bit weaker than the tendency to rotate clockwise. And this is why some experiments do show a tremble or a clockwise rotation.

VI

THE POSSIBILITY OF PROBABILITY . . . AND THEN THERE'S STATISTICS

Statistics do not lie? Maybe not, but this sure doesn't stop them from being infuriatingly confusing. As often as not, the answers lie beyond the numbers . . . let's find them.

26

A CADILLAC OR A GOAT?

The Monty Hall Paradox

In the 1960s, *Let's Make a Deal* was one of the most popular shows on American TV. On the stage were three doors. Behind one was a Cadillac; behind the other two were goats. If the contestant chose the correct door, the Cadillac would be hers.

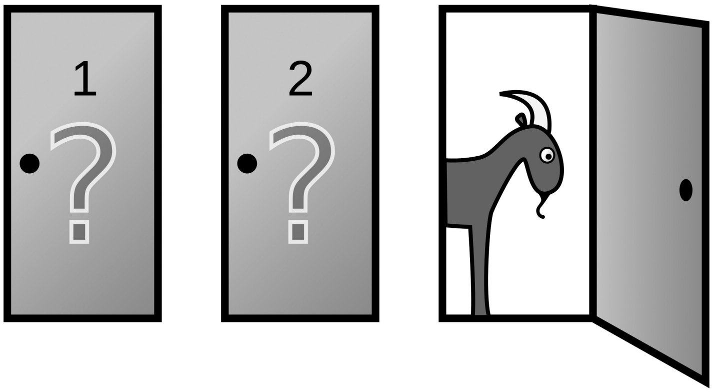
But there was an additional twist. After the contestant indicated her choice but before the door was opened, Monty Hall, the host, would open one of the other doors to reveal a goat. Then he would give the contestant a chance to revise her choice: do you want to stick with your original door, or would you like to switch to the other still unopened door?

Since there are now only two doors left, the probability of the Cadillac being behind either one, is one in two, correct? Hence, it should make no difference whether the contestant remains with the original door or switches to the other one, right?

Nope. Totally wrong!

To increase the chances of winning the Cadillac, the contestant should switch doors. (For a similar problem, see the two-envelope problem discussed in chapter 28.)

This is quite counterintuitive. There are two doors left, and the Cadillac could be behind either one. So the probability of the Cadillac being behind either one of the two doors should be



The Monty Hall problem.

Source: https://commons.wikimedia.org/wiki/File:Monty_open_door.svg.

one in two, no? The fact that Monty Hall revealed a goat hidden behind one of the other doors is irrelevant and should be ignored, should it not?

No and no.

As it turns out, if the contestant switches doors, the chances of winning the Cadillac are doubled! Even though two unopened doors remain, the probability of winning is not one in two for each door. In fact, the chances of winning the Cadillac are only one in three if the contestant remains with the original door while—surprise, surprise—switching to the other door increases the probability of winning to two in three. We'll soon see why this is so.

The seemingly paradoxical situation was raised again in the 1990s in a series of magazine articles by Marilyn vos Savant, a famously cerebral columnist. She gave the correct answer to the Monty Hall problem—that the contestant should switch doors—whereupon she received thousands of critical letters, many from readers with PhDs and many calling her an idiot.

The angry readers may be forgiven for their error, if not for the vile language, because the correct answer is indeed counterintuitive. It even threw off one of the most famous mathematicians of the time, the number theorist and probabilist Paul Erdős. He was convinced of the correct answer only reluctantly and only after a friend performed simulations on his computer showing that by changing doors, the winning probability increased to two in three.

DÉNOUEMENT

At the outset, the probability that the Cadillac is hidden behind any door is one in three. Hence, when the contestant makes a choice, say, for door 1, the chance that she will win the Cadillac is one in three. On the other hand, the chance that she will leave empty-handed—because the Cadillac is hidden behind one of the other two doors—is two in three.

Now let's say that Monty Hall reveals a goat behind door 2 and that the contestant remains with her initial choice of door 1. The probability that she will win the Cadillac does not change just because the host opened another door. It is still one in three. What does happen after the goat is revealed behind door 2, however, is that the remaining probability of two in three is now collapsed to door 3. Hence, there's a one-in-three probability of the Cadillac being behind door 1 and a two-in-three probability of the Cadillac being behind door 3. The contestant would do well to switch from door 1 to door 3!

That the switching strategy is correct can also be seen in the following manner: with their first pick, contestants will unknowingly be lucky one-third of the time. On one-third of all trials, sticking with the door will always win, and switching will always lose. So far, so good.

In the other two-thirds of the trials, contestants can guarantee themselves a win: the Cadillac is behind one of the two remaining doors. Once Monty Hall reveals which of those two doors does not hide the Cadillac, it *must* be the other door behind which the Cadillac sits. On these two-thirds of all trials, switching always wins, and sticking always loses. So, remaining with the chosen door wins one-third of the time, and switching wins two-thirds of the time.

MORE . . .

There are several possible explanations why people erroneously believe that the probability of the Cadillac being behind either remaining door is the same, that is, one in two. One reason is that people tend to believe that probability is evenly distributed across all possible alternatives, whether that is true or not.

Other explanations are behavioral. People tend to overvalue the winning probability of the door chosen first since they already "own" it. And people may regret errors of commission more than errors of omission: if they lose the Cadillac because they decided to switch doors, they will regret it more than if "fate" had decided to place the Cadillac behind the other door. Hence, thinking ahead about their possible future regret, people may prefer to stick with the choice they have already made.

• • •

In an experiment published in 2010 under the title "Are Birds Smarter Than Mathematicians?," the scientists Walter T. Herbranson and Julia Schroeder performed Monty Hall trials with pigeons. The birds were confronted with three response keys and given mixed grain as a prize. Interestingly, over time, the birds adjusted their choice of switching keys or sticking with their first choice to approximate the optimal strategy. Replication of the procedure with human participants showed that humans failed to adopt the optimal strategy even with extensive training. Start rethinking "birdbrain."

27

TO TREAT OR NOT TO TREAT?

Simpson's Paradox

A pharmaceutical company has discovered a new treatment for a disease. In clinical trials with young patients, the treatment was 90 percent effective, whereas subjects treated with a placebo had only an 80 percent chance of recovery. The results were a little less encouraging for older patients, but recovery rates were still 60 percent with treatment and only 50 percent without treatment. So, the treatment should definitely be approved for this disease.

Correct?

Well . . . this needs context.

In particular, we must look at the actual numbers in more detail, not only at the percentages. Let's say two hundred young people were treated, and eight hundred young people received the placebo. Among the trials with older patients, eight hundred were treated, and two hundred received the placebo. Recall that of the two hundred treated youngsters, 90 percent recovered; of the eight hundred who got the placebo, only 80 percent recovered. For the older people, it was 60 percent recovery with treatment and 50 percent without treatment.

Recovery by age group

	Recovery with treatment	Recovery with placebo
Younger people	180 of 200, i.e., **90%**	640 of 800, i.e., 80%
Older people	480 of 800, i.e., **60%**	100 of 200, i.e., 50%

Obviously, the treatment was more effective than the placebo, both for the young and the old. But that's not the whole story. It turns out that 740 of the one thousand people who received only a placebo recovered on their own, whereas only 660 of the one thousand who received the treatment recovered.

Overall recovery

	Recovery with treatment	Recovery with placebo
Entire trial	660 of 1,000, i.e., 66%	740 of 1,000, i.e., **74%**

The conclusion seems to be that when the patient is young, treatment should be administered. And when the patient is old, treatment should also be administered. But there's a surprise in store: when age is unknown, treatment should be withheld.

Huh? Imagine a health care hotline. Over the phone, the doctor asks the patient's age and then prescribes treatment . . . *regardless of the answer*. If, however, the patient does not reveal her age, the doctor recommends against treatment.

What a ridiculous conclusion!

The paradox is named after the British statistician Edward Hugh Simpson, who worked at the famed Bletchley Park as a code breaker during World War II. After the war, he entered the British civil service and remained there until his retirement as deputy secretary of the Department of Education and Science. Simpson described the paradox in a paper that he published in 1951 while at the University of Cambridge. However, the paradox had already been identified a half-century earlier, at the turn of the twentieth century, by Karl Pearson and G. Udny Yule, two of the founders of mathematical statistics.

The phenomenon is now often referred to as the Yule–Simpson effect or Simpson's paradox, but, once understood, this statistical puzzle is no longer considered a paradox.

DÉNOUEMENT

The erroneous conclusion—that treatment should be withheld when the patient's age is unknown—derives from the fact that the sample sizes differ so markedly. The data show that it was more difficult to treat older people. Even though the treatment was effective overall, it was effective in only 60 percent of the older population. Hence, since the trial included many more older subjects than young ones, the overall percentage of the successfully treated population was pulled down. On the other hand, younger people recovered more easily, even those who got the placebo. But since so many more young people were given the placebo, the percentage of recovery for the entire placebo population was pulled up. Hence, the overall averages for the treated and untreated populations, the so-called weighted averages, provide an incorrect picture.

Mathematically, Simpson's paradox arises because we happen to have two fractions, a/b and c/d (the proportions of recoveries for the youngsters, with and without treatment), and two fractions, A/B and C/D (the proportions of recoveries for the oldies, with and without treatment) such that

$$a/b > c/d \text{ and } A/B > C/D, \text{ but}$$
$$(a + A)/(b + B) < (c + C)/(d + D).$$

In our example,

$$180/200 > 640/800 \text{ and } 480/800 > 100/200, \text{ but}$$
$$(180 + 480)/(200 + 800) < (640 + 100)/(800 + 200).$$

So, should the treatment be approved? Or, on a more basic level, should the results of the aggregate data be used or those of the subgroup data? In the present example, the recommendation would be that the treatment should be approved and recommended . . . for everybody.

MORE . . .

In general, there is, however, no definite answer. The choice depends on the research question, sample sizes, presence of confounding variables, practical implications, and goals of the analysis. If the pharmaceutical company had performed the clinical trials but neglected to ask the subjects their age, they would have had to conclude that the treatment, with only a 66 percent recovery rate, was ineffective. The research effort would have been deemed a failure even though the treatment helped both the younger and the older test subjects.

On the other hand, a company could artificially claim success by conjuring up all kinds of stratifications of the data. Even though recovery rates might be lower overall with treatment than without, a researcher under pressure to demonstrate success at any price could claim that recovery rates were higher for, say, both left-handed and right-handed people or for people with blue eyes and for people with nonblue eyes. But, in contrast to meaningful characteristics of the test subjects, like age, sex, or medical history, handedness and eye-color are clearly confounding variables that should be ignored.

In the early days of big data, this is the sort of activity that gave data-mining a bad name. Honest science demands that a causal effect be postulated at the outset of an experiment before data are collected and the hypothesis is tested. If researchers comb through the data *after* the research has ended, in order to seek out and pick judicious tidbits—for example, to find that left- and right-handedness happen to render the results significant even though there is no fathomable reason for handedness to cause this outcome—they are guilty of hindsight bias, definitely a no-no in good science.

28

A HOLISTIC APPROACH

The Two Envelopes Problem

A TV game show presenter holds two closed envelopes in her hands and tells you that one contains twice as much money as the other. She then hands you one of the envelopes. Before you open it, she informs you that if you wish, you may switch envelopes. Should you?

You recall the Monty Hall paradox (see chapter 26) and decide to switch.

Correct?

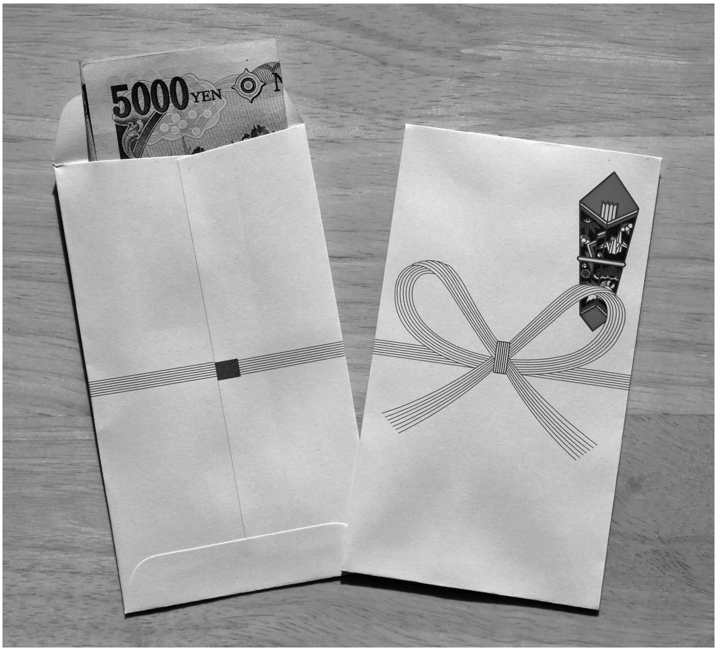
No!

Let's consider three scenarios.

Scenario 1: We designate the unknown amount in your closed envelope as X. By switching envelopes, you would get either $2X$ or $\frac{1}{2}X$. Since the chance of either happening is one in two, and since the expected result of an action is the sum of the outcomes multiplied by the probabilities that they occur, you expect to have the following after switching:

$$\tfrac{1}{2}(2X) + \tfrac{1}{2}(\tfrac{1}{2}X) = 1\tfrac{1}{4}X$$

This is more than just X! So, by switching you would expect to gain $\frac{1}{4}X$. Therefore, you should *switch*.



Source: https://commons.wikimedia.org/w/index.php?curid=85457660.

Scenario 2: Let's denote the amount in the envelope that the TV presenter keeps in her hands as *Y*. The other envelope, the one she hands you, could hold with equal probability either ½*Y* or 2*Y*. The expected content of the envelope that you hold in your hands can be expressed as follows:

$$\tfrac{1}{2}(\tfrac{1}{2}Y) + \tfrac{1}{2}(2Y) = 1\tfrac{1}{4}Y$$

This is more than *Y*, which is what you would have obtained had you switched. Hence, you should *keep* the envelope you have.

Scenario 3: Let's designate the lower amount as *Z*. Then, one envelope contains *Z* and the other 2*Z*. Let's say that the TV presenter handed you the envelope containing *Z*. By switching, you would gain another *Z*. Now let's say that she handed you the envelope containing 2*Z*. Then, by switching you would lose *Z*. Both actions by the TV presenter occur with the probability ½. Hence, the expected gain from switching would be as follows:

$$\tfrac{1}{2}(+Z) + \tfrac{1}{2}(-Z) = 0$$

Since there is nothing to gain, you may as well stick with your current envelope. Or you could switch. It simply *does not matter*.

So which is it?

• • •

The problem was first devised in 1952 by Maurice Kraitchik, a Belgian mathematician born in Russia whose field of research was numbers theory. But Kraitchik was mainly interested in recreational mathematics. In 1935, he organized the first congress on that subject in Brussels. During World War II, he lectured at the New School for Social Research in New York as an associate professor for recreational mathematics.

Naturally, the problem gained the attention of the science writer Martin Gardner, who described it in 1982 in his book *Aha! Gotcha*. In 1989, Barry Nalebuff from Yale subjected the problem to deeper analysis.

DÉNOUEMENT

First of all, let's note that if scenario 1 were correct, one could enter an ever-winning circle: after switching, one could ask oneself the same question again and come up with the same answer: switch. And again. And again. So, we can already smell that something fishy is going on here.

The confusion arises because conditional probabilities are confused with unconditional probabilities. It is not correct to say, for example in scenario 1, that you would either gain another X or lose half of X with a fifty-fifty probability. Or, in scenario 2, that you have an equal probability of holding an envelope containing either $\frac{1}{2}Y$ or $2Y$. One must consider the conditions under which circumstances these gains or losses occur.

Scenario 1 considers the content of your envelope as a given; scenario 2 considers the content of the envelope the presenter keeps as a given. Simply to compute the expected content of the other envelope is not correct. The content of the envelope that you are actually holding must be taken into consideration when computing the expected content of the other envelope.

Hence, the correct way to go about it is as described in scenario 3. There are two possibilities, each with a probability of one in two: your envelope holds Z and the other holds $2Z$, or your envelope holds $2Z$ and the other holds Z. By considering both possibilities, scenario 3 takes a holistic approach.

Then, the expected content of the "other" envelope, conditional on what is in your envelope, is as follows:

½(expected amount in the other envelope, given that your envelope contains Z)
+ ½(expected amount in the other envelope, given that your envelope contains $2Z$)
$= \frac{1}{2}(2Z) + \frac{1}{2}Z = 1\frac{1}{2}Z$

By the same calculation, the expected content of your envelope, conditional on what is in the other envelope, is also $1\frac{1}{2}Z$. Hence, each envelope contains on average the same amount, namely $\frac{3}{2}Z$. This is, of course, what one should have expected: since the total amount contained in both envelopes is 3*Z*, each envelope contains on average half that amount.

So, there's no point in switching.

MORE . . .

The flaw in the faulty reasoning can also be seen as follows. Let's say that a ten-dollar bill and a twenty-dollar bill are randomly stuffed into the two envelopes. You do not know which envelope you hold in your hand.

The naive, incorrect manner to analyze the situation, which we set out in scenario 1, is to implicitly assume that your envelope contains the ten-dollar bill. In that case, you assume that the other envelope contains either a five-dollar bill or a twenty-dollar bill. Oh, but wait a minute: five-dollar bills do not exist! Or you could implicitly assume that your envelope contains the twenty-dollar bill. Then, you would assume that the other envelope contains either a ten- or a forty-dollar bill. But forty-dollar bills do not exist either. That is why it is incorrect to compute expected amounts without regard for what is in your own envelope. The correct manner of analysis is the holistic approach of scenario 3.

29

SILVER AND GOLD?

Bertrand's Probability Paradox

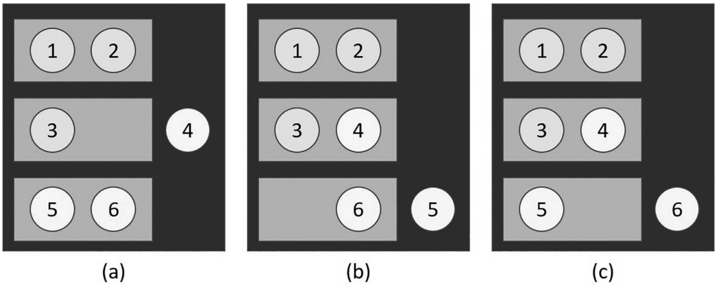
In front of you are three boxes, each with two drawers. The two drawers of one box each contain a gold coin. The two drawers of the second box each contain a silver coin. And the third box has one drawer containing a silver coin and one containing a gold coin. To summarize: there is a gold–gold box, a silver–silver box and a gold–silver box.

You choose one box at random and then one drawer at random. It contains a gold coin. What's the chance that the other drawer of that box also contains a gold coin?

Fifty percent?

If you said yes, you are in large, though not necessarily good company. Those who answer in the affirmative usually reason as follows: since they did not draw a silver coin, the box they chose could not have been the one with the two silver coins. Hence, what they chose was either the gold–gold box or the gold–silver box. Since there is a fifty-fifty chance that they chose either of those two boxes, there must be a fifty-fifty chance that the second drawer contains either a silver or gold coin. Sounds reasonable . . . but is wrong!

This is one of the apparent conundrums that the French mathematician Joseph Bertrand (1822–1900) presented in his book *Calcul des probabilités* (*Probability Calculus*) to demonstrate how naive



Three boxes, three outcomes.

Source: https://commons.wikimedia.org/wiki/File:3Outcomes.jpg.

people—or at least people not yet introduced to the mysteries of probability theory—can be misled. (By the way, Bertrand was also active in the field of economics, in which he developed a theory of competition and interactions among firms; see chapter 51.)

DÉNOUEMENT

Once one has eliminated, as one must, the silver–silver box, three possibilities of drawing a gold coin remain: one possibility of drawing the gold coin from the gold–silver box and two possibilities of drawing one of the gold coins from the gold–gold box. So, although there was an equal chance of picking either the gold–gold or the gold–silver box (since the silver–silver box has already been eliminated), once a gold coin has been revealed, there is a one-in-three chance that the other coin will be silver (panel [a] in the figure) and a two-in-three chance that the box is the gold–gold box (panels [b] and [c] in the figure). Hence, as soon as one has drawn a gold coin, the chance of drawing another gold coin is two in three.

MORE . . .

In the 1980s, two psychology professors from the Hebrew University of Jerusalem, Maya Bar-Hillel and Ruma Falk, conducted an experiment among their first-year students. They presented them with a hat that contained three playing cards: one was red on both sides, one was white on both sides, and one was red on one side and white on the other. A card was drawn out of the hat and laid on a table without the other side being revealed.

The question was, If the side facing up is red, what are the chances that the other side is also red? (The experiment also works if the side facing up is white.) And now it is obvious why the correct answer is two in three: "Clearly, an all-red card is twice as likely to show a red face up as a card that only has one red side."

Nevertheless, thirty-five of the fifty-three students answered incorrectly, and only three of the fifty-three gave the correct answer of two in three. (Unfortunately, their paper did not report what the remaining fifteen students answered.) The authors called the tendency of most people to give an incorrect answer the "fallaciousness of attributing posterior equiprobability to the remaining events."

• • •

Bar-Hillel and Falk also described another illustration of Bertrand's probability paradox. If one sees Mr. Smith—known to be a father of two children—walking down the street with a boy whom he introduces as his son, what is the probability that his other child is also a boy? Or, if Mr. Smith presents the boy as his eldest child, what is the probability that Mr. Smith's other child is also a boy?

In the first case, the possibilities are boy and boy, boy and girl, girl and boy, and girl and girl. Since girl and girl is ruled out, there is one chance out of the remaining three possibilities that if the child accompanying Mr. Smith is a boy, the other child is also a boy.

In the second case, however, the probability changes! The reason is that when Mr. Smith specifies that the boy accompanying him is his eldest child, the possibilities now are not (listing the eldest child first) boy and boy, boy and girl, girl and boy, and girl and girl. We know that girl and girl has already been ruled out. But now girl and boy also is. What remains are boy and boy and boy and girl. So, the probability that the younger child is also a boy is . . . one in two!

30

ARE MORE THAN HALF THE BABIES BOYS?

Lindley's Paradox

In a certain country in a certain year, one million babies are born: 501,200 boys and 498,800 girls. We believe that the true proportion of baby boys to baby girls is half and half but, of course, we do not expect the number of boys and girls to be *exactly* a half-million each. Some random deviation will always occur.

But does a prevalence of 50.12 percent boys indicate a bias toward male children? Or is a preponderance of up to 2,400 boys among a million babies statistically inevitable? In other words, is a birthrate of 50.12 percent versus 49.88 percent compatible with a proportion of fifty-fifty?

Yes and no, depending on whom you ask.

For the purpose of this question, statisticians can be divided into Bayesians and frequentists. The truly surprising answer to the question is that the two groups do not agree on their answers.

Bayesians would claim that even if one hypothesizes that the true distribution of boys and girls is fifty-fifty, there is—according to their methodology—a very good chance that in any given country in any given year, a deviation of up to 50.12 percent versus 49.88 percent may occur. Hence, the deviation is random, they claim, and the birthrate in that country supports the hypothesis that boys represent half of all babies born.

Source: RIA Novosti archive, image no. 450919/V. Yakovlev/CC-BY-SA 3.0, https://commons.wikimedia.org/wiki/File:RIAN_archive_450919_Maternity_Home_in_Yakutsk.jpg.

Frequentists argue differently. Their statistical methodology shows that there is only a very low chance that a preponderance of 2,400 boys would occur if the true rate were half and half. Hence, they reject the hypothesis that the deviation is random and do not accept the hypothesis that the true proportion is, in fact, fifty-fifty.

A paradox!

The problem was first pointed out by the British statistician Sir Harold Jeffreys who discussed it in 1939 in a textbook on statistics.

However, it wasn't until almost two decades later, in 1957, when the Cambridge statistician Dennis Lindley published a paper entitled "A Statistical Paradox" that it gained prominence.

The crux of the matter is a theorem devised in the eighteenth century by Thomas Bayes (1701–1761). *Bayesians* compare a hypothesis H_0 with a competing hypothesis H_1. At the outset, both hypotheses are assumed to have certain probabilities, for example a 25 percent chance that H_0 is correct and a 75 percent chance that H_1 is correct. With the gathering of additional data, the probabilities must be revised. As evidence supporting H_1 pours in, its probability is raised, or vice versa. Bayes developed an equation describing how the probabilities must be updated.

Lindley, in his paper "A Problem in Forensic Science," presented an interesting instance of the paradox in which glass shards were found on a burglary suspect's clothing. The question was whether the refractive index of the shards matched the refractive index of the window broken during the burglary. The suspect may be guilty or innocent, depending on whether the court uses the frequentist or Bayesian approach.

DÉNOUEMENT

(A) Bayesians are of the opinion that at the outset, before the babies are counted, there's a 50 percent chance that hypothesis H_0 is correct ("the true ratio of boys to girls is fifty-fifty") and a 50 percent chance that the alternative hypothesis H_1 is correct ("the true ratio can be anything"). If H_1 is correct, all boy-to-girl ratios between 0 and 100 percent are equally likely.

Once the actual proportion (501,200 boys versus 498,800 girls) is determined, Bayesians update the probability that H_0 is correct. According to Bayes's formula (the nitty-gritty details will be discussed below), the chances are updated from fifty-fifty to ninety-eight to two, which means that given the actual proportion, there is a 98 percent chance that the true proportion of boys to girls is fifty-fifty, even though the actual proportion is 50.12 to 49.88. Hence, Bayesians accept the hypothesis that the true ratio is half and half.

(B) The frequentists' argument goes as follows: the ratio of boys to girls corresponds to the so-called binomial distribution, which, for large numbers, is similar to the bell-shaped normal distribution. This means that 68 percent of cases would lie within one standard deviation of the mean and 95 percent within two standard deviations (for more on this, see any introductory textbook on statistics).

Now, the variance of a binomial distribution with a million babies is 250,000; hence, the standard deviation is 500.[1] A surplus of 1,200 boys—corresponding to a departure of 2.4 standard deviations from the conjectured 500,000 boys—would be a rare occurrence: it would happen in only 1.6 percent of all cases. Thus, frequentists would claim that the observed data disagree with the hypothesis that the true ratio is half and half.

What is the reason for the differing assessments?

Bayesians assert that the outcome of 50.12 percent versus 49.88 percent is not very far off from 50 percent versus 50 percent; hence, they conclude that the result supports the hypothesis of "half and half." The frequentists' approach is more diffuse. They compare the outcome of 50.12 percent versus 49.88 percent with *any proportion between 0 and 100 percent* and conclude that "fifty-fifty" does not explain the outcome very well. For example, 50.05 percent boys versus 49.95 percent girls would be a better explanation of the data. So

1. Variance = $np(1-p) = 1{,}000{,}000 \times 0.5 \times 0.5$. The standard deviation is the square root of the variance, i.e., $\sqrt{250{,}000} = 500$.

maybe that is the true ratio of boys to girls? Or maybe 50.20 percent versus 49.80 percent? In general, frequentists refuse a specific hypothesis more easily than Bayesians.

MORE . . .

$P(A)$ denotes the probability of A and $P(A|B)$ the probability of A, given B. When new evidence (B) comes in, Bayesians update the probabilities according to Bayes' theorem:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The theory of combinatorics says that

$$P(501{,}200|H_0) = \binom{501{,}200}{1{,}000{,}000}\left(\frac{1}{2}\right)^{501{,}200}\left(1-\frac{1}{2}\right)^{498{,}800}$$

and

$$P(501{,}200|H_1) = \binom{501{,}200}{1{,}000{,}000}\int_0^1\left((\theta)^{501{,}200}\left(1-\theta^{498{,}800}\right)\right)d\theta$$

Updating is performed with the help of Bayes' theorem:

$$P(H_0\ |501{,}200) = \frac{P(501{,}200|H_0)P(H_0)}{\frac{1}{2}P(501{,}200|H_0)+\frac{1}{2}P(501{,}200|H_1)}$$

With an a priori probability of H_0 being equal to ½ and plugging in the numbers, we get $P(H_0\ |501{,}200) = 0.98$

VII

FOOTLOOSE PHILOSOPHY

Give It Some Thought

Philosophers have a field day when it comes to paradoxes. In playing with them, their ideas shape minds and how we see the world around us. But once it's clear, it's obvious . . . or not.

31

TO SHAVE OR NOT TO SHAVE . . . ONESELF

Russell's Barber Paradox

Figaro is a barber in Seville. He must shave all of Seville's men who do not shave themselves. Does Figaro shave himself?

If he does, he mustn't. If he doesn't, he must.

Let's make this a bit more intelligible: if Figaro shaves himself, the barber must not shave him. If Figaro does not shave himself, the barber must shave him. The problem is, of course, that Figaro *is* the barber. So, what is Figaro—the barber—to do?

This famous paradox is based on a discovery in 1902 by the philosopher Bertrand Russell. Actually, the paradox had been identified before, but Russell formulated it in terms of the then newfangled theory of sets that had been introduced in the late nineteenth century by the mathematician Georg Cantor and further developed by the logician Gottlob Frege.

Using the word *class* instead of *set*, Russell pointed out what would turn out to be the crux of the paradox: "Some classes are members of themselves, some are not: the class of all classes is a class, the class of not-teapots is a not-teapot. Consider the class of all the classes not members of themselves; if it is a member of itself, it is not a member of itself; if it is not, it is."

At about the same time, Frege had just finished writing the second volume of *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*) in

The Barber paradox (in a different literal *and* metaphorical sense). Seen at Rennweg 32 in Zurich, Switzerland.

which he formalized his theory of logic. His aim had been to derive all laws of arithmetic from just a few axioms that he considered self-evident. The two volumes were to be the culmination of his life's work. In fact, the second volume was just about to go to press in 1903 when an ominous letter arrived from across the Channel. In it, Russell informed Frege of the paradox. The letter threw into doubt the totality of Frege's work.

The logician was devastated. But in a remarkable display of intellectual honesty, he added an appendix to his book in which he described his quandary: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion." Poor Frege recognized and admitted the problem and subsequently abandoned many of his mathematical and logical beliefs.

DÉNOUEMENT

The barber's paradox is one of the notorious problems in set theory that occur when elements of a set refer to themselves. In this theory, sets are defined as collections of items with a common property, for example, the shirts in a closet, the books in a library, the odd numbers. An item either belongs to the set, or it does not. The blue jeans do not belong to the set of shirts, the Elvis Presley CD does not belong to the set of books, and the numbers two, four, and six do not belong to the set of odd numbers. That said, the items need not have a property that is common in the usual sense. For example, my sunglasses, a copy of the Bible, a pebble, and the banana in the fridge can make up a set of four items. The only thing they have in common is that I imagined them as members of a set.

A set of sets is also a set, the common property being that the items in this set are not shirts or books but sets. A library, for example, can have a set of French books and a set of Italian books. The library itself is the set that contains the two sets of books.

Now let's construct a "library catalog." We define it as a booklet that lists all the sets that do not contain themselves. Obviously, "library catalog" contains the sets "French books" and "Italian books." Neither of these two sets contains itself, so the definition is satisfied. How about the booklet "library catalog" itself? Since it is a set that does not list itself as one of its elements—its sole elements are “French books” and “Italian books”—it should be listed in “library catalog.” But as soon as it is listed, “library catalog” *does* contain itself, thereby violating the booklet’s condition. So, it should not be listed.

It’s the annoying Figaro all over again, the barber who auto-shaves . . . or not.

• • •

In the statement “Figaro must shave all of Seville’s men who do not shave themselves,” the subject (Figaro) and its object (hetero-shavers) belong to the same set of people (the community of men, which, in turn, is composed of auto-shavers and hetero-shavers). The paradox arises because one cannot assert whether Figaro is an auto-shaver or a hetero-shaver.

The culprit here is the word *all*, as in “all of Seville’s men.” One way out of the dilemma is to assume that Figaro hails not from Seville but from another town. Or that the barber is a woman. “Susanna must shave all men who do not shave themselves” presents no problem because Susanna does not belong to the community of men. But these would be cop-outs. In fact, the answer to the dilemma is simple: such self-referential statements are neither true nor false but meaningless. There simply is no such barber: Figaro does not, and cannot, exist.

MORE . . .

Whenever a series of reasonings leads both to a conclusion and to its opposite, we have a problem; something has to give. Luckily, it is usually possible to pinpoint the culprit in logic and mathematics. And so it was with the barber problem: Frege had grounded his basic laws of arithmetic on several axioms; since their combined use leads to a paradox, one of them must be incompatible with the others. It soon became apparent that Frege's "basic law V" was the culprit.[1]

As it eventually turned out, the infamous basic law V is not even required to prove the laws of arithmetic. Instead, one may use something called "Hume's principle," which asserts that *for any concepts* F *and* G*, the number of* F*s is equal to the number of* G*s if and only if there is a one-to-one correspondence between the* F*s and the* G*s.* Suffice it here to say that by using Hume's principle, the paradox of self-reference is avoided.

• • •

Russell's barber paradox can come in many guises. All that is required is a suitable transitive verb (e.g., "to *blurb*") and its substantive form ("the *blurb*er"). Then one can ask the question, *Does the blurber who blurbs all who don't blurb themselves blurb himself?* Try it with, say, "to teach," "to paint," or "to love," and voilà: you have a paradox.

1. Let's not even try to understand basic law V. The *Stanford Encyclopedia of Philosophy* states it as follows:

$$\varepsilon' f(\varepsilon) = \alpha' g(\alpha) \equiv \forall x[f(x) = g(x)]$$

32

I DON'T BELIEVE IT

Moore's Paradox

We would not be overly surprised if some master of alternative facts tweeted, "The climate is getting warmer, but I do not believe it is." Or "The climate is getting warmer, but I believe that it is not."[1]

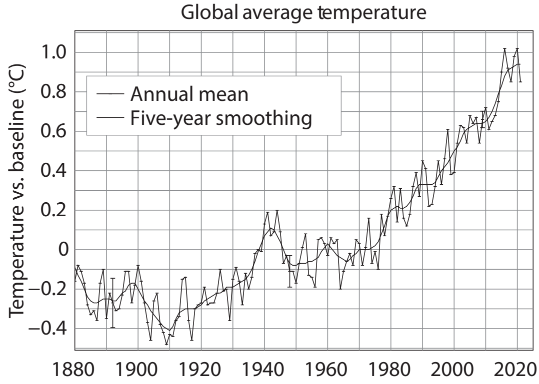
Whoever makes declarations like these is very confused—or must have been smoking something. Such statements simply don't make sense. They are contradictions given the data. Right?

Nope, no contradictions here. Whatever one may think of "alternative facts," these tweets are not contradictory.

Huh? On the one hand, said Twitter user contends something; on the other hand, he refuses to believe in the very same thing. The proclamations sound downright irrational or at the very least patently absurd. How can one simultaneously assert a fact and, in the same breath, deny one's belief in it?

But, if these statements are not contradictory, why do we have the queasy feeling that they are nevertheless paradoxical? Isn't that in itself a paradox?

1. When asked about the findings of his own government's climate report, President Donald Trump retorted, "I don't believe it." "Trump on Climate Change Report: 'I Don't Believe It,'" *BBC News*, November 26, 2018, https://www.bbc.com/news/world-us-canada-46351940.



The climate is getting warmer.

Source: https://commons.wikimedia.org/wiki/File:Global_Temperature_Anomaly.svg.

The seeming paradox derives from George Edward Moore (1873–1958), a distinguished British philosopher at the University of Cambridge. An influential thinker, he was, together with Bertrand Russell and Ludwig Wittgenstein, one of the trinity of philosophers at Trinity College who made Cambridge one of the centers of analytic philosophy. Wittgenstein once remarked that the most important philosophical discovery made by Moore was of the oddity of sentences like those in the tweets cited at the beginning of this chapter. It was Wittgenstein who named the paradox after Moore.

Such puzzling statements represent a different take on the adage "seeing is believing." To wit, the stunned remark "I can't believe what I am seeing" translates into "I see it, but I don't believe it," and from there into the Moorean sentence "It exists (or it is happening), but I do not believe it." Absurd, yes, but not illogical.

By the way, there is a slight difference between the two versions of Moorean sentences. The first version—for example, "It is raining, but I do not believe it is"—expresses the negation of a belief, whereas the second—for example, "It is raining, but I believe it is not"—expresses a belief in the statement's negation. The first version is called the omissive form of the paradox because it reports the omission of a belief; the second is called the commissive form because the speaker commits to a belief.

DÉNOUEMENT

First, why are declarations like those in the Moorean tweets not paradoxes? Because they contain no contradictions, that's why. The declarations are perfectly logical and admissible.

All such declarations consist of two statements: first, an assertion ("the climate is getting warmer") and second, a belief ("I do not believe it is" or "I believe it is not"). Both the assertion and the belief can be true, each on its own. And they can remain true even when they are combined into one statement. For example, if I sit in a windowless room and the weather channel announces that it is raining, I may well say, "It is raining, but I do not believe it is." Such a statement is not at all irrational or absurd; it simply says something about my faith in the weather channel.

Obviously, had we declared that "the climate is getting warmer, and the climate is not getting warmer," we would have uttered a contradiction. And the statements "I believe that the climate is getting warmer, and I do not believe that the climate is getting warmer," and "I believe that the climate is getting warmer, and I believe that the climate is not getting warmer," would also be contradictions. But these are not the statements cited at the chapter's outset.

This begs the second question: Why do we have that queasy feeling that we are being confronted with irrational statements? Why do we feel intuitively that such sentences are absurd even though—as I just showed—they are not?

The reason is that, in general, once we assert something, we should also believe that it is true. Hence, when we utter a statement of fact like "The climate is getting warmer," we implicitly suggest that we believe that this claim is true. Hence, the sentence at the top of the chapter, in a more extensive guise, seems to imply, "I believe that the climate is getting warmer, but I believe that it is not." This statement is truly absurd, of course, because the speaker's beliefs are contradictory.

In reality, Moorean sentences in their pure form, without the addition of bells and whistles, are neither contradictory nor illogical nor paradoxical. This becomes apparent when such a sentence is transposed to the past tense: "I did not believe it was raining, but as a matter of fact it was" is not contradictory. A Moorean sentence's inoffensiveness also becomes obvious when it is transposed to the third person: "The climate is getting warmer, but climate deniers think it is not" is a perfectly logical and rational, if profoundly regrettable, statement by any standard.

MORE . . .

Roy Sorensen, a philosopher who did much work on Moore's paradox, created the witticism "My atheism angers God." This bon mot implies the Moorean sentence "God exists (otherwise, who would be angered?), but I believe that God does not exist (because an atheist does not believe in God)."

This points to intriguing Moorean sentences that permit reticent interviewees to wriggle out of testifying to their belief systems. By asserting that "There is no God, but I believe there is," or "God exists, but I do not believe it," one attests to all and to nothing. (However, by capitalizing "God," one implicitly admits to one's faith.) On a more down-to-earth level, the statement "Communism/

socialism/capitalism (take your pick) is good for all of us, but I do not believe in it" allows the speaker to remain noncommittal.

• • •

To end the chapter, here's an anecdote showing that even highly rational people may be subject to Moore's paradox. It is said that one day Albert Einstein came to the house of Niels Bohr, an equally renowned scientist, and noted a horseshoe, believed by simple people to fend off bad luck, nailed to the wall above the front door:

> EINSTEIN: Herr Bohr, you don't believe in such nonsense, do you?
>
> BOHR: Of course not, Herr Einstein! But I have been told that it works even if you don't.

33

KNOWN KNOWNS, KNOWN UNKNOWNS

Fitch's Paradox

"As we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say, we know there are some things we do not know. But there are also unknown unknowns—the ones we don't know we don't know."

Huh?

Right! "Huh?" was the reaction of many a journalist present at Donald Rumsfeld's Department of Defense obfuscation masquerading as a news briefing on February 12, 2002. What the secretary said was befuddling. But was it wrong?

Note that Rumsfeld omitted *unknown knowns* from his pronouncement, and right he was because that really would have been a contradiction. But other than that, what he said made perfect sense, though "making sense" may not have been foremost on the secretary's mind that day.

Some members of the press may have realized that the pronouncement contained a deep epistemological question: Is every truth in principle knowable? Can the truth value of every statement be determined if sufficient effort is invested? In an era of fake news and "alternative facts," many members of the press would answer no. More investigatively minded journalists may answer yes.

Donald Rumsfeld during a news briefing.

Source: https://commons.wikimedia.org/wiki/File:Defense.gov_News_Photo_020221-D-9880W-080.jpg.

The latter would be said to subscribe to the so-called knowability thesis, which states that, in principle, every truth is knowable.

One may accept the knowability thesis or not. The investigatively minded journalists could say that with enough muckraking, all truths can be exposed. Mathematicians who subscribe to the thesis would claim that every correct mathematical statement can, in principle, be proven. All it takes is sufficient talent and time. They point to the famous dictum of David Hilbert, the foremost mathematician of the early twentieth century: "*Wir müssen wissen—wir werden wissen*!" ("We must know—we will know!")

But acceptance of the knowability thesis leads to an absurdity, namely, to the conclusion that we are potentially omniscient. If every truth is potentially knowable, we can, indeed, know everything. However, since we are not omniscient, not even potentially, we cannot maintain that every truth is knowable.

The paradox derives from a little-known paper published in 1963 by Frederic Fitch, then a Sterling Professor of philosophy at Yale University. Hidden in its midst, and considered rather insignificant by the author himself, was theorem 4: "For each agent who is not omniscient, there is a true proposition which that agent cannot know." The converse of theorem 4 is that "if all true propositions can be known, there are agents who are omniscient." As we shall see, the converse leads to a paradox.

Fitch did not come up with the theorem on his own. In a footnote, he disclosed that the theorem came about because of a comment by an anonymous referee. Half a century later, it was revealed that this anonymous referee was the celebrated Princeton logician Alonzo Church.

DÉNOUEMENT

To illustrate, let me draw on a celebrated mathematical proposition, the Poincaré theorem. For an entire century, the theorem was only a conjecture, and nobody knew whether it was true. Then, in 2002, the Russian mathematician Grigori Perelman published an extremely complex proof demonstrating that the theorem is true; the proof has since been verified by the world's foremost mathematicians.

I know that the Poincaré theorem is correct because the experts say so. This is the knowability thesis. Unfortunately as a mere mortal, I cannot follow, let alone verify, the proof . . . even though I wrote a book about the history of the theorem. Hence, I personally cannot vouch for the theorem's correctness. Let's call this the "ignorance thesis." Can the two theses coexist?

In mathematical notation, I will denote a theorem as *p* and "I know that . . ." as *K* . . . Thus, *Kp* means "I know that theorem *p* is correct." On the other hand, if I do not know whether *p* is correct, I write ~*Kp*. ("Not" is denoted by the prefix ~.)

From the knowability thesis, I know in principle (a) that *p* is correct since experts say so. Since I am not omniscient, however, I also know (b) that I don't know that *p* is correct. Hence, $K(p \,\&\, {\sim}Kp)$.

Once we know that the combination of two things is correct, each on its own must also be correct. (For example, if I know that it rains and that the road is wet, then I know that it rains, and I also know that the road is wet.)

Therefore, $K(p \,\&\, {\sim}Kp)$ implies both Kp and $K{\sim}Kp$.

The second part of the implication, $K{\sim}Kp$, can be shortened simply to ${\sim}Kp$ because "I know that I do not know . . ." is tantamount to saying, "I do not know . . . ."

And here's the rub: from the combination of the knowability thesis and the ignorance thesis, we have derived both Kp and ${\sim}Kp$. But that *is* a contradiction! We cannot know that a statement is true and at the same time *not* know that the statement is true. Hence, we are left with two options. Either the knowability thesis is correct, and every truth is, in principle, knowable. In this case, we must abandon the ignorance thesis, and we are omniscient. Or the knowability thesis is incorrect, and we are ignorant. In this case, some truths are unknowable. Take your pick.

• • •

Another example is the following. Let's say that our pocket calculator tells us that the product of two large numbers, *A* and *B*, is *C*. We could verify this ourselves by multiplying the two numbers by hand, but we don't want to waste our time. So, in principle, we accept the proposition (let's call it *m*) that $A \times B = C$ without actually knowing it: $K(m \,\&\, {\sim}Km)$. This time, we subscribe to the knowability thesis and discard the ignorance thesis. In arithmetic, we are omniscient.

MORE . . .

The confrontation between the two theses may seem trivial: if you know everything, you are omniscient; if there are things you don't know, you are ignorant. But it actually makes an important philosophical point. One school of thought asserts that only truths that are verifiable by the human senses are legitimate truths. The knowability thesis, in contrast, asserts that truths can be established through the rules of logic, purely by thought processes.

Of course, the knowability thesis prompts the question about the existence of God. Billions of people believe that God exists, yet I don't know if He does. In monotheism, it is the very essence of a supreme deity that He be unknowable, even in principle. So, in the case of God, the knowability thesis must be rejected, and we remain ignorant.

34

NO ATM IN THE DESERT

Parfit's Hitchhiker

Imagine the following scenario: during a stroll through the desert, Lucinda is held up by bandits and robbed of all her belongings. Left to perish in the desert, she has only the clothes on her back and her wallet, empty except for the credit card that the brigands had not bothered to take. Suddenly, seemingly out of nowhere, an SUV drives up. The driver leans out the window: "Lost?" he asks, somewhat superfluously. Barely able to keep her head up, she simply nods in the affirmative, whereupon he makes her an offer: "I can give you a lift to the next town, but it's quite out of my way, so you'll have to pay me a thousand dollars for my troubles."

Dehydrated and lost, but relieved and thankful, Lucinda looks up and accepts the offer, promising to pay him with cash from an ATM as soon as they reach civilization. The driver takes a good look at her, verifies that she seems to be a rational individual . . . and drives off, leaving her stranded.

Was the driver rational?

Yes, the driver was perfectly rational, albeit quite ruthless. And Lucinda was also quite rational and conveyed that to the driver. And that was her undoing! Had he taken her to town, she would have been saved. But then, being perfectly rational, she would have said

Source: https://commons.wikimedia.org/wiki/File:Hitchhiker%27s_gesture.jpg.

to herself, "Why pay him?" Rather, she would tell him to get lost. Serves him right for being an extortionist.

Oh, but wait! When he came upon her in the desert, the driver confirmed that Lucinda was a rational person and willing to pay a thousand dollars to have her life saved. But he also realized that once in town, there would no longer be any reason for her to pay him. Being rational himself, he correctly predicted how Lucinda would react the moment she was out of danger. So why bother making the detour to the town? He was better off just driving on and leaving her stranded in the desert. And that's what he did.

So, even though Lucinda was perfectly willing to pay a measly thousand dollars to save her life and assured the driver of this, he departed, and she was left to die. The very fact that she seemed rational is what did her in.

A paradox.

• • •

The problem has become known as "Parfit's hitchhiker" after the British philosopher Derek Parfit (1942–2017), who discussed the situation in his book *Reasons and Persons*. In a profile about Parfit, the *New Yorker* described him as "the most important moral philosopher in the English-speaking world" and the book, together with his three-volume *On What Matters*, as "the most important works to be written in the field in more than a century." The subject matter of his philosophical inquiries were questions about ethics, rationality, and personal identity. "We have reasons for acting. We ought to act in certain ways, and some ways of acting are morally wrong. Some outcomes are good or bad, in a sense that has moral relevance," he wrote.

DÉNOUEMENT

The reason for Lucinda's undoing was the fact that she appeared to the driver to be a rational person. Although rational people lost in the desert would gladly pay a thousand dollars to save their lives, once saved, they would not become philanthropists. "Nothing that I do now will change what happened in the desert," they would say to themselves after reaching town. "My paying a thousand dollars wouldn't provide me with any further advantage." Of course, Lucinda knew this before she found herself lost in the desert, and the driver knew it, too. And so, by driving off, he did the rational thing.

The root cause of the problem is that the relationship between cause and effect, in which one event is the result of another, is reversed. In general, causes entail effects. For example, a shot is fired, and a target is hit. Or one waters the flowers, and the flowers grow. In the scenario described here, the cause–effect situation is reversed. The driver wants to give Lucinda a lift (effect) if he gets money (cause). But those two events occur in the wrong order! (See also chapter 4 for the related good service paradox.)

The problem could be solved by enacting laws and regulations and by instituting mechanisms to enforce them. True, a lack of enforcement mechanisms may very often be to the advantage of debtors; but in this case, it is to Lucinda's *detriment*. If there were a way for her to make a binding commitment in the desert—that she would pay the one thousand dollars upon arrival in town—she would have been saved. But since no such instrument exists in this scenario, the driver abandons her.

Another enforcement mechanism would be the enhancement or diminution of a person's trustworthiness. But this is only of significance in a scenario that repeats itself. Since the two protagonists are not likely to cross each other's paths again, trustworthiness plays no role.

MORE . . .

If only electorates were as rational as the SUV driver. "Rational politicians," who make campaign promises only to forget them once elected, would never be picked by "rational voters." Fortunately, there is an enforcement mechanism of sorts: politicians want to get reelected.

Of course, enforcement mechanisms and contract law are no panaceas. Insolvencies may leave creditors stranded. Apparently, that was ex-president Donald Trump's modus operandi: get the work done, and then stiff the businesses by declaring bankruptcy. That's what makes America great again.

Oh, by the way, buses are more rational than taxis. One must pay for bus trips in advance, whereas naive taxi drivers charge only at the end.

35

PLUS, OR QUUS?

The Kripkenstein Paradox

1) 25 + 14 = 39
2) 68 + 57 = 5
3) Any number divided by itself equals one.

True?

1) True.
2) Could be true.
3) Not always.

What?

For starters, we'll deal only with (1) and (2).

Samira is a smart girl. Whenever she is asked to add two numbers, she gives the arithmetic sum: 25 + 14 = 39; 68 + 57 = 125.

Quentin is a bit dim. True, whenever asked to add two numbers, both of which are smaller than or equal to 57, he gives the arithmetic sum. But whenever one of the summands is larger than 57, he answers 5. His teacher calms Quentin's frantic parents; their boy is not unintelligent but quirky. Instead of performing "plus" operations, Quentin performs "quus" operations, which say that whenever one of the summands is greater than 57, the answer is 5.

Thus, for summands smaller than or equal to 57, the results of the plus operation and the quus operation coincide. But in Quentin's world, while 25 quus 14 equals 39, 68 quus 57 equals 5.

Xaviera comes along for an IQ test. She is asked to add 35 and 12. Her answer is 47. Is Xaviera smart by having performed a plus operation, or is she quirky by having performed a quus operation?

• • •

The paradox goes back to Ludwig Wittgenstein's investigations into language and meaning. In 1982, the American logician Saul Kripke, a distinguished philosopher with a dozen awards and honorary doctorates, reinvestigated and reinterpreted Wittgenstein's analysis. Since he did not quite follow Wittgenstein's lead, the paradox described in this chapter is an amalgamation of Kripke's and Wittgenstein's thinking on the subject, which became known as the "Kripkenstein paradox."

DÉNOUEMENT

How can one tell whether Xaviera did a plus or a quus operation? Is she herself aware of which operation she performed? We would need to know in order to predict what answer she would give when the summands are larger than 57.

True, the typographical symbol "+" seems to have been used by her in the past as a symbol for the addition operation. But so far in her life, she has only been asked to add numbers that were smaller than 57. So, based on her history, we cannot tell which operation she performed, and therefore we won't know which operation she will perform with numbers larger than 57. Her previous answers provide no indication whether she performed a plus or quus operation.

One could ask Xaviera how she arrived at her result, 35 + 12 = 47. If she answers that whenever she encounters the "+" symbol, she performs a procedure (an algorithm)—she places 35 marbles on a table, then places another 12, and then counts the lot—one may conclude that this corresponds exactly to the plus operation. But for larger numbers, the table may be too small to hold all the marbles, and some may roll off, leaving only 47. In that case, Xaviera's procedure would correspond to the quus operation. We are again left in the lurch.

Let's look into Xaviera's mind—into what Kripke called her "disposition." Though she may never have been asked, "How much is 68 plus 57?" before, Kripke would suggest that if asked, she would be disposed to answer, "125." This explanation works slightly better than ascribing her meaning based on her past usage, but it must be justified—because Xaviera might be prone to, say, dyscalculia, or, as Kripke pointed out, she might be in the grip of a frenzy or under the influence of LSD and thus disposed to making errors. So, why would we say that she is disposed to using the plus operation and giving the arithmetically correct answer?

For this, one must look beyond Xaviera's own mind—and turn to sociology. Since Xaviera is part of a community, her use of a term is justified if it matches the meaning that the members of the community ascribe to it. Once the community agrees on the meaning that one ought to ascribe to the term, Xaviera is justified in asserting that the meaning of the plus operation is arithmetic addition.

MORE . . .

Let's now return to statement (3), which says that any number divided by itself equals one.

Neither Wittgenstein nor Kripke would have believed that one day, there would be personal computers that would behave like

Quentin. When asked to divide a number by itself, most PCs using double-precision arithmetic would answer correctly 1.0000 or 0.999 . . . (with a further fifteen 9s behind the decimal point). Given the inevitable rounding errors of PCs, the latter number is legitimately considered equal to one (see also chapter 17 on rounding crooked numbers).

But in 1994, when certain numbers larger than 824,633,702,441 were divided by themselves, some PCs produced answers with only eight 9s after the decimal point, followed by another ten random-looking digits. Though that may seem a minute error, it was a major scandal: the specifications of the PCs guaranteed an accuracy of eighteen digits, but the rounding errors were a billion times larger than they should have been.

The conclusion was that some PCs computed, whereas others *quomputed*! And the problem was that even if one obtained eighteen 9s after the decimal point when dividing small numbers by themselves, one could not be sure whether one had a computer or a quomputer on one's desk.

It took a while, but eventually it was discovered that processors produced by Intel Corporation did, in fact, quompute. It was the famous Pentium bug. Intel eventually spent half a billion dollars to replace the faulty quomputer chips. Now, ask yourself this alarming question: Is your home computer computing or quomputing?

VIII

LOOPY LOGIC

Making Sense of Seeming Nonsense

Where's the logic? The Ancient Greek philosophers invented it, but then boggled the logical mind by inventing paradoxes at the same time. Modern logicians are no less provocative.

NB: This part of the book contains two of my favorite paradoxes, namely Meno's and Hempel's.

36

GOD EXISTS AND THE MOON IS MADE OF CHEESE

Curry's Paradox

If this statement is correct, the moon is made of cheese.
If this statement is correct, Germany borders China.
If this statement is correct, twelve is a prime number.
If this statement is correct, twelve is not a prime number.
If this statement is correct, God exists.
Are these assertions true?

Surprisingly, yes!

In formal logic, a conditional assertion (A) is a statement of the form "if the antecedent (B) is true, it follows that the consequence (C) is true." Let's take, for example, the conditional assertion "if it rains, the street is wet." If this assertion (the entire sentence) is true, and if the antecedent ("it rains") is true, then the consequence ("the street is wet") is true. This rule of inference—used to draw logical conclusions—is called *modus ponens* (loosely translated from the Latin: method of affirming) and can be expressed in mathematical notation as follows: *if P is true, and if P implies Q, then Q must be true*.

Note that the assertion does not claim "*only* if it rains is the street wet." The street may also be wet if it does not rain. After all, someone could have spilled water onto the street, or a dog may have peed onto the pavement.

Is the moon made of cheese?

Source: https://commons.wikimedia.org/wiki/File:The_cheese_circle.JPG.

Note also that *modus ponens* does not preclude the entire conditional assertion from being true just because the antecedent is false. In fact, an assertion is true even if its antecedent is false. This means that the assertion "if it does not rain, the street is wet" is also true. (To wit: spilled water or dog pee.) So, in addition to the assertion "if it rains, the street is wet," the two other assertions—"if it does *not* rain, the street is wet" and "if it does not rain, the street is *not* wet"—are also true.

Of course, this does not imply that the street is actually wet if it does not rain; it only means that the assertion (A) is true even though the consequence (C) as a standalone statement might be false. On the other hand, the consequence itself is true only if both the assertion and the antecedent are true.

In summary, even if the antecedent (B) is false (i.e., it does *not* rain), and regardless of whether the consequence (C) is true or not (i.e., the street may or may not be wet), the assertion (A) itself is still true. The only assertion that is definitely false in this context is "if it rains, the street is *not* wet." This accords with intuition since the street *must* be wet if it rains.

• • •

The problem with the conditional assertions posed at the beginning of this chapter (e.g., "if this statement is correct, twelve is a prime number") is that the antecedent (B)—"if this statement is correct"—refers to the entire assertion (A)—"if this statement is correct, twelve is a prime number." We are once again confronted with a vexing case of self-reference since the word *statement* in the antecedent refers to the entire assertion itself. (See also chapters 9, 14, and 31.)

Let's analyze the conditional assertion "if this statement is correct, twelve is a prime number" in detail. We know that twelve is *not* a prime number since it can be divided by two, three, four, and six (and, of course, by one and twelve), but let's see what happens.

First, let's assume that the assertion (A) is *true*. Since the assertion and the antecedent are one and the same, this is tantamount to saying that the assertion's antecedent (B) is true. Now, since both the assertion and the antecedent are true, the consequence—"twelve is a prime number"—is also true. Hmmm!

Second, let's assume that the assertion (A) is *false*. Again, since the assertion and the antecedent are one and the same, this is tantamount to saying that the assertion's antecedent (B) is false. Now comes the crucial point: as explained earlier, even if the antecedent is false, the conditional assertion remains true. (Oh, the joys of those who trust self-reference: this second assumption, that the assertion (A) is false, is false.)

Whether we start out by assuming that the assertion is true or that it is false, the implication in both cases is that twelve is a prime number. Similarly, the moon is made of cheese. And Germany borders China. And twelve is not a prime number. And God exists.

• • •

The paradox is named after Haskell Curry (1900–1982), an American logician who obtained his PhD in 1930 in Göttingen, Germany, from the then undisputed high priest of mathematics, David Hilbert. Haskell Curry must not to be confused with the New York magician Paul Curry (1917–1986), who invented the "missing square puzzle," a geometric riddle often also referred to as Curry's paradox, which turns out to be simply an optical illusion. *Our* Curry's paradox is one more example of self-reference paradoxes.

DÉNOUEMENT

The crux of the matter is that the assertion "if this statement is true . . ." does not make clear what is meant by the word *statement*. If "this statement" refers to the sentence "if this statement is true, then twelve is a prime number," then . . .

"If 'if this statement is true, then twelve is a prime number' is true, then twelve is a prime number."

And this means that . . .

"If, 'if, "if this statement is true, then twelve is a prime number" is true, then twelve is a prime number,' is true, then twelve is a prime number."

And so on, and so on. Hence, "this statement" does not refer to an actual statement but to an infinite recursion of statements and is therefore undefined. So, how can an undefined statement be true? It cannot. It makes no sense to claim the veracity of statements that are undefined.[1]

MORE . . .

This paradox is especially vexing since it seems to show that any statement one can think of can be proved to be true. "God exists" can be shown to be true, as can "God does not exist." Ludicrous statements like "Germany is made of cheese" and "the moon borders China" can be proved. The number twelve can be revealed—to paraphrase King Hamlet—to be and not to be a prime number.

The disconcerting conclusion would be that in contrast to some politicians' conviction that "all is fake," Curry's paradox is able to show that all is true. True—but truly ridiculous!

1. The contrapositive formulation is "If the moon is not made of cheese, then this statement is false," which, since the moon is indeed not made of cheese, is just the liar's paradox.

37

TO KNOW NOTHING

Socrates's Paradox

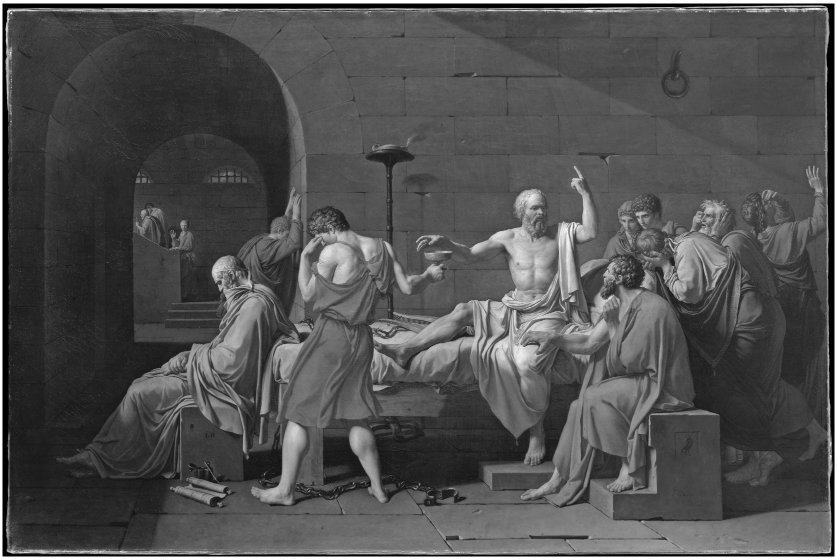
"I know that I know nothing." This self-effacing, humble admission of his own intellectual limitations was supposedly uttered by Socrates. But can one know that one knows nothing?

Obviously, the sentence is a contradiction; the first part refutes the latter part. Namely, the speaker admits, on the one hand, that she knows nothing, and, on the other hand, asserts that she does know at least one thing: that she knows nothing.

A paradox!

It's similar to saying, "I won't say anything." By uttering the sentence, the speaker contradicts what he has just said.

Plato, who put down Socrates's dialogues in writing, never actually quotes the great philosopher as having uttered this exact sentence. Rather, in *Apologies*, the written version of the speech that Socrates gave in front of the court that was to condemn him to death, he tells a story of the Oracle of Delphi. A friend asked the oracle if there was a man wiser than Socrates. The oracle responded that there was not, and Socrates, modest as he was, sought out people wiser than he to refute the divination. He found a supposedly wise man, a politician, who, he soon found out, knew nothing—just as he knew nothing. But in one respect, this man differed from him:



The Death of Socrates, painting by Jacques Louis David.

Source: Metropolitan Museum of Art, https://commons.wikimedia.org/wiki/File:The_Death_of_Socrates_MET_DT40.jpg.

he did not know that he knew nothing. Socrates, on the other hand, knew that he knew nothing: "I am wiser than this human being. For probably neither of us knows anything noble and good, but he supposes he knows something when he does not know, while I, just as I do not know, do not even suppose that I do. I am likely to be a little bit wiser than he in this very thing: that whatever I do not know, I do not even suppose I know."[1]

In this respect, he truly was the wisest man around: he alone was prepared to admit his own ignorance. But—and this is the point of this chapter—the contradiction in his statement persists.

1. Socrates, Apology 21d.

DÉNOUEMENT

As it stands, the utterance "I know that I know nothing" makes no sense because it contains both an assertion and its negation. By admitting to knowing nothing, one admits to knowing something: that one knows nothing.

In a similar vein, the statement "I am not saying anything" is nonsensical. If someone says something like "I won't say anything about politics," the utterance makes sense. But to say that you won't say anything is a contradiction. In the same manner, "I know that I know nothing about analytic geometry" makes sense. But to *know* that you *don't know* and, further, that you don't know *anything* is a contradiction.

So, how can we reconcile the eminent thinker's statement with our logical intuition? On the one hand, "I know that I know nothing" is contradictory and sounds illogical to our ears. On the other hand, the statement may make sense if the second "know" refers to something different from the first "know." For example, the second "know" could be a synonym for "understand," as in "I know that I understand nothing." Or it could be used in the sense of "comprehend," as in "I know that I comprehend nothing." If Socrates had meant to say something like "I know that I comprehend nothing about the purpose of life," we would have no problem with his statement.

MORE . . .

One can conjure up assertions similar to Socrates's famous but self-contradictory quote. "I am not thinking straight" serves as an example. If a drunkard's thinking is confused, and he describes his thought process at that moment accurately as "not thinking straight," he is, in fact, thinking straight—at least in describing his current state of mind.

"I can't remember anything" is another example. To see this, let's expand the statement into a fuller declaration: "I recall that I do not remember anything." Obviously, "recall" and "not remember" contradict each other. But "I can't remember anything about my childhood" would be perfectly legitimate.

And Judge Kavanaugh's argument during his 2018 Supreme Court confirmation hearing—while defending himself against accusations of sexual assault—that he could not recall whether he had blacked out during his drinking sessions sounds very suspect since "recall" and "blacked out" can well be considered contradictions.

Food for thought: Does the pronouncement "I won't say anything about Jill and Jim's extramarital affair" fall into the category of nonsensical statements? While we just argued that the statement "I won't say anything" is nonsensical, the statement "I won't say anything about Jill and Jim" is a legitimate utterance. But "I won't say anything about Jill and Jim's extramarital affair" is contradictory because one has just divulged the fact that there is an extramarital affair.

• • •

A final point: in contrast to "*I am not saying anything*," the vernacular declaration "I ain't sayin' nothin' " does make perfect sense, though most probably not the sense the speaker intended. Nevertheless, the double negative produces a legitimate statement. (See also chapter 6 on double negatives.) But how about "I ain't sayin' nothin' to nobody no more"?

38

IS THERE A POINT IN ASKING THE QUESTION?

Meno's Paradox

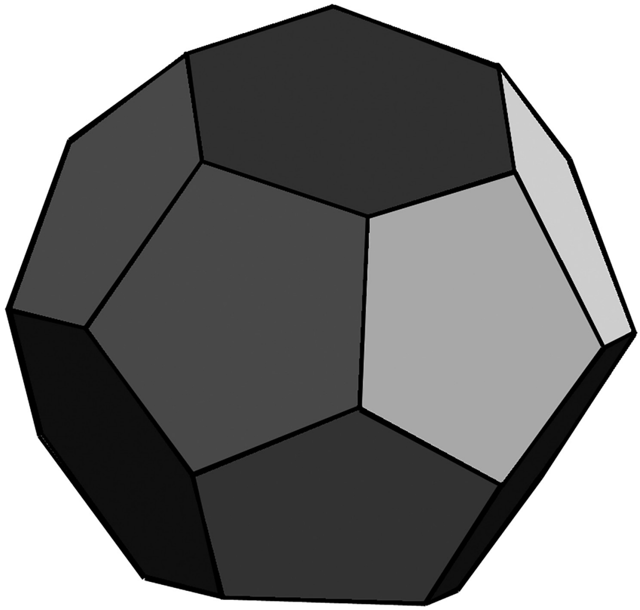
What is a hexakaidekahedron?

Is it an insect with more than a dozen pairs of legs? Is it an incantation used in witchcraft? If I told you that it is an outer planet of the solar system, would you believe me? If you were told that it is a sixteen-faced polyhedron, would that be correct?

More generally, is there a point in asking the question?

If you opted for the polyhedron, you were right. But how would you be able to know that it was the correct answer? Maybe you studied advanced geometry in high school. Or you read my book on Kepler's conjecture in which the hexakaidekahedron makes an appearance. Maybe you had heard the term before and knew what it was. In all these cases, you did not need to ask about the term's meaning because you already knew the answer.

But what if you did not know beforehand what a hexakaidekahedron is, and someone simply told you, "It is a sixteen-faced polyhedron"? How would you realize whether this is the correct answer? What if someone told you, "It is a sixteen-sided polygon"? How would you know that this is an incorrect answer? You would not. Hence, if you didn't know the answer beforehand, you would not be able to recognize a response as correct, even if it hit you over the head.



A sixteen-faced polyhedron.

Source: https://commons.wikimedia.org/wiki/File:Tétrakitétraèdre_tronqué.png.

The upshot of all this is that if you know the answer to a question, there's no need to search for it. Hence, you cannot learn anything by asking. And if you don't know the answer, you have no way of recognizing a correct answer even if it is given to you. Therefore, there's no point in asking questions.

All this invites the question, of course, how we can trust what we read on the internet. How can we know if gossip reported on Facebook, information found via Google, an answer given by ChatGPT, or news reported on shady websites is true or fake?

• • •

As reported by Plato, the question whether there is a point in even asking a question arose in a debate between Socrates and a young general by the name of Meno. Discussing what "virtue" is, Socrates challenges Meno to define the term. After several unsuccessful attempts, Meno is about to give up but then has an inspiration. In an attempt to upstage Socrates, he challenges the philosopher in return. How would he, Socrates, recognize the correct answer if he did not already know what virtue was?

According to Meno, a thinker who searches for answers finds himself in a dilemma: he cannot search for what he knows—if he knows it, there is no need to search for such a thing—and he cannot search for what he doesn't know—if he doesn't know it, he does not even know what he's searching for.

DÉNOUEMENT

Meno, pleased with his brainwave, gloats, "Well, doesn't this argument seem to be finely stated, Socrates?" Socrates's rejoinder came immediately: "Not to me!"

Of course, Socrates does not agree. The hallmark of his method of inquiry, the so-called dialectical method, is to elicit truth by dialogue until the interlocutor "gets it." To illustrate his method, Socrates draws some geometric figures on the ground and then leads one of Meno's ignorant slaves through a series of steps until this uneducated man realizes a geometric truth. The slave did not know the answer beforehand, nor did he know what to ask. Nevertheless, when the correct answer became obvious in the course of the to and fro between Socrates and the slave, it just hit him. Hence, knowledge can be acquired through reasoning rather than through empirical investigation, and truth can be made explicit through dialectical questioning.

But there is a missing link. Socrates's argument presupposes that the slave is not totally ignorant. He does not know enough to find the answer on his own but knows enough to recognize it as correct when it is presented to him. How does Socrates explain that?

Well, says Socrates, the slave, like everybody, possesses prenatal knowledge, an understanding that is imparted to the immortal soul of every human being before they are born. In rational inquiry, a human being can call on this knowledge when needed through the process of recollection. Thus, one comes to know what one did not know previously.

MORE . . .

The dénouement does not sound very convincing. Socrates's appeal to the immortal soul and to prenatal knowledge seems far-fetched. Indeed, one may be excused for thinking that it is a cop-out. After all, the slave did not draw on his alleged prenatal knowledge all by himself to recognize a truth. By guiding the slave to the correct answer with leading questions, it was the philosopher who played the role of what he termed "prenatal knowledge" to elicit the knowledge that was supposedly latent within the slave.

In the ensuing discussion, even Socrates admits that he is not quite sure about his theory of prenatal knowledge and recollection. But one thing he *is* sure of: he vehemently emphasizes that "we shall be better, braver, and more active men if we believe it right to look for what we don't know."

But how should one go about doing that? We ask questions every day, and we get answers. Can we trust them? Can we believe search engines, media, artificial intelligence chatbots, social networks—and friends, teachers, politicians, and doctors? No magic path leads around Meno's dilemma, and in the end, only the proven strategies remain: examine answers critically, compare answers with known facts, ask for second opinions, listen to experts, consult experts, and trust historically reliable sources.

• • •

Socrates's own doubts notwithstanding, his theory of prenatal knowledge was revived with Noam Chomsky's theory of language acquisition. How is it, the MIT linguist asked himself, that children supposedly born as blank slates learn to speak? And how is it that as they grow older, they learn to express things of which they have no empirical knowledge? Chomsky holds the view that the human faculty of language is innate, hardwired in the brain, a view that hearkens back to Socrates's notion of prenatal knowledge.[1]

1. It should be noted that Chomsky's theory of language acquisition is not universally accepted. Further, the idea that humans possess an innate ability for mathematics was proposed much earlier by the German thinker Immanuel Kant (1724–1804).

39

THE WHOLE TRUTH AND NOTHING BUT THE TRUTH

Gödel's Incompleteness Theorem

One plus two equals three. Three plus two does not equal seven. These statements are easy to prove, just like everything in arithmetic: statements about the integers (. . . −3, −2, −1, 0, 1, 2, 3 . . .) are either true or false and can always be proved to be either one or the other. Correct?

No!

It turns out that there are arithmetic systems in which some statements are true but cannot be proved.

To illustrate, let's look at an example not from a mathematical system but the English language. Take the declaration "this statement is unprovable." Can we prove that the statement is, in fact, unprovable? If we could, then we would actually have proved a contradiction: we would have *proved* that the statement is *unprovable*. That cannot be. Hence, we must be unable to prove the statement. And since we are unable to prove it, the statement ("this statement is unprovable") is, of course, true.

The example shows that in the English language, there can be statements that are true but cannot be proved. As in several other paradoxes cited in this book, the crux of the problem is self-reference. By referring to "this statement," the declaration refers to itself and, as so often, creates a paradox.

At the International Congress of Mathematics in Paris in 1900, David Hilbert, then the world's foremost mathematician, posed a series of problems that he hoped would be solved during the following century. One of them was to find a complete and consistent set of axioms for arithmetic.

Three notions must first be clarified. First, a *set of axioms* is a collection of statements that are self-evidently true and from which other true statements can be derived. Second, the set of axioms is said to be *consistent* if it does not allow both the proof of a statement and its contradiction. This is a reasonable demand; after all, a system in which one can simultaneously prove a statement and its negation is useless. Third, the set of axioms is said to be *complete* if it permits the proof of all true statements and the disproof of all false statements. As in a court of law, a complete and consistent set of axioms can prove all truths and nothing but truths. That's what Hilbert wished for arithmetic.

Unfortunately, three decades after Hilbert announced his list of problems, his hope that someone would find a complete and consistent set of axioms for arithmetic was utterly crushed. Not only did nobody find such a set . . . it was much worse: the task turned out to be impossible. In 1931, the Austrian mathematician Kurt Gödel published a paper that proved that a set of axioms for arithmetic cannot be both complete and consistent. In particular, a consistent set of axioms (and we are interested only in such systems[1]) cannot be complete: not every true arithmetic statement can be proved, and not every false arithmetic statement can be disproved. To the immense consternation of mathematicians, the edifice built upon the foundation of such axioms will always contain truths that cannot be proved.

1. Gödel proved another theorem, the so-called second incompleteness theorem, which says that a consistent system cannot prove that it is, indeed, consistent.

Kurt Gödel (1906–1978).

Source: https://commons.wikimedia.org/wiki/File:Kurt_g%C3%B6del.jpg.

Gödel's proof that every axiomatic system for arithmetic is inherently incomplete has profound consequences for mathematical logic and the philosophy of mathematics. And apart from the importance for the foundations of mathematics, Gödel's incompleteness theorem may have consequences for concrete mathematical questions. Famous unsolved problems in number theory, like the twin prime conjecture (which says that there exist an infinite number of pairs of numbers *n* and *n* + 2 such that they are both prime), Goldbach's conjecture (that every even number is the sum of two primes), and Riemann's hypothesis (that the nontrivial zeros of the Riemann zeta function have real part ½) could be affected. Maybe these problems are unprovable in the axiomatic system that we are familiar with?

DÉNOUEMENT

There is no dénouement. The incompleteness theorem is here to stay. Gödel claimed not only that true but unprovable statements exist. He also showed how such a statement can be produced in any arithmetic axiomatic system. Using an intricate method of combining prime numbers, he developed a method to "translate" axioms and arithmetic statements into code numbers (so-called Gödel numbers). One can then prove the truth of a mathematical statement, he said, by verifying whether its Gödel number can be factorized into the axioms' Gödel numbers. Using this scheme, Gödel then produced a self-referential statement similar to the one cited earlier ("this statement is unprovable"), which, though true, is neither provable nor refutable in this axiomatic system about arithmetic.

This does not mean that the statement is unprovable in all axiomatic systems. One can devise a new system of axioms in which the statement's truth *can* be proved. To illustrate, let's take an example outside arithmetic, namely from geometry. In an axiomatic system that incorporates four of the five Euclidean axioms, the statement "parallel lines never cross" cannot be proved. Even though we intuitively know the statement to be true, we cannot prove it with the help of the first four axioms. But with the addition of the parallel axiom ("given a line *L* and a point *P* not on *L*, there is exactly one line through *P* that is parallel to *L*," or, in other words, "parallel lines never cross"), the statement is now mathematically and trivially true.

However, as Gödel showed for arithmetic, even after adding the offending statement as an axiom, one can then construct another true statement that cannot be proved in that new system.

For the unsolved conjectures in number theory (e.g., the twin prime conjecture), this does not mean that they are false but simply that they may not be provable within the axiom system to which we are used. Maybe additional axioms or an entirely different axiomatic system is required. Many mathematicians who base their proof of a theorem on an as yet unproven hypothesis, as, for example, on the unproven Riemann hypothesis, use the cop-out of simply adding it as an axiom, as Euclid did with the parallel axiom. They present mathematical proofs that hold "under the assumption that the Riemann hypothesis is correct."

MORE . . .

To calm jittery readers' nerves, let me say that the so-called Zermelo–Fraenkel set theory with the addition of the so-called axiom of choice (see chapter 16), commonly denoted "ZFC," suffices to prove all commonplace arithmetic problems required in day-to-day life. So, no worries about bridges collapsing, airplanes crashing, or elevators getting stuck because of incomplete arithmetic. But some mysterious conjectures may very well be true, though not provable in the ZFC axiomatic system to which we are used.

40

ARE ALL RAVENS BLACK?

Hempel's Paradox

Hypothesis: if the sun shines, it does not rain.

According to strict logic, this hypothesis is equivalent to its so-called contraposition: if it rains, the sun does not shine. One can prove the hypothesis by proving its contraposition (a.k.a. the "contrapositive"), and vice versa.

Now let's find some evidence for the hypothesis. On Monday, the sun shone, and it did not rain. That is evidence that the hypothesis is correct. But on Tuesday, it rained, and the sun did not shine. No problem: this is evidence for the contraposition and hence represents additional evidence for the original hypothesis. To summarize: both sunny Monday and rainy Tuesday are evidence for the original hypothesis. So far, OK.

Another hypothesis: all ravens are black.

And its logically equivalent contraposition: all nonblack things are not ravens.

Again, let's seek some evidence. We go to Central Park and spot a raven. It's black, which is good evidence for the hypothesis. We also spot a bed of red flowers. The flowers are not black, and they are not ravens. This is equally good evidence, this time for the contraposition. Recall that evidence for the contraposition is also evidence for the original hypothesis.

A black raven.

Source: https://commons.wikimedia.org/wiki/File:Raven,_Tower_of_London.JPG.

To summarize: both the black raven and the red flowers are evidence that all ravens are black. OK?

Hmm!

How can the spotting of red flowers be additional evidence that all ravens are black?

Now have a look at your brown shoes. They are not black, and they are not ravens; hence, they are evidence for the contraposition. (On the other hand, if your shoes were black that would obviously not be a disproof of "all ravens are black.") But there's more: your baseball cap is green, and your T-shirt is blue. Do the brown shoes, the green baseball cap, and the blue T-shirt represent additional evidence that all ravens are black? By strict logic, yes.

By intuition, nope. Though the logic seems impeccable, it somehow does not agree with our intuition.

A paradox!

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The paradox was raised in 1945 by Carl Gustav Hempel (1905–1997), a German philosopher who emigrated from his homeland in the mid-1930s and taught at the University of Chicago, Yale, Princeton, the Hebrew University of Jerusalem, and the University of Pittsburg. His thinking about what is evidence for a statement or how one can confirm a hypothesis had a profound influence upon more than a generation of philosophers of science. One of his best-known contributions is the paradox that I describe here, namely, the paradox of the raven or the paradox of confirmation.

Let's denote two statements as P and Q and their negations as $\sim P$ and $\sim Q$. Then, the statement "P implies Q" is written as "$P \rightarrow Q$," where P is the antecedent and Q the consequent. By inverting and flipping the two, we arrive at the contraposition of the hypothesis, expressed as "not Q implies not P" and written as "$\sim Q \rightarrow \sim P$." The original hypothesis and its contraposition are equivalent; they are different formulations of the same hypothesis. If one is true, then the other is also true.

Going from the sublime to the ridiculous, a bird-watcher could sit at home in an armchair instead of braving the outside weather and practice ornithology simply by looking at the furniture. Every piece that happens not to be black confirms the hypothesis that all ravens are black: the wooden table is a nonblack nonraven, and so are the yellow carpet, the blue aluminum bookcase, and the brown wicker chair. The observations are evidence for the hypothesis that all ravens are black. Even more strangely, the furniture can serve as evidence for the hypothesis "all ravens are purple" (since the pieces of furniture are nonpurple nonravens).

DÉNOUEMENT

Hempel was aware that all this seemed paradoxical, but he remained adamant. There is nothing wrong with the logic, he maintained; it is intuition that is misguided. In fact, every observation of a nonblack object that is a nonraven does provide evidence, though minute, minuscule, and insignificant, for the hypothesis that all ravens are black.

Evidence, even if abundant, is not proof, however. It was the philosopher Karl Popper (1902–1994) who stated that confirmations of general statements like "all ravens are black" are impossible. No matter how many black ravens one sees, the hypothesis "all ravens are black" cannot be said to be true because—the next thing you know—a white albino raven may suddenly appear in the sky. The only thing one can do, therefore, is to refute the hypothesis by spotting a single nonblack raven.[1] That would conclusively *disprove* the hypothesis. Seekers of truth must therefore make strong attempts to falsify hypotheses. If no evidence can be found to refute a hypothesis, in spite of serious attempts, it is likely that the hypothesis is not wrong—but it cannot be said to be unequivocally true.

Irving John Good (1916–2009), a cryptologist who helped Alan Turing decrypt coded Nazi messages at Bletchley Park, went further by disputing that even spotting a black raven is necessarily evidence for the hypothesis. He provided an example: there are two worlds with birds, world A with 950 finches and fifty black ravens and world B with fifty finches, 949 black ravens, and one white raven. An observer spots a bird which, upon further inspection, turns out to be a black raven. Which world is she in?

1. The fact that one is able to conceive of evidence that would disprove a scientific hypothesis is called *refutability*. In contrast, religious beliefs or ideological dogmas are nonrefutable; hence, they are not science.

By Bayes' theorem, she most likely finds herself in world B (since there are 949 black ravens in world B and only fifty in world A). In fact, the probability that she is in world B is 95 percent. But now comes the clincher: recall that in world B there is a white raven. Therefore, the bird-watcher's spotting of a black raven is evidence that in her world *not* all ravens are black.

MORE . . .

Bayes' theorem shows how to compute the probability of finding oneself in world B given that one spotted a black raven:

$$
\begin{aligned}
P(\text{world B} \mid \text{black raven}) &= P(\text{black raven} \mid \text{world B}) \times \\
&\quad P(\text{world B})/P(\text{black raven}) \\
&= (949/1{,}000) \times \tfrac{1}{2} \,/\, (999/2{,}000) = 0.9499
\end{aligned}
$$

IX

QUESTIONS OF FAITH

The Small Print of Holy Texts

Believers or doubters, saints and sinners, all fall prey to confusions when it comes to principles of faith. Come to think of it, even some commandments may be paradoxical. Thou shalt read on!

41

IN THE NAME OF THE LORD, YOUR GOD

The Third Commandment

The Bible's third commandment (third in the Jewish count, number two according to the catechism of the Catholic Church) commands, "You shall not take the name of the Lord, your God, in vain." And the punishment: "For the Lord will not hold him guiltless who takes His name in vain."

Now, here's a tricky question: What do you answer when someone asks, "*What* name should I not take in vain?" Since no human being is allowed to utter it, how can one instruct ignoramuses not to use it?

One can't!

With idols forbidden and God being without body or form, awe for the Supreme Being expresses itself in the dread in which His name is held and in the stringent limitations of its utterance. Blasphemy is considered one of the most evil sins a Jew can commit.

But blasphemy is not the only instance in which the forbidden appellation is uttered. Profanities and curses are infamous for invoking the name of God and are, of course, also strictly prohibited. Another occasion is the swearing of oaths and vows; they are always made in the name of God, not in the name of any other being or thing. Note that it is forbidden not only to swear to falsehoods—that is prohibited by the ninth commandment—but also to take an oath for which there is no need since that would mean taking God's name in vain.

So, on the one hand, God's name must never be taken in vain; on the other hand, if one must testify under oath, it must be in God's name.

But how can ignorant people avoid pronouncing the forbidden name by mistake if they do not know what it is? Well, if they are Jewish, they're simply supposed to know what is meant; nobody will spell it out for them.

A paradox!

The problem is that we once again have an instance of self-reference, a statement in which a subject refers to itself in order to exclude itself. As so often in philosophy—from Bertrand Russell's "the set of all sets that are not members of themselves,"[1] to Groucho Marx's "I don't want to belong to any club that would have me as a member," to the barber who shaves everybody who does not shave himself—self-reference leads to a paradox.

And so it is with the pronunciation of a name that must not be pronounced. Commonly written "YHWH" in the Bible and in prayer books in Hebrew letters (from right to left: Yod, Heh, Wav, Heh), the so-called Tetragrammaton *is* God's name. Only the high priest was permitted, and only once, on the holiest day of the Jewish year, on Yom Kippur, to utter God's name.

Orthodox Jews tiptoe their way around the difficulty by substituting another word for the forbidden name. Whenever the Tetragrammaton is encountered in Jewish liturgy—for example, when reading the Bible, reciting a prayer, or just wishing someone good or bad luck—the congregation substitutes "*Adonai*" ("Our Lord"), "*Elohim*" ("God"), "*Shaddai*" ("the Omnipotent"), or—the name that says it all—"*HaShem*" ("The Name"). And even that may be too close for comfort. Many faithful substitute "*Ha*'" for "*HaShem*." (In English, the Tetragrammaton is most often translated as "the Lord.")

1. Such a set is a member if and only if it is *not* a member (see chapter 31).

The Tetragrammaton.

Source: https://commons.wikimedia.org/wiki/File:YHWH_pronunciation.svg.

DÉNOUEMENT

Let's leave aside curses and profanities, which are anathema in any case. But how about blasphemers? How can they be brought to justice?

In biblical times, the penalty for blasphemy was death. However, a Jewish blasphemer could not be condemned to death unless a witness had heard the person pronounce God's name and would testify to the fact. Thus, courts were confronted with a dilemma: on the one hand, the judges needed to hear the witness describe what he had heard; on the other hand, uttering God's name, even by a witness, was forbidden. How could a trial proceed?

The rabbis' dilemma, indeed the paradox, could not be avoided. All the sages could do was reduce collateral damage as much as possible. To diminish the fallout of this capital sin, they devised an elaborate procedure. During the interrogation of a witness, the common forename "Yossi" was substituted every time God's name was meant. That was good for a while, but at some point the inescapable moment could no longer be avoided; the witness would be obliged to repeat what he had heard word for word. It was the dramatic high point of the proceedings. Everybody, except for the judges and the witness, was sent out of chambers. Then the judges called upon the witness to repeat verbatim what he had heard. As soon as he uttered the hallowed Tetragrammaton, the judges tore their clothes in shock and mourning, and the blasphemer's fate was sealed.

In spite of the dire consequences, blasphemy became so widespread with time that the quaint tradition of tearing one's clothes was discontinued by Rabbi Hiyya, a sage who lived about 1,800 years ago. He decreed that "he who hears the Divine Name blasphemed nowadays is not obliged to rend his garments, because otherwise his garments would be nothing but tatters."[2]

MORE . . .

When Moses encountered the burning bush and asked God what his name was, God answered, "I am what I am," or, since the conjugation of the Hebrew for "to be" is identical in the present and future tenses, "I will be what I will be" ("*Eheye asher eheye*").

Apart from the tautological answer not being very informative, the sentence's exact pronunciation is not known since in Hebrew script, only consonants are written. Which vowels the Israelites used in their spoken language was inferred only much later by Jewish scribes.

Thus, the original pronunciation of God's utterance "I am what I am" has been lost; what is left are the four consonants that many scholars believe may designate the first- or third-person singular form of "to be" in the present, past, and future tenses.

• • •

Food for thought: journalists must never use racial slurs (for example, the *n-word*) in their writing, and guests must never utter curses (for example, *f****) in polite society. So how does one explain to a six-year-old which words not to use when they become adults?

2. The Talmud, Sanhedrin 60a.

42

A STONE SO HEAVY . . .

The Omnipotence Paradox

Can an omnipotent being create a stone so heavy that He or She cannot lift it?

Can an omnipotent being square circles?

Can He or She make two plus two equal five?

Can an omnipotent God sin or lie?

An omnipotent being should be able to create anything, even a stone that is so heavy that He or She cannot lift it. On the other hand, an omnipotent being should be able to lift everything, including a stone that is so heavy that even He or She cannot lift it. It's a vicious circle.

To square a circle means to devise a geometric procedure—using only a straightedge and compass—that creates a square with the same surface area as a circle. Unfortunately, no such procedure exists because it would mean constructing something that involves the number pi (π). Since π is a transcendental number, it is impossible to accomplish this with a straightedge and compass. In fact, the phrase "squaring the circle" is often used to designate something that is impossible to do. So, can an omnipotent being construct a transcendental number with geometric tools?

Next, one may envisage the creation of a number system in which two plus two would equal five but only after smoking something weird or while intoxicated. The absurdity of such a number system is illustrated by Big Brother in George Orwell's dystopian novel *1984*. In fact, according to Orwell's "English Socialism," "war is peace" and "two plus two equals five." But let's forget about Big Brother. Can an omnipotent being force two plus two to be equal to five?

A final example: in contrast to Greek mythology, in which gods can commit sins, just like human beings, God is envisaged by the Abrahamic religions as both omnipotent and infinitely good. So, could an omnipotent God lie? Can He or She be evil? Is He or She able to commit a sin?

No, no, and no.

A notoriously vexing question that concerns itself with the same paradox is, What happens when an irresistible force meets an immovable object? More such questions can be asked, but you certainly get the paradox: the being would simultaneously be omnipotent and *strictopotent* (of limited capabilities, that is, not omnipotent).

These questions, and many like them, have been discussed among religious scholars since the Middle Ages, for example, by the Islamic jurist Averroes (1126–1198), the Jewish medical doctor Maimonides (1138–1204), and the Catholic priest Thomas Aquinas (1225–1274).

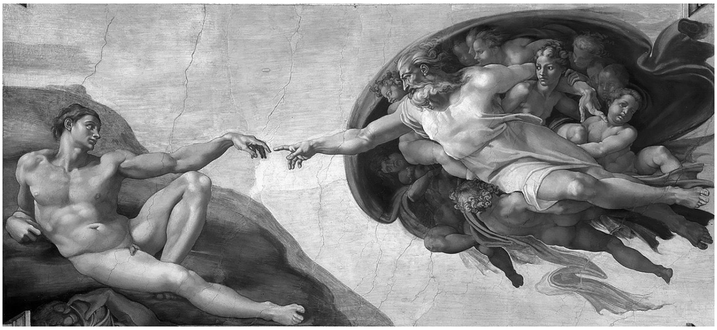
One of the perennial problems discussed in this context by adherents of various faiths is the question whether more than one omnipotent being could exist. Obviously, the answer is no; at most, one omnipotent being can exist, if at all. If there were more than one, each could be thwarted in their activities by the other and would therefore not be omnipotent. Actually, this insight could reconcile the competing monotheistic religions: the Jewish Elohim, the Islamic Allah, and the Christian God must be one and the same being, albeit assigned different names.

DÉNOUEMENT

An omnipotent being (generally considered to be God) can indeed do anything. That is the tenet usually held by religious scholars. After all, this is the definition of the word: *omni* meaning "all," and *potent* meaning "powerful." But not everything that can be expressed in words exists. Skeptics, even religious skeptics, questioned whether omnipotence should mean that everything, absolutely *everything that can be uttered*, should be doable by an omnipotent being. This would reduce the paradox to a semantic problem.

Philosophers point out that omnipotence and logic are clearly incompatible. Since *P* and *not P* cannot be true simultaneously, omnipotence—the ability to perform both *P* and *not P* at the same time—is simply inconceivable, if not absurd.

As a consequence, some doubters held that the term *omnipotence* must be restricted such that it would refer only to actions that are logically possible. So, an omnipotent being can indeed do anything—except break the rules of logic. Squaring the circle,



Michelangelo's *The Creation of Adam*.

Source: https://commons.wikimedia.org/wiki/File:Creation_of_Adam,_Michelangelo_(1475%E2%80%931564),_circa_1511.jpg.

making two plus two equal five, and performing both *P* and *not P* simultaneously are off limits.

Hard-core believers retort that logic, nature, and reality need not coincide. Since God, the omnipotent being, created the universe and everything in it, including us, He or She also created the laws of logic as we know them. Hence, He or She could also alter them if He or She so wanted.

The consensus among rational believers (pardon the oxymoron) seems to be that an omnipotent being can perform anything that is logically possible but that not even God can perform logical absurdities. That was also the view, seven centuries ago, of the Jewish sage Levi ben Gershon (1288–1344). He believed that God's ability to perform miracles was limited to whatever could exist in nature. For example, God was able to spontaneously turn Moses's staff into a snake because snakes can exist in nature. But He cannot generate something that could not exist in nature in the first place.

Conversely, rational skeptics, a.k.a. modern atheists, come to a different conclusion: they argue that the paradox clearly proves that no omnipotent being can exist.

MORE . . .

In at least one respect, mortal human beings are more capable than an omnipotent being: a strictopotent being could amputate His leg but would then be unable to walk, pull out teeth but then be unable to chew, or plug His nose but then be unable to smell. The key word here is *unable*. An omnipotent being, on the other hand, should not be able to do anything to Himself or Herself that would render Himself or Herself unable to do something (e.g., walk, chew, or smell). Paradoxically, the omnipotent being is, by definition, strictopotent.

• • •

One suggested solution to the paradox is to stipulate that an omnipotent being be able to remove His or Her ability to lift that heavy stone temporarily and then reinstate the ability. But that won't work if the question at the outset is rephrased as "Can an omnipotent being create a stone so heavy that He or She can *never* lift it?"

43

ACCUMULATE WEALTH . . . BUT DON'T SPEND IT

The Paradox of Asceticism

When Jesus preached, "Woe to you who are rich," in the Sermon on the Plain, he made the case for frugality and against extravagance. That lifestyle found its apogee in Catholic cloisters and convents, where monks and nuns forewent all riches and lived in poverty. Asceticism was in; enjoyment of luxuries was out.

With the emergence of the Reformation in Northern Europe, however, wealth was no longer sneered at; it became cool to pursue riches. Protestants, in particular Calvinists, subscribe to the so-called Protestant work ethic, which advocates hard work, discipline, diligence, punctuality, conscientiousness, and reliability, the goal being to earn one's living and accumulate wealth. Whatever one was predestined to do—be it to manage a corporation, cobble shoes, or serve food in a restaurant—one was obliged to do it as God's calling. It was the birth of capitalism.[1]

Some of the essential traits that Protestantism inherited from Catholicism were and are still considered virtues. Frugality and thrift are advocated, whereas profligate spending on

1. Some historians trace the roots of capitalism to the commercial centers of Venice, Florence, and Flanders in the fifteenth century.

Franciscan Martyrs, painting by Bernardino Licinio (1489–1565).
Source: https://commons.wikimedia.org/wiki/File:Bernardino_Licinio_-_Franciscan_Martyrs_-_WGA12986.jpg.

luxuries is frowned upon. This imagery has endured throughout the centuries.

Hold on! How can the accumulation of wealth and frugality coexist? On the one hand, God-fearing people should be industrious and hardworking to amass wealth, while, on the other hand, they should be frugal and thrifty and not squander their wealth. One must earn one's living by the sweat of one's brow, but this hard-earned money must not be used wastefully. Self-denial is virtuous; spending money on personal luxuries is sinful.

So, what to do with the amassed riches? Hoard them? Give them away? Why amass them in the first place if self-discipline, asceticism, and abstention from all forms of indulgence are advocated?

A paradox.

• • •

Members of Catholic mendicant orders—the Augustinians, Franciscans, Dominicans, and Carmelites—follow a lifestyle of work and poverty. Personal possessions are rejected, and the monks sustain their livelihood by whatever they receive for their manual labor and by donations and alms. Beginning with the teachings of the Augustinian monk Martin Luther (1483–1546), the driving force behind the Protestant Reformation, work was seen as a duty toward God that would benefit both the individual and society as a whole.

Four centuries later, the German political philosopher and sociologist Max Weber (1864–1920) was one of the first to point out the close relationship between the Protestant religion and the rise of capitalism. In his seminal work *The Protestant Ethic and the Spirit of Capitalism*, Weber noted that Protestants were religiously obliged to engage in professions and trades or to develop enterprises. Considered callings by God, these tasks had to be performed by the faithful as zealously as they possibly could. Simultaneously, they were prohibited from spending money on anything but the bare necessities. It stands to reason that people who go about their daily work in a diligent fashion and don't spend their money will eventually amass wealth. The character traits to which the faithful adhere—industry and frugality—cannot but produce riches.[2]

But religion was not the only factor in the development of capitalism. Weber theorized that the Protestant ethic remained "the spirit of capitalism" even when religious worldviews waned. Benjamin Franklin (1706–1790) is a case in point. One of the Founding Fathers of the United States, he emphasized hard work, frugality, and thrift without basing these ideals on spirituality.

DÉNOUEMENT

The thought may arise that the newly wealthy would donate surplus money to the poor and the destitute, but no, charity is frowned

2. See also Mandeville's paradox, discussed in chapter 54.

upon because poverty is seen as a consequence of laziness. To give to charity would only promote mendicancy, and beggars are a burden to others, which, in turn, is an affront to God. By not working, one fails to glorify God.

As a consequence, Protestants and capitalists are faced with a dilemma: they have lots of money but mustn't spend it. The manner in which this dilemma is resolved, according to Weber, is to invest one's wealth. But invest in what? Obviously not in jewels, yachts, vacations, or sundry luxuries, which are off limits, but preferably in the means to create even more wealth, for example in factories and infrastructure. And the profits of these endeavors should be reinvested and even more wealth created. It was capitalism par excellence.

MORE . . .

Karl Marx (1818–1883) held that human institutions like religion evolved according to the economic conditions of the time. Max Weber's *Protestant Ethic* held that it was the other way around: it was religion that fostered capitalism.

44

THOU MAYEST STEAL

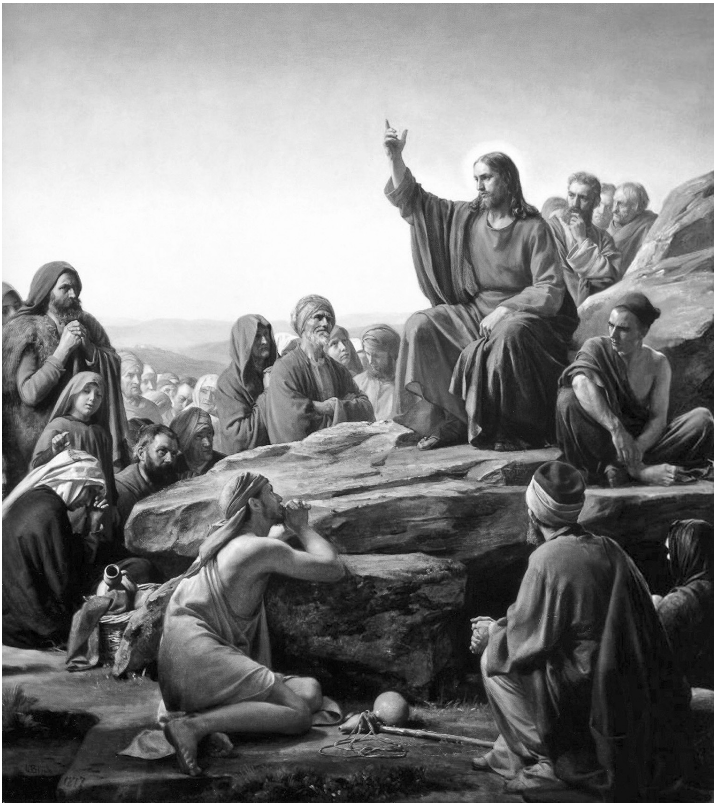
The Sermon on the Mount

The Ten Commandments handed to Moses on Mount Sinai were written on two tablets. The first listed five commandments about the holiness of God, the Sabbath, and the need to honor one's parents. The second tablet set out the laws that would permit a society to live together without too much strife. They were as follows:

6) Thou shalt not kill.
7) Thou shalt not commit adultery.
8) Thou shalt not steal.
9) Thou shalt not bear false witness.
10) Thou shalt not covet thy neighbor's house, wife, etc.

About twelve centuries after Moses received the tablets, Jesus of Nazareth wandered around the area near the Sea of Galilee. He preached the law, healed the sick, and performed miracles. One day, so the Gospel of Matthew recounts, Jesus decided to give a sermon about Moses's commandments from the top of a hill in today's northern Israel. As word got around, crowds converged.

To placate doubtful listeners, Jesus assured the audience that he would not revise God's words but just reiterate and explain the



Carl Bloch's *Sermon on the Mount.*

Source: https://commons.wikimedia.org/wiki/File:Bloch-SermonOnTheMount.jpg.

laws. And then he set forth: "You shall not murder . . . do not commit adultery . . . anyone who marries the divorced woman commits adultery . . . do not break your oath . . . if someone strikes you on the right cheek, turn to him the other also . . . love your enemies . . . give to the needy . . . pray . . . fast . . . do not worry . . . do not judge . . ."

With this and more, Jesus exhausted all that God in His infinite wisdom had imparted in the second tablet—and then some.

All?

No. There was one glaring omission: Jesus never mentioned, "Thou shalt not steal."

Did he forget commandment eight? Or did he, perish the thought, condone theft? The only reference to the offense occurs when Jesus points out the futility of amassing riches because they may be stolen by thieves: "Do not store up for yourselves treasures on Earth, where moths and vermin destroy, and where thieves break in and steal," Jesus gleefully remarks—not to criticize robbers but to blame the victim.

True, on other occasions Jesus included theft as an evil thought (Matthew 5:19), together with thoughts about murder, adultery, sexual vice, false testimony, and slander. And when asked by a man what commandments one must keep, Jesus replied, "You shall not murder, you shall not commit adultery, you shall not steal, you shall not give false testimony, honor your father and mother, and love your neighbor as yourself" (Matthew 5:18–19). But these seem like afterthoughts. In the all-important Sermon on the Mount, the much-anticipated lecture on the laws . . . nothing!

Mind you, this was not a one-off omission. In a sequel to the Sermon on the Mount, the so-called Sermon on the Plain (Luke 6), Jesus did not mention theft as a crime either. It was "Love your enemies . . . turn the other cheek . . . do to others as you would have them do to you" and more. But theft? Not a word.

• • •

Theft had been considered a crime long before Jesus roamed the countryside of Galilee, ever since laws had been given to the Babylonians by Hammurabi (in the eighteenth century BCE), to the Jews by Moses (in the fourteenth century BCE), to the Greeks by Solon (in the seventh century BCE), to the Chinese by Confucius (in the sixth century BCE), to the Romans by the Decemviri (in the fifth century BCE), and to the Indians by Kautilya (in the fourth century BCE).

In fact, the triad of serious crimes—murder, theft, and adultery—remained central to every law code in all cultures until the twentieth century when adultery was no longer considered everywhere to be illicit sex.

DÉNOUEMENT

The apparent omission of theft as a crime is nothing if not consistent with the teachings of Jesus, however. To wit, before embarking on a discussion of the commandments, Jesus issued a preamble. "Blessed are the poor in spirit," he announced, "the mourners, the meek, the hungry and thirsty . . ." On the plain, he again declared as blessed the poor, the hungry, the weeping, and the hated. These preambles may be the key to understanding the omission of theft from the list of crimes.

Since poverty is praised—"blessed are the poor"—and hoarding discouraged—"woe to you who are rich"—private property seems to have been of little concern to Jesus. Granted, on the mount, Jesus does not necessarily refer to the destitute but to "the poor in spirit." But on the plain, it is clear: Jesus refers to those "who are poor," that is, those who are materially deprived.

The point is reinforced by the exhortation "Do not worry about your life, what you will eat or drink; or about your body, what you will wear" (Matthew 6:25). Since everything will be provided for, there is no need to own possessions; the only apparent reason to accumulate wealth is to give alms. Since private property is superfluous, there is no need to discuss theft.

The point is reinforced by yet another quotation from the plain: "And of him that taketh away thy goods ask them not again"

(Luke 6:30). Not only is theft not denounced, but whenever a thief steals something from you, you are to let him keep. Private property is not untouchable, not yours to keep forever.

MORE . . .

What one may take away from the Sermon on the Mount and the Sermon on the Plain is that one must love one's neighbor and must not covet the neighbor's house or wife but that one may steal the neighbor's property, correct?

Well, not quite. In another gospel, the apostle Paul explains that the command to "love your neighbor as yourself" actually encompasses all the other commandments. He specifically includes the one we discussed here: "You shall not steal" (Romans 13:9).

45

ALL IS PERFECT

The Smarandache Paradox

Everything is possible.
Everything is perfect.
All is good.
Nothing is certain.
Every rule has an exception.

So . . .

. . . if everything is possible, is the impossible possible?
. . . if everything is perfect, is the imperfect also perfect?
. . . if all is good, is the bad also good?
. . . if nothing is certain, is the uncertain certain?
. . . if every rule has an exception, does this one, too?
Hmm . . . the questions are paradoxical!

These paradoxical clichés are similar to the questions of what occurs when an irresistible force meets an immovable object and whether an omnipotent being (i.e., God) can create a stone so heavy that He or She cannot lift it (see chapter 42).

It is named after one Florentin Smarandache, a mathematician born in Romania in 1954 who now lives and works in the United States. I was hesitant to name this paradox because in the scientific community, Smarandache is considered a controversial figure by polite people and a clown by the not so polite. Smarandache studied mathematics at the University of Craiova (ranked 1777th in the world in 2014) before fleeing the Ceauşescu regime for the United States. There he began to style himself as an all-round genius, not only in mathematics and theoretical physics but also in his roles as author, poet, playwright, and artist.

Employed at a two-year college, he has disseminated hundreds of papers in internet depositories, obscure periodicals, and self-published books and bulletins, practically all without having undergone serious refereeing. Rife with typos and grammatical errors, the papers are usually either nonsensical or trivial. His coauthors more often than not hail from obscure, unaccredited colleges.

At the time of this writing, his only Wikipedia entry is in German, and its "talk page" is telling: largely written by Smarandache himself, the page is filled with grudges, railings, and rants against Wikipedia, the entry's authors, and some obscure scientific mafia.

As an artist, Smarandache is not even controversial but simply ignored. He strives to make art as ugly as possible, as wrong as possible, or as bad as possible—anything to make people talk about him. Unfortunately for Smarandache, nobody does.

Smarandache's real gift lies in self-promotion. Somehow, he managed to have his name attached to an obscure mathematical constant, a function that he claims to have discovered though it has been known since the nineteenth century, and to a prime number whose claim to fame is that it may not even exist . . . and to this paradox.

DÉNOUEMENT

This paradox occurs when a statement connects two extremes that express opposites. One should realize that, in fact, one is actually speaking about two different universes: one where both extremes may occur and another where one extreme does not even exist. For example, in a universe where all is possible, it makes no sense to talk of impossibility. In a godless world, it makes no sense to ask people to which faith they belong. And it is useless to talk to flat-earthers about the globe.

To illustrate, let's consider the group of even integers. The statement "all members of the group are divisible by two" is clearly true. On the other hand, the statement "all members are divisible by two, also those that cannot be divided by two" is not only false but nonsensical—because the phrase "cannot be divided by two" makes no sense within the group of even integers.

One can conjure up more examples, for instance, statements about prime numbers that are divisible, or irrational numbers that are expressable as fractions. Whenever one tries to apply a notion that does not exist in that universe, one arrives at the paradox.

Hence, if *everything* in a group has an attribute, then, by definition, anything that has the opposite attribute is excluded. It does not exist. In universes where all is possible, perfect, good, and certain, the very notions of impossible, imperfect, bad, and uncertain are inconceivable.

Mathematically speaking, the paradox can be expressed thus: let *A* be some attribute (e.g., possible, perfect, good, or certain); then, "if everything is *A*, then *nonA* must also be *A*." Obvious nonsense.

MORE . . .

The paradox reminds of the famous exclamation by the fictitious Dr. Pangloss in Voltaire's eighteenth-century novel *Candide*. With the sarcastic announcement "All is for the best in the best of all possible worlds" (French: "*Tout est pour le mieux dans le meilleur des mondes possibles*"), the French philosopher, in his usual ironic style, attacked the optimism of all those who trust in God's benevolence and believe that He created a perfect world.

X

LEGAL LIABILITIES

Terms and Conditions Apply

As the saying goes, justice is blind. But who would have thought that staid judges, prosecutors, and lawyers may find themselves caught in paradoxical conundrums. This is quite concerning since a criminal's fate or a victim's vindication can be at stake.

46

WHEN TWO RIGHTS MAKE A WRONG

The Blackmail Paradox

Alec, a senior engineer at Blimey Inc., seeks a raise in salary. He makes an appointment with Erika, the head of human resources, to discuss the matter. The evening before the meeting, Alec spots Erika on a date with a gentleman who is not her husband. At the meeting the next day, while discussing the possible raise, Alec casually informs Erika that he saw her the day before and intends to tell her husband about the obviously illicit affair.

Erika gets the message. The salary increase is a done deal, and everybody is happy: Alec gets a raise, Erika is off the hook, and the husband is none the worse for it. All OK!

All OK?

No!

Why not? Seeking a salary raise is perfectly legal. It is also legal to reveal or to conceal information about an illicit affair. So, if both of Alec's actions are legal, what's the problem?

The problem is that the combination of these two perfectly legal activities is called blackmail, and it is illegal. So why would Alec's "Give me a raise or I'll tell your husband" be illegal? And would Erika's "I'll give you a raise if you don't tell my husband" also be illegal?

The paradox that two rights can make a wrong has been and still is being pondered by legal scholars and philosophers. "Why is

Source: https://commons.wikimedia.org/wiki/File:Blackmail_(1947_film)_poster.jpg.

it illegal to threaten to do what you can do legally anyway?" the law professor James Lindgren asked. If it's legal to do it, it should be legal to threaten to do it.

But we are loath to accept blackmail as acceptable behavior. Indeed, in the thirteenth century, England outlawed extortion, a closely related concept, by royal officials. Scotland extended the prohibition in 1567 to any perpetrator, making it a crime to obtain property by threats of physical harm to person or property.

In the United States, the first statute that prohibited threats to expose crimes was passed in New Jersey in 1796. Blackmail as such was deemed a crime in Illinois in 1827 with a law that prohibited written threats to publish any "infirmities or failings, with intent to extort." An 1835 statute in Massachusetts extended the ban to verbal threats. And threats to expose evidence of embarrassing, albeit noncriminal, behavior, were outlawed in the United Kingdom in 1843.

DÉNOUEMENT

In declarations such as "I'll sue you unless we settle this matter," "I'll quit if you don't increase my bonus," and "We'll strike if you don't agree to a thirty-five-hour work week," both the means and the ends are legal. In divorce proceedings, too, quid pro quos are common practice.

They are normal negotiation tactics in bargaining situations and do not constitute blackmail. Banning agreements based on such discussions could be considered an infringement of the freedom of consenting adults to engage in voluntary transactions. One could even make a tenuous argument that blackmail benefits society as a form of private law enforcement—tenuous because, in effect, it would legalize vigilantism.

• • •

To determine whether a crime is being committed, one must consider who is being harmed. Is it the recipient of the blackmail threat, a third party, or society as a whole?

First, if the harmed party is the blackmailee, the next question to be asked is whether the alleged blackmail is a threat or an offer. Obviously, giving in to blackmail is less harmful to the victim than if the threat were carried out. So, in a bad situation, it may be considered a helpful concession to let the victim choose the lesser of two evils.

However, the act of blackmail must definitely be considered wrongful if the threat would be illegal were it carried out. Thus, if the blackmailer threatens physical harm to a balking blackmailee or the destruction of property, the act of making the threat is illegal.

Second, if it is a third party that is being harmed, the question is whether a breach of duty is involved. Citizens are obliged to report crimes and serve as witnesses. Withholding such information may harm a third party. If Mrs. Smith witnessed a Mercedes rear-ending a Fiat and then told the Mercedes driver that she would not testify if he paid her a certain amount of money, the Fiat driver would be harmed, and Mrs. Smith would have committed a crime.

On the other hand, government officials are obliged to keep classified documents secret; doctors, lawyers, priests, and accountants must withhold privileged information; company executives must not divulge insider knowledge and parties to a nondisclosure agreement must not reveal anything. When the American whistleblower Edward Snowden, an employee of the National Security Agency (NSA) with a high-level security clearance, found out that the NSA was eavesdropping on American citizens, he fled to Russia and asked for asylum. Had the Russians then threatened him with extradition unless he revealed everything he knew, Russia would have been the blackmailer, Snowden the blackmailee, and the NSA the harmed third party. Had Snowden demanded a ransom from the NSA to withhold the information instead of fleeing, the American public would have been the harmed third party. Both cases would be considered illegal blackmail (though Russia would be difficult to prosecute).

But there is also information that one is not obliged to disclose even if a third party is harmed. Erika's husband, the third party in the case presented at the beginning of this chapter, is not *entitled* to information about his wife's secret rendezvous. Many would argue that withholding this information, even in exchange for payment, would not be considered a crime.

So what about Erika's action? In principle, Erika may legally agree to grant Alec more money in exchange for his discretion. But since Erika is not an owner but an employee of Blimey Inc., the salary raise is not hers to grant. By giving in to Alec's blackmail, she would hurt Blimey, Inc. and would be guilty of corruption.

Finally, some theorists argue that society as a whole would be harmed if blackmail were legal, mainly through the economic consequences. Legalizing blackmail would lead to fraud and lack of trust, create improper behavioral incentives, result in an inefficient allocation of resources, and be costly to society.

MORE . . .

Today, the United States Code declares blackmail a federal crime. Paragraph 873 states, "Whoever, under a threat of informing, or as a consideration for not informing, against any violation of any law of the United States, demands or receives any money or other valuable thing, shall be fined under this title or imprisoned not more than one year, or both."

The code covers blackmail only with respect to federal crimes, however. On the state level, all states have statutes banning blackmail, though they vary in their definitions of what constitutes blackmail and in their exceptions to the statues.

47

GUILTY UNTIL PROVEN INNOCENT

The Prosecutor's Fallacy

The prosecutor at a murder trial in Manhattan sums up her case to the jury. The accused's blood type matches the very rare type found at the crime; its incidence among the general population is only one in one hundred thousand. Hence, she claims, the probability that the blood would match the accused's if he were innocent is 0.00001, that is, one thousandth of 1 percent. "There is only a one-in-one-hundred-thousand chance that the accused is innocent," she exhorts the jury. "Therefore, you must convict him."

Her argument is persuasive. But is it correct?

No!

Admittedly, it is very unlikely that an innocent person's blood type would match the one at the crime scene. But that is true only for people picked at random.

Consider this: approximately 1.6 million people live in Manhattan. If the odds are one in one hundred thousand, then there are about sixteen people in Manhattan whose blood type matches the sample found at the crime scene. One of them must be the culprit. But if just one of them is arrested, there is still a 94 percent chance (fifteen out of sixteen) that he is innocent.

Or this: let's say that one out of every one hundred Manhattanites (i.e., sixteen thousand) is tested at random. The probability of finding a match is nearly 15 percent, and there is no guarantee that this match is the guilty person.[1]

The term "prosecutor's fallacy" was coined in a 1987 paper by William Thompson and Edward Schumann, professors at the University of California, Irvine. They ran experiments with undergraduates that showed that the majority failed to detect errors when exposed to fallacious arguments about the interpretation of statistical evidence. Their conclusion was that people's tendency to draw erroneous conclusions casts doubts on a jury's ability to judge guilt or innocence.

DÉNOUEMENT

The apparent paradox arises because of a confusion between conditional probabilities, that is, probabilities of something happening given that something else occurs. The prosecutor in this chapter's example confused the statement "The accused is guilty, given that the blood types match" with the statement "The blood types match, given that the person is guilty." She changed the true condition and the doubtful conclusion around.

The correct way to judge the probability is, first of all, to determine what the *prior* probability is, that is, the probability of guilt before one has evidence. Let's call that $P(A)$. When evidence arises—for example, the blood results—the prior probability must be updated to give the *posterior* probability. We'll denote this by $P(A|B)$, where B indicates the availability of new evidence. In words: the probability of A, given B. Similarly, $P(B|A)$ indicates the probability of B, given A.

1. The probability of finding a match with one test = 1/100,000 = 0.00001. The probability of not finding a match after one test = 1 – 0.00001 = 0.99999. The probability of not finding a match after sixteen thousand tests = $0.99999^{16,000}$ = 0.852 . . . The probability of finding a match after sixteen thousand tests = 1 – 0.852 . . . = 0.147 . . .

The way to update probabilities when new evidence becomes apparent was developed by Thomas Bayes (1701–1761). Bayes' theorem is as follows (see also chapter 30):

$$P(A|B) = P(A) \times P(B|A) \;/\; P(B)$$

Let's illustrate. Dennis must choose one of three cookies on the table: a round cookie (*R*), a square cookie (*S*), and a triangular cookie (*T*). One of these cookies contains a hidden diamond. The prior probability that a cookie contains the diamond is ⅓ for each. Now let's say that Annie grabs the triangular cookie, bites into it . . . and finds nothing. Given that Dennis now knows that there's no diamond in cookie *T*, what is the posterior probability that the diamond is hidden in the round cookie? We can figure this out easily: since only the round and the square cookies are left, the probability is ½ for each.

Let's see how this works out with Bayes' theorem. We denote "there's a diamond in the round cookie" as event *A*, which has a prior probability of ⅓, and "there's no diamond in the triangular cookie" as event *B*, which has a probability of ⅔. Now let's plug the numbers into Bayes' formula to see what $P(A|B)$ is, that is, the probability that the diamond is hidden in the round cookie, given that it is not in the triangular one.

To apply Bayes' theorem, we still need $P(B|A)$, that is, the probability that there's no diamond in the triangular cookie, given that the diamond is in the round cookie. But that is a certainty: if the diamond is in *R*, it cannot be in *S*. So, $P(B|A)$ is 1. Plugging these numbers into Bayes' formula, we get

$$P(A|B) = \tfrac{1}{3} \times 1/\tfrac{2}{3}$$

which equals ½, as expected.

• • •

The prosecutor committed the fallacy because she confused $P(innocent|blood\ match)$ with $P(blood\ match|innocent)$. The correct manner to update the prior probability, $P(innocent)$, to the posterior probability, $P(innocent|blood\ match)$, according to Bayes is as follows:

$$P(innocent|blood\ match) = P(innocent) \times P(blood\ match|innocent) / P(blood\ match)$$

We know that $P(innocent)$ is 1,599,999/1,600,000 = 0.99999938. We also know that the probability that an innocent person will have a blood match is $P(blood\ match|innocent) = 0.00001$.

We still need $P(blood\ match)$. This is the weighted average of the probabilities of getting a blood match if the suspect is guilty and if he is innocent (note that $P(blood\ match|guilty) = 1$):

$$\begin{aligned} P(blood\ match) &= P(blood\ match|innocent) \times P(innocent) + \\ &\quad P(blood\ match|guilty) \times P(guilty) \\ &= 0.00001 \times 0.99999938 + 1 \times 0.00000072 \\ &= 0.00001072 \end{aligned}$$

Putting all this into Bayes' formula, we get

$$\begin{aligned} P(innocent|blood\ match) &= 0.99999938 \times 0.00001 / 0.0000107 \\ &= 0.93457\ldots \end{aligned}$$

and

$$P(guilty|blood\ match) = 1 - P(innocent|blood\ match) = 0.06543\ldots$$

The upshot is that the probability of the suspect's innocence is increased from one in one hundred thousand to more than 6,500 in one hundred thousand. It's by no means a ticket to exoneration, but it significantly weakens the prosecutor's argument to the jury.

MORE . . .

A similar fallacy is the defense attorney's fallacy. Consider this: blood of a type that occurs in only 1 percent of the population is found at a crime scene in a wooded area on the outskirts of a city. A suspect is picked up nearby, and it turns out that he has that blood type. But in a city of one million people, there are ten thousand with that blood type. So, the defense attorney claims, there is only a one-in-ten-thousand chance that the suspect is guilty.

This argument is fallacious because it assumes that the suspect was drawn at random from among the one million citizens; all other evidence is ignored. Obviously, the vast majority of the ten thousand people with this blood type could not have committed the crime because they were elsewhere in the city at the time. The fact that the suspect has the same blood type as the guilty person *and* was picked up near the crime scene makes it much more likely than a one-in-ten-thousand chance that he is the perpetrator.

48

THE RIGHT TO REMAIN SILENT

The Fifth Amendment

During the holdup of a convenience store, a masked burglar kills the cashier. Several people witness the crime. Some see the burglar draw a gun, others hear shots, and still others see a masked man run away. Within minutes, police officers arrive on the scene and catch and arrest the suspect.

The day of the trial arrives. Witnesses are examined and cross-examined. They contradict each other: some saw the gun, others did not; some recognize the suspect as the burglar, others do not. One witness refuses to testify, claiming he is the suspect's cousin. No luck. Just like any other witness, he is subpoenaed and forced to describe what he saw.

Just like any other witness?

No! There is one other witness—the one who is most intimately familiar with what happened—who refuses to testify. And by law, nobody at court can make him talk: the suspect himself.

Why?

• • •

It is a criminal court's duty to determine the guilt or innocence of the accused in a fair trial. There are many limits, however, on how a fair trial is to be conducted. One is the Fifth Amendment of the U.S. Constitution, which specifies that "no person shall be . . . compelled in any criminal case to be a witness against himself." Hence, suspects cannot be forced to testify if they believe that their testimony may cast doubt on their innocence. The United States is not alone in this; the statutes of many countries contain such a stipulation, and the European Court of Human Rights has found the privilege to be an implicit requirement of the right to a fair trial.[1]

Something seems amiss. At court, nobody is allowed to lie; witnesses swear oaths to tell the truth, the whole truth, and nothing but the truth. But some are permitted to withhold the truth. Why not compel suspects to describe what they witnessed or what they did? Their testimony would be ever so helpful in finding out the truth.

So why keep this clause of the Fifth Amendment on the books? A paradox!

One reason for the enactment of the Fifth Amendment goes back to ancient Jewish law. According to the Talmud, nobody can be convicted of a capital crime by his or her own confession. Maimonides, the twelfth-century scholar explained that a confused person or someone with a death wish may falsely admit to a capital crime in order to be executed. (He thus anticipated Freud's psychoanalytic theories by eight centuries.) Since human life and the human body are sacred according to Jewish law, suicide is forbidden; accordingly, one may not provoke one's own death by confessing to a crime.

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1. The Latin motto *nemo tenetur se ipsum accusare* ("nobody can be held to accuse himself") was coined, although no trace of the privilege to remain silent during a criminal trial has been found in Roman law. Note that a prosecutor could argue, tongue in cheek, that the Amendment does not apply to female persons since the wording specifies "a witness against himself."

Source: Nick Youngson, CC BY-SA 3.0, https://pix4free.org/photo/4523/fifth-amendment.html.

And since the ninth commandment forbids lying, remaining silent is the only option.[2]

Another more pertinent reason goes back to fifteenth-century England. During the reign of Henry VII, lower courts were reluctant to prosecute and convict powerful members of the nobility. Therefore, in 1487, the Court of Star Chambers was created; it would call anyone to justice regardless of their social or political standing. Initially acclaimed for its fairness and competence, the court eventually developed into a tool for the king's whims and became an instrument of oppression. (It was abolished in 1641.)

2. The full commandment says, "Thou shalt not bear false witness against thy neighbor" (Exodus 20:16), which has been interpreted over the centuries to mean "do not lie."

But it was not the oppressiveness of courts like the Court of Star Chambers that induced the Founding Fathers to enact the Fifth Amendment. It was the court's inquisitorial method of truth-seeking that did it: instead of the prosecution having to bear the burden of proof, it sufficed to elicit a confession from the accused—by any means deemed necessary, including torture. And unimaginable cruelty was used to draw out admissions of guilt, be they true or false. By the eighteenth century, the English judiciary had realized that coerced admissions to a crime were inherently unreliable. Henceforth, confessions obtained through torture before or during a trial could no longer be used as evidence.

And this was the reason the Founding Fathers appended the Fifth Amendment to the Constitution.

DÉNOUEMENT

So, in olden times, the right to remain silent had its raison d'être, though it was questioned even then. Indeed, the English philosopher and social reformer Jeremy Bentham (1747–1832) called the rule against self-incrimination "one of the most pernicious and irrational rules that has ever found its way into the human mind. . . . If all criminals of every class had assembled and framed a system after their own wishes, is not this rule the very first they would have established for their security?"

But nowadays, in a constitutional democracy operating under the rule of law with orderly judicial procedures, fair courts, and an absolute ban on torture, the privilege against self-incrimination may be questioned. In 1937, Justice Benjamin Cardozo delivered an opinion of the Supreme Court in which he challenged its usefulness: "Indeed, today as in the past there are students of our penal system who look upon the immunity as a mischief rather than a benefit, and who would limit its scope, or destroy it altogether. No doubt there would remain the need to give protection against torture, physical or mental. . . . Justice, however, would not perish if the accused were subject to a duty to respond to orderly inquiry."

Hence, the paradox could be resolved by insisting on testimony by the accused, albeit while safeguarding them from physical and mental torture.

MORE . . .

Such an amendment to the Fifth Amendment would not even require a change to the text's wording. All it would take is a change in the interpretation of the clause "shall not be compelled to be a witness against himself." At present, the wording signifies that the accused can under no circumstances be legally forced to testify against him- or herself. But "not compelled" could be interpreted as meaning "shall not be tortured," which would give a more modern meaning to the Fifth Amendment; as a consequence, the accused could be legally compelled to testify.

49

WHEN IN DOUBT, ACQUIT

The Unspecified Offense Paradox

Ernie is known throughout the county as a brawler. A month ago, he was involved in a fistfight at the Red Dog Bar, a week later in a knife fight at the Blue Cat Club. The local sheriff has had enough of Ernie's shenanigans and arrests him; the prosecutor decides to charge him with disorderly conduct in the first case and with bodily harm in the other. Sentencing guidelines specify that initiating a fistfight carries a mandatory sentence of one hundred hours of community service; the sanction for a stabbing is much harsher: two years in jail.

Ernie's day in court arrives. The judge decides to try both cases together. The prosecution produces nine witnesses who testify that Ernie was the attacker in the fistfight. The defense produces one witness who says he is not sure whether Ernie started the fight. In the case of the stabbing, four of five witnesses point to Ernie as the culprit; the other claims it was someone else.

The judge's instructions to the jury are clear: "There are two cases before you, a fistfight and a stabbing. You must unanimously determine the defendant's guilt in each case. If you are not at least 95 percent certain that the accused is guilty of the offense, you must acquit him."

After several hours, the jury comes back . . . without a verdict. The jury is hung. Has justice been done?

The discussion in the jury chamber had been lively. The facts were clear: there was a 90 percent probability that Ernie was the attacker in the fistfight. After all, nine out of ten witnesses had identified him. But according to the judge's instructions, the hurdle for conviction is 95 percent, so Ernie should be cleared of the charge. In the stabbing case, the evidence for Ernie's culpability was even weaker: only four out of five witnesses identified him, an 80 percent probability of guilt. So, the jury's foreperson suggested, "Let's just acquit Ernie of the offenses and go home." Ten jurors nodded in agreement.

"Not so," Mrs. Ipswich piped up. "This guy is obviously a no-goodnik," she claimed emphatically. "He got involved in not one fight but two. He belongs behind bars, but if he doesn't go to jail, he should at least be sentenced to community service. That would teach him a lesson."

A discussion flared up. The foreperson and the ten jurors tried to convince their recalcitrant colleague of their reasoning by explaining the judge's guidelines to her. But timid Mrs. Ipswich stood her ground and would not budge. Her eleven exasperated colleagues saw that she wasn't giving up and sent a note to the judge informing him that they could not reach a unanimous verdict. The judge was forced to declare a mistrial.

Did Mrs. Ipswich serve or harm justice?

• • •

There are two theories about criminal sanctions. Sentences pronounced by a judge can punish wrongdoers, or they can express moral outrage. Those who believe in punishing wrongdoers, generally called retributivists, are of the opinion that the commission of a crime alone is sufficient reason to inflict a sentence. Lawbreakers should not be put away solely in order to be reeducated, isolated from society, or deterred from committing future crimes. No, criminals simply deserve to be punished.

Proponents of the moral outrage theory, called expressivists, want to convey the condemnation or disapproval of an act. They emphasize the importance of the expressive, educational, and communicative aspects of the criminal sanction. Under expressivist theories, sanctioning a wrongdoer is a public manifestation of condemnation and disapproval of their deeds.

DÉNOUEMENT

Was Ernie an innocent bystander who had simply spent two relaxing evenings at the Red Dog Bar and the Blue Cat Club? Or is he a no-goodnik who should be taught a lesson?

The probability that he was innocent of the fistfight is 10 percent, leaving enough doubt to acquit him. And the probability that he was innocent of the stabbing is 20 percent, much more than required to get him off the hook. But what is the possibility that he was innocent of both? Or, what is the probability that he was guilty of at least one of these offenses?

Basic probability theory provides the answer. The aggregate probability of two unrelated events occurring is the product of their probabilities. In Ernie's case, this is 10 percent times 20 percent, which equals 2 percent.

Hence, the probability that he was innocent of both offenses is only 2 percent, and the probability that he was guilty of at least one is 98 percent. This is sufficiently high to declare Ernie guilty, albeit without specifying which offense. As Mrs. Ipswich claimed, Ernie should at least be sentenced to community service.

This is the opinion that would be expressed by proponents of the retributivist theory. If an agent committed a wrong, they say, there is reason to impose a sanction on that person, even if the nature of the wrong remains unspecified; the person clearly deserves to be punished.

But current legal systems, based largely on expressivist theory, reject the aggregation of probabilities in criminal law. They require the unambiguous identification of the act being condemned: "Punishing a person for an offense she may or may not have committed (simply because it is highly probable that she committed either this offense or a more serious one) rather than for the offense she committed dilutes the important expressive, educational, and communicative message of punishment. . . . The sanction meted out to a convicted offender should reflect disapproval of a particular act and the act needs to be identified so that the disapproval is sufficiently concrete," as two law professors wrote in the *Minnesota Law Review* in 2009.

The discussion in legal circles is ongoing. While retributivists would support the aggregation of probabilities, expressivists are reluctant to accept it, except with very limiting restrictions.

MORE . . .

A similar question may arise in the following case. Two identical twins are caught at the scene of a gruesome crime: a woman was raped by one and then killed by the other. The sentence for rape is twenty years' imprisonment; for murder, it is death by execution. The prosecution does not know which twin did what.[1] Shouldn't both at least go to prison for twenty years?

1. The twins themselves are confronted with the "prisoner's dilemma" from game theory. By the way, lately it has become possible to distinguish between the DNA of identical twins.

50

CAN TWO WRONGS MAKE A RIGHT?

The Holistic Trial Paradox

Cuthbert stands accused of not one charge but two. A man was murdered with a knife on Tuesday, and a woman was raped on Wednesday. Fingerprints on the knife match Cuthbert's fingerprints with a probability of 96 percent. The DNA collected from the woman's body matches Cuthbert's DNA, also with a probability of 96 percent.

Ms. Rosewall, the prosecutor, has an open-and-shut case and offers the accused a deal: fifty years in prison without parole. Cuthbert, a hardened criminal, decides that he may as well take his chance with a jury. He pleads not guilty and demands to be tried by a jury.

Unfortunately, jury trials are expensive and to save the taxpayers' money, the exasperated Ms. Rosewall opts to pursue a holistic approach by trying the two cases together. After all, the two crimes are equally serious, and the accused is obviously guilty of both. "Let's not waste everybody's time with two trials," Ms. Rosewall declares. "A holistic trial will be more efficient and serve justice just as well." The judge agrees to try both crimes as one case.

The trial takes several days, and after final summations, the judge sends the jury off to decide on their verdict. His instructions are clear: "The suspect before you is accused of murder and rape. You must unanimously determine the defendant's guilt. If you are not at least 95 percent certain that the accused is guilty of the offenses, you must acquit him."

The prosecutor smirks with satisfaction: this will be a piece of cake! The results of the fingerprint test and the DNA test passed the 95 percent hurdle.

After several hours, the jury comes back with their verdict: innocent!

Has justice been done?

Yes!

Whoa! Cuthbert, by common sense and by any reasonable measure, both a murderer and a rapist, now goes free. What went wrong with the holistic approach to criminal trials?

Let's see what happened in the jury chamber. At first, the jurors were unanimous in their opinion. Cuthbert is a no-goodnik who belongs in jail for the rest of his life. But just before the jurors were about to vote, Mr. O'Connor spoke up. "I've been thinking," he said. "Maybe, just maybe, there is a chance that one of the situations is a case of mistaken identity. After all, 95 percent is not 100 percent, even if it happens twice." So, the jurors began discussing again. And the more they discussed, the less sure they became. In the end, they voted to acquit.

A miscarriage of justice? Maybe, maybe not. But certainly a paradox!

• • •

Probabilities have played an important role in criminal trials at least since the conviction and condemnation to death of Socrates in 399 BCE pronounced by a simple majority of the senators present. In the eighteenth century, the eminent mathematician and probabilist Pierre-Simon Laplace (1749–1828) thought deeply about probability in jury trials. If a majority consisting of only one more

Statue of Lady Justice on the Old Bailey, London.

Source: Lonpicman, https://commons.wikimedia.org/wiki/File:Artists-impressions-of-Lady-Justice,_(statue_on_the_Old_Bailey,_London).png.

than half the jury members is required to convict an accused, he explained, then the culpability of the accused is somewhat in doubt since the verdict could have been arrived at by coincidence.

On the other hand, demanding unanimity among jury members—as is the case for criminal trials in the United States today—also poses problems. Though it would all but guarantee that guilty verdicts were just, such a stringent requirement would often result in failures to convict. A single holdout could impose her will on all the others. Many truly guilty people would go free and remain menaces to society just because the jury could not bring itself to render a unanimous verdict.

Laplace recommended a compromise. If society wants guilty verdicts to be pronounced unanimously, a limit should be placed on the size of the jury. (After all, getting, say, thirty-one judges to agree on a guilty verdict would be difficult.) If, however, society prefers a large number of judges, the requirement of unanimity should be abandoned, but a majority larger than just one more than half the judges should be demanded in order to balance the presumption of innocence with the danger of letting criminals go free. Basing himself on probability calculations, Laplace suggested a majority of nine judges out of twelve to condemn an accused, instead of the five out of eight, as was customary at the time.

• • •

In modern times, the discussion has turned toward questions of *probability assessments* and how much faith to put into them, but also to questions of the *theory of probability*, for example, whether and how to combine probabilities in a trial's evidentiary stage, how to use probabilities in sentencing guidelines, how to assess the chance of recidivism, and, as discussed in this chapter (and in chapter 49), whether it is permissible or advisable to combine several crimes in a single trial.

DÉNOUEMENT

Mr. O'Connor had a valid point. If Ms. Rosewall had asked for a trial of the rape case, Cuthbert would have been convicted, no doubt about it. The fingerprints on the murder weapon matched his with a 96 percent probability, which was higher than the threshold specified by the judge. And if she had asked for a murder trial, Cuthbert would also have been convicted since the DNA matched his with a probability of 96 percent. He would probably have had to serve two life sentences.

But Ms. Rosewall made a fundamental mistake. She asked for the two crimes to be tried together as one case. This effectively raised the bar: by combining the cases, the prosecutor asked the jury whether Cuthbert was guilty of *both* crimes.

This is the crucial point. While Cuthbert was guilty with a probability of 96 percent of each crime, the probability of his being guilty of both crimes is computed as the product of the two probabilities: 96 percent times 96 percent. And that comes out to be only slightly more than 92 percent (0.96 × 0.96 = 0.9216)—lower than the bar set by the judge. Cuthbert was still guilty, of course, of the murder, and he was still guilty, of course, of the rape. But the jurors had not been asked, "Is Cuthbert guilty of the rape?" and "Is Cuthbert guilty of the murder?" They had been asked "Is Cuthbert guilty of the murder *and* the rape?"

The jurors had no other option than to acquit.

MORE . . .

This chapter is the counterpoint to the unspecified offense paradox (see chapter 49). In that chapter, a holistic trial would have put Ernie in jail because—by combining the two cases—the probability of his innocence would have dropped below 5 percent, even though each case by itself had a probability of guilt of less than 95 percent. In this chapter, the probability of Cuthbert's guilt in a holistic trial is less than 95 percent when the two cases are combined, even though each case by itself has a probability of guilt of more than 95 percent.

Had the judge instructed the jury "If you feel that there is even just a 1 percent chance that the accused is innocent of the offenses, you must acquit him," Cuthbert would have been put away because the probability that he was innocent of the murder *and* the rape was only 0.16 percent (4 percent multiplied by 4 percent). But that is not how justice works. Cuthbert did not need to prove his innocence because an accused is innocent until proven guilty.

XI

THE ECONOMICS OF THE UNEXPECTED

It Stacks Up . . . but Does It Balance?

Economists and accountants pride themselves on the idea that they go strictly by the numbers. But sometimes the numbers don't make sense . . . or don't tell the whole story.

51

SELL A LOT AND MAKE NO PROFITS

Bertrand's Economics Paradox

Two companies, Toastee and Presso, produce and sell coffee makers that also toast bagels. They are what is known in economics as a duopoly because they are the only two corporations in the world that produce that contraption.

The cost of producing and distributing each unit is $100. Toastee is the first to go to market. Pricing its device at $115, the corporation makes a nice $15 profit on each unit. Presso comes next and offers their product for $110. True, they make only $10 profit per unit, but wow, do they rip into Toastee's market share. Not to be outdone, Toastee adjusts its unit price to $105. Now Presso is faced with a dilemma: it can lower its price again, to $100, at which price it would capture the entire market. But profitability would be zero. And Toastee would have no choice but to draw level.

The logical end to this price war is that Toastee and Presso realize that they are no longer earning any money. So they might as well stop producing and go out of business.

Correct?

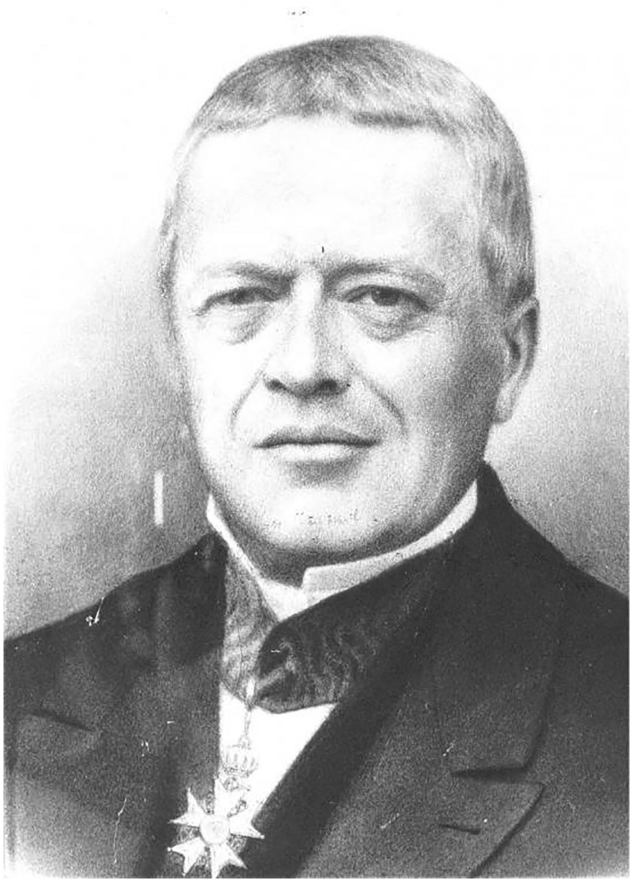
Well, that depends on whom you want to listen to. According to Joseph Bertrand (1822–1900), a French mathematician who was also interested in economics, the answer is yes. According to his

Joseph Bertrand.

Source: https://commons.wikimedia.org/wiki/File:Joseph_bertrand.jpg.

compatriot, the mathematician and philosopher Antoine Augustin Cournot (1801–1877), however, the answer is no. Who is correct?

We know, of course, that in most industries dominated by only two competitors (duopolies) or just a few competitors (oligopolies), the firms remain in business and are profitable. So, paradoxically,



Antoine Augustin Cournot.

Source: https://commons.wikimedia.org/wiki/File:Antoine_Augustin_Cournot.jpg.

Bertrand cannot have been quite correct, even though his analysis does not seem wrong. Cournot's model, which will be explained, corresponds more closely to reality.

In the early days of economic inquiry, some thinkers thought deeply about models of the economy. However, apart from arithmetic illustrations and examples, their work consisted mainly of words: they described their observations, recounted anecdotes, and explained their conclusions.

Thus, compared with, say, physics, medicine, or chemistry, economics was not considered a science until mathematics entered the scene. The discipline became a serious discipline only after mathematical models were developed that suggested how to optimize something, be it wealth, profits, or the utility for money. This occurred only in the late nineteenth century when neoclassical economists, Cournot and Bertrand among them, began to make use of mathematical methodology and tools. (This changed again in the late twentieth century with the emergence of behavioral economics.)

DÉNOUEMENT

Cournot had a different idea from Bertrand's about how duopolies and oligopolies operate. In Cournot's analysis, the companies compete on quantity, not price.

Both firms know the shape of the demand curve for a product (i.e., the lower the price, the higher the demand). Each firm contemplates what quantity the other firm might produce to maximize its profits and then decides for itself what quantity it should produce to maximize its own profits.

In our example, Presso knows that Toastee must make a decision about the quantity that it will produce. But Toastee could choose from a whole range of quantities. So, Presso determines—given each of Toastee's possible output levels—what quantities it should produce. Toastee, for its part, does the same. After some to and fro, the quantities of the two firms converge to a combined output level that determines the price that the firms can charge.

(According to Cournot, all this occurs simultaneously.) The resulting price will be somewhat lower than the exorbitant monopoly price but higher than the punishing perfect-market price. In this manner, both Toastee and Presso can be profitable and remain in business.

For a market dominated by a duopoly, Cournot's model is more realistic than Bertrand's. But there are other reasons that Bertrand's model does not describe reality. For example, both Toastee and Presso have only limited capacities for production. Toastee may be located on the West Coast and delivery to the East Coast may be prohibitively expensive. Presso may launch a branding campaign claiming that its product is superior to Toastee's.

MORE . . .

The most advantageous market for consumers is one in which there are many producers and perfect competition drives the price down nearly to the level of marginal cost. In general, the producers make a small but reasonable profit, and consumers are guaranteed nearly the lowest possible price.

On the other hand, in a monopoly, the sole producer is able to charge any price it wants. Its profits are enormous, and consumers pay through their noses.

A duopoly is somewhere in between: the quantity produced is greater than in a monopoly but lower than under perfect competition. And the price charged is lower than in a monopoly but higher than under perfect competition.

What if Toastee and Presso decide to play nice with each other, thus becoming a monopoly? Ah, that's illegal! To protect consumers from corporations that wield and abuse monopoly powers, monopolies and cartels are usually outlawed. The only exceptions are utility companies, like power companies or, in times long gone, telephone companies, who are sometimes allowed exceptionally high monopoly prices in lucrative locations in order to entice them to operate also in unprofitable geographic regions without forcing them to go bankrupt.

So, playing nice is against the law. What a paradox!

52

DOING MORE WITH LESS

Jevons's Paradox

Under the Obama administration, federal rules required automakers to improve the average fuel consumption of their new-car fleets to more than fifty miles per gallon. It would be win–win all around: total oil consumption would be reduced by billions of barrels, greenhouse gas emissions would go down by millions of tons, and American drivers would save thousands of dollars.

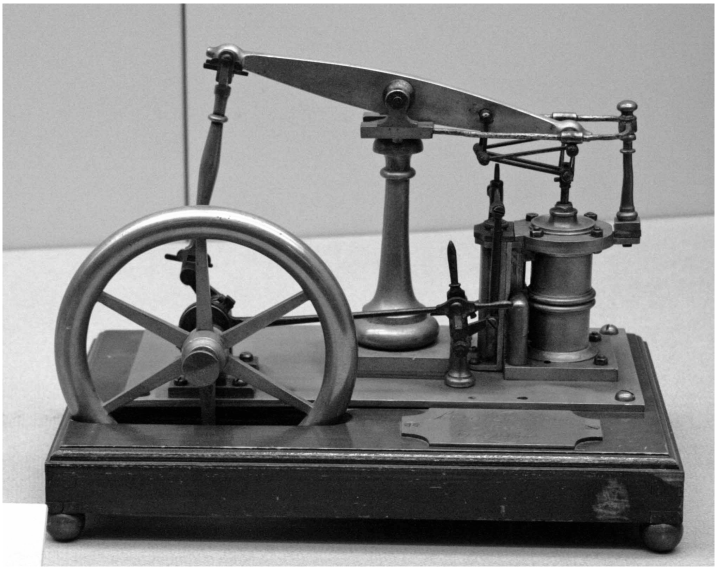
As usual, greater efficiency preserves scarce resources, protects the environment, and saves us money. Correct?

Surprisingly, no. At least, not always. It turns out that greater efficiency sometimes leads to more, not less, exploitation of resources.

In 1775, James Watt vastly improved the efficiency of the steam engines that had been devised six decades earlier by the English inventor Thomas Newcomen. Did England's consumption of coal, needed to create steam, decrease as a result?

No! To the contrary, it skyrocketed.

After the Arab oil boycott in 1973, more energy-efficient cars were created. Nevertheless, the consumption of motor gasoline in the United States has increased by 36 percent (from 2.5 billion barrels in 1973 to 3.4 billion in 2017).



A Watt steam engine.

Source: https://commons.wikimedia.org/wiki/File:JamesWattEngine.jpg.

Air travel has shown a similar trend. The per-seat fuel efficiency of jet airliners more than tripled between 1960 and 2016. But in spite of this tripling, seventeen times more fuel is being burned than previously.

As another example, take air conditioning. Between 1993 and 2005, the energy efficiency of residential air-conditioning equipment improved by 28 percent. But energy consumption for air conditioning by the average air-conditioned household rose by 37 percent.

The seeming paradox was first proposed in 1865 by the twenty-nine-year-old Englishman William Stanley Jevons. One of the first economists to use mathematical techniques, he published a book entitled *The Coal Question*, in which he studied the repercussions of

Britain's dependence on coal. Because the availability of this source of energy was limited, he discussed whether the country could sustain its economy beyond another century.

DÉNOUEMENT

It is in *The Coal Question* that Jevons formulated what would become known as Jevons's paradox. A key idea presented in the book is as follows: "It is wholly a confusion of ideas to suppose that the economical use of fuel is equivalent to a diminished consumption. The very contrary is the truth." Increases in the efficiency of energy production may lead to more, not less, consumption.

The prime example of this phenomenon is, as suggested earlier, James Watt's steam engine. The increased efficiency of his engine brought about the Industrial Revolution, which led to a surge in the use of coal. The more economical the use of coal in engines became, the more the overall consumption of not only coal but also iron and other resources increased.

It was, and is, a classic catch-22 situation. Increasing the productivity of any resource is tantamount to reducing the cost of using it. Reducing the cost means that demand goes up. Demand goes up, and resource use increases. This is why seventeen times more fuel is being burned for air travel nowadays than previously. Since fuel efficiency increased threefold, air travel became cheaper, and global air travel increased fifty-fold.[1]

1. Global air travel is measured in passenger-kilometers. The increase resulted not only from greater fuel efficiency and hence lower airfares but also from increasing incomes and growing populations. A fifty-fold increase in travel, offset by a three-fold increase in efficiency, means that about seventeen times as much fuel is used.

Air conditioners, successively cheaper to operate and hence ever more abundant, remain in continuous use, rather than just during heat spells. And modern LED light bulbs, vastly more efficient than the successors of Thomas Alva Edison's incandescent bulbs, are often left on around the clock.

MORE . . .

The gist of the story is that in spite of improved efficiency, greenhouse effects and environmental pollution may increase. The irony is that this occurs not in spite of increased efficiency but because of it!

Thus, the sad news is that efforts to improve efficiency by well-meaning environmentalists may boomerang. Environmental gains resulting from increased efficiency may be offset by the loss of scarce resources and increased pollution owing to increased demand. Economists call this the "rebound effect." Only a fraction of the efficiencies results in salvaged resources because the population uses more of the cheaper resource. For example, instead of using public transportation, many people may decide that the cost of gasoline is so low that they may as well drive their car to work.

Indirect rebound effects occur when some energy savings in one sector (for example, less expenditure on gasoline for cars) leads to increased outlays for energy elsewhere (for example, more electricity for air conditioning). On a macroeconomic level, more efficient (and hence cheaper) energy leads to faster economic growth—a good thing—which, in turn, increases energy use throughout the economy—not necessarily a good thing.

The situation can be even worse. When gains in energy efficiency lead to greater total energy use than was the case previously, the rebound effect may become what environmental economists call the "backfire effect."

• • •

Since gains in efficiency, according to Jevons, may be counterproductive, government intervention may be required to limit air pollution, the greenhouse effect, and the depletion of scarce resources. Hence, say environmentalists, efficiency gains should be combined with policies like quotas, rationing, taxation, environmental standards, and laws.

53

OPTIMAL LIBERALISM

Sen's Paradox

In the Western world, most societies prefer liberalism to socialism or communism. Everybody should be allowed to choose for himself or herself which of several options he or she prefers. And if all members of a community have made their optimal choices in a manner such that nobody can be made better off without hurting somebody else—this situation is called "Pareto optimality"—then we have the best of all worlds.

Correct?

Yes, but unfortunately, it's not always possible for a society to be both liberal and Pareto optimal.

Let's consider the following scenario: a mother prepared a bowl of spinach that she wants her son to eat; otherwise, it will go bad and have to be thrown out. Hence, there are three options: the son eats the spinach (S), the mother eats it (M), or it is thrown in the garbage (G). The mother's preferences are as follows:

> Mom: son eats spinach > mom eats spinach > spinach goes in garbage ($S > M > G$).

The son hates spinach but wants to please his mother. Hence, his preferences are as follows:

> Son: spinach goes in garbage > son eats spinach > mom eats spinach ($G > S > M$).

The mother and son belong to a liberal family. Neither will force the other to eat the spinach. Each can decide only for himself and herself whether to eat the bowl of spinach or throw it out. Since Mom cannot *make* her son eat the spinach, and given the options at her disposal, she prefers to eat the spinach herself rather than throw it out ($M > G$). The son, given the options at his disposal, prefers to throw out the spinach rather than eat it himself ($G > S$). Combining their preferences, we have $M > G > S$. So, Mom eats the bowl of spinach.

But hey, wait! Both of them, according to their own inclinations, indicated that they would prefer the son to eat the spinach ($S > M$). If he had eaten the spinach, both would have been happier.

So, we have a problem: liberal mom and liberal son announced their preferred choices from among the alternatives available to them. Nevertheless, the choices did not lead to Pareto optimality. A different choice would have benefited both. (See also Condorcet Cycles in chapter 56.)

A paradox!

The paradox was devised by Amartya Sen, an Indian economist awarded the Nobel Prize in Economic Sciences in 1998 for his work on welfare economics, social choice theory, and social justice.

It was in 1970, in an article in the *Journal of Political Economy* entitled "The Impossibility of a Paretian Liberal," that he showed that liberalism—if an individual prefers one option over another, then society should let him choose it—and Pareto optimality—if everybody

Amartya Sen.

Source: https://commons.wikimedia.org/wiki/File:Amartya_Sen_no_Fronteiras_do_Pensamento_S%C3%A3o_Paulo_2012_(7110299309).jpg.

prefers *a* to *b*, then society as a whole should choose *a* over *b*—are incompatible. Liberalism and Pareto optimality are contradictory.

As an example, Sen imagines two people, a prude and a lewd, who are considering reading an erotic novel. (The novel Sen used in his example, D. H. Lawrence's *Lady Chatterley's Lover*, would not be considered explicit nowadays but was half a century ago.) I adapted Sen's example to spinach eating.

DÉNOUEMENT

There's no dénouement. Sen proved mathematically that, given certain preferences of the individuals, liberal values conflict with the Pareto principle. They may be incompatible.

What is the moral of the story? Sen's answer is nothing if not depressing: "If someone takes the Pareto principle seriously . . . then he has to face problems of consistency in cherishing liberal values." Or, "if someone does have certain liberal values, then he may have to eschew his adherence to Pareto optimality."

There is one way out, Sen claims: "The ultimate guarantee for individual liberty may rest not on rules for social choice but on developing individual values that respect each other's personal choices." In other words, one can be a liberal as long as one's values conform to the community's laws.

Hmm, that seems to be something of a contradiction . . .

MORE . . .

One criticism of Sen's paper is that it ignores the intensities of each individual's preferences. This is related to the fact that it is impossible to compare preferences between individuals. In our example, the son may loathe spinach, and the mother may not mind throwing it into the garbage. Taking this into account, the choices of mother and son may turn out differently.

The implication is that if the intensities of preferences are considered, the social choices of a community should be affected accordingly. For example, if the owner of a house really, really wants to paint her front door yellow, she should be allowed to do so, even if the neighbors don't like that color.

Thus, whenever a choice is likely to affect one person profoundly but is very unlikely to significantly affect anybody else, society should agree that this choice should be left entirely to that person.

• • •

Sen's paradox uses similar mathematical techniques and is related to Arrow's impossibility theorem, which states that no rank-order electoral system can be designed that always satisfies some reasonable axioms of fairness.[1]

1. However, Sen's paradox does not depend on the so-called axiom of the Independence of Irrelevant Alternatives (IIA), as does Arrow's theorem. Hence, relaxing IIA does nothing to escape Sen's paradox. See also my book *Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present* (Princeton, NJ: Princeton University Press, 2010).

54

PRIVATE VICES, PUBLICK BENEFITS

Mandeville's Paradox

Virtues are good; vices are bad.

Solidarity is good; self-interest is bad.

Honesty is good; fraud is bad.

Generosity is good; greed is bad.

Yes and yes. Yes and yes. Yes and yes. Yes and . . . no.

Oh, really?

A key scene in the 1987 movie *Wall Street* shows Gordon Gekko, the movie's villain (played by Michael Douglas), pronouncing at the annual general meeting of Teldar Paper: "Greed, for lack of a better word, is good." He implores the shareholders, "Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit. Greed, in all of its forms, greed for life, for money, for love, knowledge, has marked the upward surge in mankind, and greed, you mark my words, will not only save Teldar Paper but that other malfunctioning corporation called the USA. Thank you very much."

So, here we have it: greed, a vice by any reasonable standard, is not bad; to the contrary, it is good. And so are the other vices. And the virtues mentioned earlier may be bad. A paradox!

Gordon Gekko was preceded nearly three centuries by the Dutch medical doctor and social philosopher Bernard Mandeville (1670–1733) who published an anonymous poem in England in 1705 entitled

"The Grumbling Hive: or, Knaves Turn'd Honest." The fable, which in 1714 became part of the book *The Fable of the Bees or Private Vices, Publick Benefits*, recounts the story of a hive and its inhabitants, the bees. Guided solely by self-interest, lust, avarice, and vanity, they cheat, steal, bribe, and murder. "Thus every part was full of vice, yet," Mandeville wrote, surprisingly, "the whole mass a paradise." It was precisely the bees' moral failings that made their hive superior to those of others: "Their crimes conspir'd to make them great. . . . The worst of all the multitude / did something for the common good." It was their cunning and dishonesty, their willingness to cheat, swindle, and deceive, that provided them with an advantage over other hives.

But one day, the rogues had enough. "Good gods, had we but honesty!" they cried, deciding on the spot to become virtuous: As "honesty fills all their hearts," debtors paid their bills, lawyers became honest, doctors stopped being quacks, and the clergy were roused from laziness.

And that was the beginning of the end. As soon as the hive's inhabitants had to live only off their salaries and repay their debts, they were forced to pawn their goods and sell their coaches, horses, and stately country houses. Extravagant palaces, fancy clothes, and other vain expenses were henceforth shunned. As a result, the building trade was destroyed, craftspeople lost their jobs, and inns and taverns closed. "As pride and luxury decrease. . . . All arts and crafts neglected lie."

With the crash of the economy and social structures, the hive's population is decimated. Survivors fly into a hollow tree, and the once formidable hive is reduced to an insignificant entity. The moral of the story? "Fools only strive to make a great an' honest hive. . . . Fraud, luxury, and pride must live; whilst we the benefits receive."

At first, the doggerel was published anonymously, and a good thing that was because the praise for vices and the rebuke of morals kicked off a controversy that lasted for more than two centuries and influenced even modern-day economists.

DÉNOUEMENT

At the time, the central tenet of the poem, that private vices create social benefits, was abhorrent to the general readership; it went against all Christian teachings. But, of course, there is something to the principle that was expressed in the poem. The desire to improve one's economic situation, to enjoy comforts, and even to indulge in luxury stimulates trade and industry. Striving to move upward on the social ladder increases productivity.

So, are pride and vanity, vices by any definition, boons to the economy and of public benefit? Without pride and vanity, people would not be motivated to buy new clothes, expensive cars, prime real estate, and overpriced pieces of art. Without indulgence, there would be little consumer spending. As a result, companies would go bankrupt, unemployment would increase, industries would collapse, and economies would be devastated. To drive home his point, Mandeville maintained that even thieves, burglars, and sundry evildoers spur on the economy. Without them, there would be no locksmiths, no police officers, no lawyers. In remarks that he later amended to the poem, he also advocated for brothels . . . you begin to get the point.

Mandeville's ideas influenced the Scottish thinker Adam Smith (1723–1790), generally considered the father of modern economics. He believed that economies are guided by an invisible hand: in a society in which everybody acts according to his or her self-interest, production and consumption are allocated in the best possible manner, as if guided by the proverbial invisible hand. In less evocative terms, this situation is described as a market-based system of free resource allocation. Prices, wages, and costs are not determined by some controlling authority but approach equilibrium on their own, purely through the workings of individuals' self-interest. Without self-interest, the invisible hand disappears.[1]

MORE . . .

In the twentieth century, the fable of the bees found an expression in the laissez-faire economics of Friedrich Hayek (1899–1992), the 1974 recipient of the Nobel Prize in Economic Sciences, and Milton Friedman (1912–2006), the winner of the same award in 1976, among many others.

The principle of laissez-faire economics is that the less the government interferes in free markets and the more individual entities (people but also corporations) are left to look out for their self-interest, the better the economy functions. Competition among economic entities with very little regulation will allow the markets to self-regulate.

Unfortunately, unrestrained laissez-faire economics more often than not leads to excesses and long-term damage. A century and a half after the publication of "The Grumbling Hive," Karl Marx saw Mandeville's observations as providing important insights into the nature of an emerging capitalist system. "Only Mandeville was of course infinitely bolder and more honest than the philistine apologists of bourgeois society," he wrote.

Except for die-hard free-market apologists, the consensus among today's economists is that the invisible hand must be paired with a visible hand that balances private initiative and public intervention to limit collateral damage.

1. The counterpart to the invisible hand is a centrally controlled economy, that is, communism, which is notoriously inefficient.

55

TIGHTENING ONE'S BELT

The Paradox of Thrift

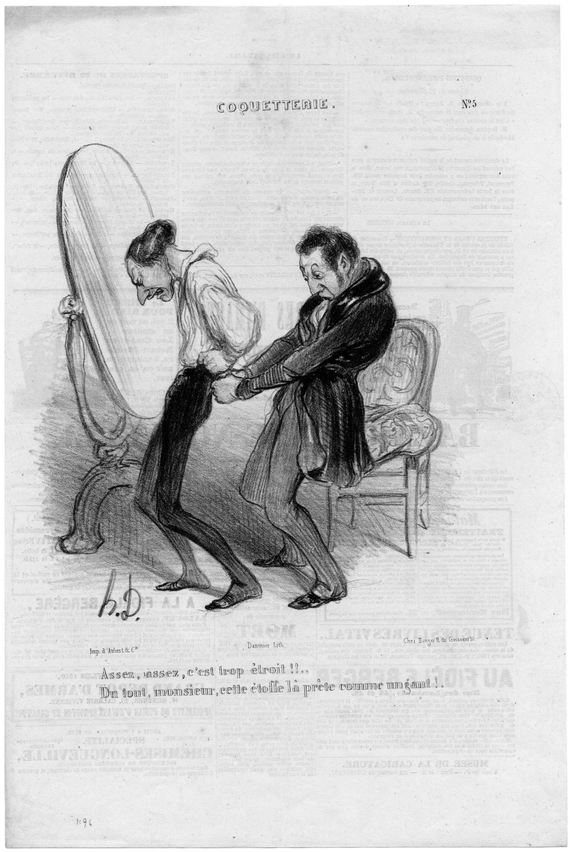
Economically, the prudent way to behave is to look ahead and make provisions for the future. Saving for a rainy day is especially commendable. And when the rainy day comes and the financial situation is dire, it is time to tighten one's belt and reduce expenditures.

Correct?

Yes, if you're an individual. No, if you're the government.

The financial crisis of 2007 and 2008 in the United States, caused by the collapse of the real estate market, excessive granting of subprime mortgages, unsafe lending practices by banks, imprudent risk-taking by corporations, and failures of regulators, led to an economic decline in markets worldwide and a recession considered the worst economic crisis since the Great Depression of the 1930s.

Conventional wisdom might have suggested that then was the time to tighten the belt. But instead of doing that, which a "prudent" decision-maker would have done, President Barack Obama signed a stimulus package into law, the cost of which was estimated at about $800 billion. It encouraged extravagant expenditures across the board. Tax incentives (nearly $300 billion) were granted to embolden individuals and corporations to spend more. Additional outlays were approved for health care ($155 billion); education ($100 billion); aid to the unemployed, retirees, and individuals



"Enough, enough, it's too tight!" (1839). Comic by Honoré Daumier.

Source: https://commons.wikimedia.org/wiki/File:Assez,_assez,_c%27est_trop_%C3%A9troit!_(Enough,_enough,_it%27s_too_tight!)_(BM_1910,0324.67).jpg.

living on low incomes ($82 billion); infrastructure ($105 billion); renewable energy ($27 billion); and housing ($15 billion).

Was that prudent? This was the worst economic crisis that had hit the United States in eight decades, and, instead of tightening its belt, the government decided to *spend* billions of dollars. It seemed truly mind-boggling.

Thriftiness, often considered a core Christian virtue, made an appearance as a paradox in the masterful work *The General Theory of Employment, Interest and Money* by John Maynard Keynes (1883–1946), one of the most influential economists of the twentieth century. His thesis, which is quintessential to what was henceforth called Keynesian economics and which starkly contradicted received wisdom, stated that in times of economic downturns, government should not engage in belt-tightening. To the contrary, they should encourage the population to spend their money and increase government outlays. His reasoning for this counterintuitive, seemingly paradoxical recommendation will be described in the dénouement.

• • •

Keynes was not the first to propose this theory. Keynes himself cited the book *The Fable of the Bees or Private Vices, Publick Benefits* by the political scientist and author Bernard Mandeville (see chapter 54). Mandeville claimed that vices such as vanity and greed are beneficial to the public as a whole. As was to be expected, Mandeville's provocative book created quite a scandal in its time.

But Mandeville was not the first to propose this opinion either. In fact, a similar idea had been pronounced in the Bible, in Proverbs 11:24: "One person gives freely, yet gains even more; another withholds more than is right, but comes to poverty."

DÉNOUEMENT

What could be more reasonable and virtuous than saving money, both by people and by governments? Whereas the Bible speaks about the behavior of individuals, Mandeville and Keynes proposed their theory on a macro scale: a population's thriftiness is counterproductive to its well-being, especially in tight periods. How can that be? How can the most virtuous behavior—spending little and saving money—be bad?

According to Keynes, consumption drives economic growth. Saving, on the other hand, reduces the demand for goods and services. Money is withdrawn from circulation, and, with demand decreasing, businesses produce less, thus impeding economic growth. If, for example, people save money instead of going to restaurants, there will be less demand for restaurants, and the restaurant business will suffer. So will the businesses that supply restaurants. Waiters, chefs, and other staff are laid off, and unemployment goes up. The negative effects are felt especially strongly during a recession since they prolong the downturn. It's a vicious cycle.

Thus, if people save less and spend more, businesses reap additional profits, which they then reinvest to expand their operations. This, in turn, requires hiring more workers, who in turn drive up demand. Unemployment decreases.

Of course, it is not only upon the people to do their part by spending more but also upon the government to do its part. Lowering interest rates and increasing government spending are the tools at the administration's disposal. The former discourages saving and encourages investment. The latter increases employment. It's win–win–win all around—at least according to Keynes and his followers. But unfortunately, things are not as unambiguously rosy as Keynesians pretend.

MORE . . .

We saw that by discouraging savings, people will spend more, and the demand for goods and services will increase. This will encourage corporations to invest in additional means of production, which allows the economy to grow. So far, so good.

But to invest, companies must take out loans. And for banks to be able to grant such loans, they must have the funds. And where do these funds come from? Well, from the people's savings . . . which we just discouraged. It is a paradox within a paradox.

Economics is like a complex machine with many moving parts. It is rarely entirely clear what will happen as a consequence of certain government policies or why. For example, increased spending, a good thing according to Keynesians, may lead to excess demand that cannot be met by current supply, thus causing prices to rise. The unwanted consequence is inflation.

On the other hand, increased saving, a not-so-good thing according to Keynesians, may provide banks with the necessary funds to loan money to businesses for increased investments. And although the revenues of corporations will decline because of increased savings and less spending, proponents of the so-called Austrian school of economics[1] argue that the additional savings will be invested in things like new plants and machinery, which will increase production and lead to a growing economy. So, the opposite of what the Keynesians propose would lead to the same desired outcome.

Hence, the circle need not be vicious. If demand decreases because people prefer to save rather than to spend, prices will fall, which will encourage people to spend more. Thus, decreased demand will be followed, after a while, by increased demand, thus letting the economy grow. (What will suffer are the profit margins of the producers and the wages of laborers.) So, we could end up with a virtuous circle.

1. This school of thought holds that individuals and their personal preferences determine economic processes.

XII

Puzzling Politics

The Vexing Mathematics of Democracy

Some of the very foundations of democracy—the majority vote, for example—are founts of paradoxes. This is all the more worrying since they may have important policy implications.

56

WHO SHOULD WIN?

Condorcet Cycles

In most modern Western countries, citizens are justly proud of their democracy. All have their say in who will lead and administer their country. Every few years, nationwide elections are held—one person, one vote!—and the candidate who obtains the most votes is elected. Similarly, in board meetings, on committees, in businesses, in schools, and among friends, we simply take a vote, and the majority decides. It is the fairest way.

Is it really? Does our beloved majority rule really reflect the true will of the voters?

Let's see for ourselves. Peter, Paul, and Mary must decide what to order for their after-dinner drinks. Peter prefers amaretto to grappa and grappa to limoncello. Paul prefers grappa to limoncello and limoncello to amaretto. Mary prefers limoncello to amaretto and amaretto to grappa. With ">" indicating "preferred to," we can express the situation thus:

Peter:	amaretto > grappa > limoncello
Paul:	grappa > limoncello > amaretto
Mary:	limoncello > amaretto > grappa

True democrats, the three diners decide to go by the majority opinion. A majority prefers grappa to limoncello (Peter and Paul), and a majority prefers amaretto to grappa (Peter and Mary). Based on these two rounds, they can make their decision: order a bottle of amaretto.

But surprise, surprise: Paul and Mary point out that they would prefer even limoncello, the lowest-ranked option, over amaretto. How come? Here is the clincher: had the three campers had a third round of voting, between limoncello and amaretto, a majority would have preferred limoncello (Paul and Mary). So let them order limoncello and get it over with. But wait a minute. Order limoncello, and Peter and Paul will protest just as vigorously. They prefer grappa to limoncello. A paradox.

It was the eighteenth-century French nobleman Marquis de Condorcet (1743–1794) who first identified the problem. One of the foremost intellectuals before and during the French Revolution, the marquis was a mathematician, economist, political scientist, and defender of human rights.

Condorcet wrote important works that combined mathematics and social issues. Some of his most intriguing texts were his contributions to the theory of voting and elections. In 1785, he wrote a two-hundred-page pamphlet entitled *Essay on the Application of Probability Analysis to Majority Decisions* in which he described the paradox.

One way to overcome it was suggested by Condorcet's contemporary and compatriot, Jean-Charles de Borda (1733–1799). Also a nobleman and a scientist of note, Borda found his calling not in politics but in the military and distinguished himself in the Navy as a formidable maritime engineer.

Borda's scholarly achievements include important advances in experimental physics and engineering, geodesy, cartography, and other areas. He also showed interest in the subject of voting and elections and wrote an article in 1781 entitled "Essay on Ballot Elections."

DÉNOUEMENT

To understand the mathematical reasoning of the paradox, let's compare majority rule with weights and measures. If Marc is taller than Nancy, who is taller than Oscar, then Marc is definitely taller than Oscar. In mathematical lingo, the measurements of the people's heights are transitive. But in the context of elections, the statement "majorities prefer A to B and B to C" does not imply that "the majority prefers A to C." In other words, the Condorcet paradox arises because majority opinions are not transitive.

• • •

What did Borda suggest to overcome the paradox? In his proposal, voters rank the candidates according to their tastes and then award the candidates points according to the ranks they gave them. The candidate at the bottom is given one point, the next-lowest two, the next three, and so on, until the top-ranked candidate is awarded the most points by each voter. Then, the total number of points are added up for each candidate, and the one with the most points is elected. With many voters awarding their points, it is rare for two or more candidates to obtain the same number. Hence, there is usually a well-defined "Borda winner," which is why Borda counts are often used in games and TV shows.

MORE . . .

Borda's proposal has problems of its own, however. First of all, the winner of the Borda count may be nobody's favorite. Let's say that thirty students are electing their class president. Their preferences are as follows:

Eleven students:	Paul > Mary > John > Peter
Ten students:	Peter > Mary > John > Paul
Nine students:	John > Mary > Peter > Paul

In the traditional voting method, Paul would be elected—though only by what is called a plurality (eleven votes), not a majority, which would have required sixteen votes. But in a Borda count, Paul would get sixty-three points, Peter sixty-nine, and John seventy-eight. Mary, whom nobody really likes, would win with ninety points.

Then, there is the problem that the winner of a Borda count depends crucially on the number of points awarded at each rank. After all, there is no intrinsic reason that each subsequent rank should be rewarded with exactly one additional point. The Borda winner would vary depending on the system used. In the example of the class president election, for example, if ten points are awarded to the top-ranked candidate, six to the second, five to the third, and zero to the last, John would be the winner with195 points and Mary the runner-up with 180. Or why not give each voter a certain number of points, say one hundred, which they can then allocate in any manner according to the intensity of their feelings for the candidates?

Then, there is the Bozo problem. By entering the race, an irrelevant candidate could change the outcome. Even though he would be ranked low on every voter's list, his addition may influence the election's outcome. For example, let's assume that fifty-one electors prefer Ginger to Fred and that forty-nine prefer Fred to Ginger:

Fifty-one electors:	Ginger > Fred
Forty-nine electors:	Fred > Ginger

With 151 points and Fred with 149, the Borda count declares Ginger the winner. But now Bozo appears on the scene. Nobody really likes Bozo, but his entry persuaded three of Fred's voters to rank Ginger even behind Bozo:

Fifty-one electors:	Ginger > Fred > Bozo
Forty-six electors:	Fred > Ginger > Bozo
Three electors:	Fred > Bozo > Ginger

This time, Ginger receives 248 points, Fred 249, and Bozo 102. Bozo's entry caused Fred to win. (The Bozo problem is related to Kenneth Arrow's notorious axiom of the independence of irrelevant alternatives. In our example, Bozo was an irrelevant alternative.)

• • •

Condorcet proposed his own solution. It was as simple as it was impractical. Every candidate is paired against every other candidate in a series of showdowns. In each round, the voters express their preferences, and the candidate who receives the majority of the votes is considered to be superior to the others. Once all pairings have been performed, the candidates are ranked. The candidate who comes out on top, who beat all the others, is declared the winner.

But things are not quite so simple. First of all, mathematical combinatorics implies that Condorcet's proposal with N candidates would require $N(N - 1)/2$ showdowns. For twenty candidates, that would mean 190 pairings! Second, in general, no unambiguous ranking can be drawn up because the results of the pairings are, again, not transitive. Cycles appear, the very cycles that led to the amaretto–grappa–limoncello paradox in the first place.

57

MORE SEATS OR FEWER?

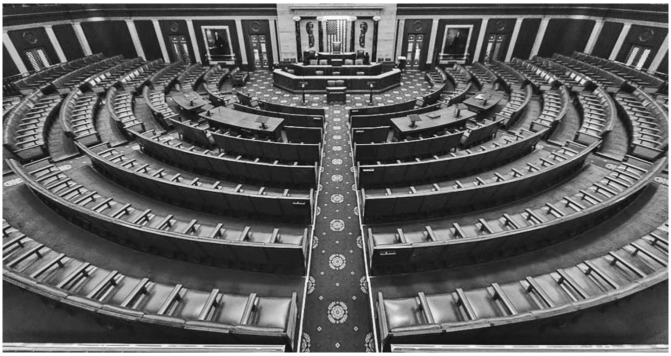
The Alabama Paradox

Seats in a parliament are allocated according to the size of the voting districts' populations. Unfortunately, the proportions are nearly never integer numbers. So, how many seats in a one-hundred-seat parliament should districts get if their populations are, say, 10.2 percent and 16.8 percent of the nation's population? Should the number of seats be rounded to ten and seventeen?

Unfortunately, rounding to the nearest integer won't do because that usually ends up with one seat short or long. In the U.S. House of Representatives, rounding to the closest integer would most probably result in 433, 434, 436, or 437 seats instead of the desired 435.

In 1850, to avoid squabbles, Senator Samuel Vinton from Ohio, relying on a previous suggestion by Alexander Hamilton, the first secretary of the treasury, proposed a simple method. First, an appropriate divisor is sought, such that when the states' populations are divided by this divisor and all results are rounded down, only a few seats are left over. Next, the leftover seats are allocated to the states with the highest leftover fractions. So far, so good.

Now, to accommodate the nation's ever-growing population, the size of the legislature needs to be increased from time to time. After the U.S. census of 1880, for example, Congress considered an increase in the size of the House of Representatives from what was



The floor of the U.S. House of Representatives.

Source: https://commons.wikimedia.org/wiki/File:United_States_House_of_Representatives_chamber.jpg.

then 293 seats. Obviously, whenever the House is enlarged by a seat, one lucky state will gain an additional representative.

Obviously?

• • •

For decades, after every census, there followed squabbling and wrangling about the apportionment of seats in the House. To give the congressmen the necessary ammunition for the infighting that would undoubtedly precede the 1880 apportionment of the House, C. W. Seaton, the chief clerk of the Census Office, did some computations. Using the census results, he did the long divisions to work out the apportionments according to the Hamilton–Vinton method for all House sizes between 275 and 350 seats. Starting with 275 representatives, everything worked out just fine all the way up to 299. Whenever he increased the House by one, the additional seat was picked up by some lucky state.

Let's look at the actual numbers of the 1880 census. The total population of the United States was 49,713,370. If the House had

299 seats, the appropriate divisor would be 165,120, and the number of representatives for the states of Alabama, Texas, and Illinois would be as shown in the following table:

Allocation of seats to Alabama, Texas, and Illinois with 299 seats in the House

	Alabama	Texas	Illinois
Population	1,262,505	1,591,749	3,077,871
"Raw" allocation	7.646	9.640	18.640
Seats in first round	7	9	18
Fractional part	0.646	0.640	0.640
Additional seats	1	0	0
Total seats	8	9	18

But when the chief clerk of the Census Office reached 300 seats, a bombshell exploded. If the size of the House were increased by one and 300 seats were to be allocated, the appropriate divisor would be 164,580, and the calculations would be as shown in the following table:

Allocation of seats to Alabama, Texas, and Illinois with 300 seats in the House

	Alabama	Texas	Illinois
Population	1,262,505	1,591,749	3,077,871
"Raw" allocation	7.671	9.672	18.701
Seats in first round	7	9	18
Fractional part	0.671	0.672	0.701
Additional seats	0	1	1
Total seats	7	10	19

Lo and behold! The delegation of Alabama was *decreased* by one representative, from eight to seven. In its stead, *two* states, Illinois and Texas, each gained one additional seat. A paradox!

Congress went into a tizzy. The Hamilton–Vinton method of apportionment, which had by now become dear to most of them, was in danger. A short-lived method proposed by Senator Daniel Webster in 1842—finding a divisor such that the results, when rounded *up or down* to the nearest integer, would give the desired number of seats—was raised again. Tempers ran high. One congressman accused another of "committing a classic rape on a cloud of statistics, right in the face of the House."

DÉNOUEMENT

The reason for the Alabama paradox becomes apparent when we delve a little deeper into the numbers. When the total number of seats increases from 299 to 300, the states' "raw" numbers of seats grows on average by about one-third of 1 percent. But Texas and Illinois start out with larger populations and therefore gain more in absolute numbers. Thus, the number of "raw" seats grows by only 0.025 in Alabama (from 7.646 to 7.671), by 0.032 in Texas (from 9.640 to 9.672), and by 0.061 in Illinois (from 18.640 to 18.701). As a consequence, the larger states creep past Alabama.

The problem is created by the increase in the size of the House. The obvious solution is to keep the size of the House unchanged, and since 1911 the number of seats has remained fixed at 435, thus avoiding the Alabama paradox. To allocate seats, the so-called Huntington–Hill method is used nowadays which calculates an initial divisor and adjusts it iteratively to ensure a fair allocation.

MORE . . .

So, what happened in 1880?

To avoid the contest between the proponents of the two methods from turning even uglier, Congress decided not to decide and instead resolved to enlarge the House to 325 seats. With this size, the congressmen did not have to take sides because the Webster and the Hamilton–Vinton methods agreed, and the problem could be postponed for at least another ten years. Maybe a wholly different apportionment method would be found in the meantime? Or the methods would again agree? Or the congressmen would no longer be in Congress and could let their successors worry about the Alabama paradox.

They were right. All it took in 1890 was an increase to 356 seats and the same compromise could be forged. With a House of that size, both methods agreed, and no state would lose a seat as compared to the previous apportionment.

Ten years later, no such luck. When tables on the apportionment were prepared in 1901 for sizes of the House between 350 and 400, Maine's apportionment oscillated between three and four seats, and Colorado would receive three seats for every size of the House except 357, at which point it would be allocated just two. Which number do you think the chairman of the Select Committee on the Twelfth Census, no friend of Colorado's, chose? Yes, of all the numbers he could have chosen, he suggested fixing the size of the House at 357. Tempers rose, and the atmosphere again became ugly. This time, Congress did take a stand, and the Hamilton–Vinton method was abandoned for the 1900 census data in favor of Webster's, which at least did not suffer from the defect of the Alabama paradox. In addition, the House was enlarged to 386 seats, which ensured that no state would lose a seat.

58

TO ABSTAIN FROM ELECTIONS

The Nonvoting Paradox

There's a hotly contested election on for class president. You don't like any of the candidates, so you decide not to participate in the election. By not casting a ballot, you remain out of the fray.

Correct?

No!

Declarations like "I'm not participating in the election" and "I'm not voting for anybody" are logical contradictions. By refraining from voting, you *do participate* in the election, and you *are casting* a ballot, though virtually, for one of the candidates. To state that you are not contradicts what you are doing.

BACKGROUND

It is a puzzle why people bother to vote. After all, it is a hassle to go to the polling station, stand in line, show ID, and so on. Given that your ballot, one of very many that are cast, will not make any difference to the outcome, it may just be an enormous waste of time. So, should one bother to vote?

An "I voted" sticker.

Source: Dwight Burdette, https://commons.wikimedia.org/wiki/File:I_Voted_Sticker.JPG.

There are arguments both pro and con. On the one hand, for a democracy to work, citizens have certain civic duties like payment of taxes, compulsory education, jury duty, military service, and, of course, participation in elections. To formulate policy, the government must consider the will of the entire electorate. But only if everyone participates in elections, plebiscites, surveys, and the census are the desires and aspirations of all people reflected in the results. Whoever abstains from voting leaves it up to those who do vote to decide which candidate is elected or whether a proposal is accepted or rejected. Some countries have laws that make voting mandatory.

On the other hand, many people consider voting a right, not a duty. They claim that it is undemocratic and an infringement of liberty to force people to vote. Just as one has the right to free speech, one has the right to withhold one's vote. Further, if people are forced to participate in an election or plebiscite, they may opt for a random candidate or a random issue just to get it over with. Finally, those who are illiterate or ignorant, if forced, indeed, if allowed to vote, may be misled by their spiritual leaders, teachers, or spouses. (After all, this is why the mentally impaired are usually prevented from voting even in countries where it is, in principle, mandatory.)

While mandatory voting may or may not be a good idea, suppressing people's ability to participate in elections or plebiscites, for example, by making it difficult to register as a voter or by unreasonably forcing people to prove their eligibility, is definitely a bad idea.

Now we know why one should participate or want to ignore elections. So, why does a statement like "I don't participate in elections" present a paradox?

DÉNOUEMENT

Among the entire voting population, your individual vote may not make a difference. True, you're only one person. But so is everybody else who votes, and if many people think like that, it may influence the outcome.

Obviously, in a close election—and nobody knows beforehand whether an election will be a landslide or a cliff-hanger—every vote counts. In the 2000 presidential election, for example, Al Gore won the popular vote with 50,999,897 votes cast, and George W. Bush lost it with 50,456,002. The difference was only about half a percent, but, in any case, it is the Electoral College that decides the presidency, and, crucially, in Florida, which awarded the decisive twenty-five votes in the Electoral College, 2,912,790 people voted for Bush, while 2,912,253 voted for Gore. A mere 538 nonvoters could have changed history! By staying away, they may have handed the victory to George W. Bush.

• • •

Let's take another example. Let's say there are eighteen students in your class at high school. Marty, the current class president, is disliked by many of his classmates; they want to impeach him. Odelia challenges Marty for the post, and an election is called. The rules of an out-of-turn election specify that Odelia must attain a two-thirds majority to become the new class president. If she does not, Marty remains in his post.

You don't care at all who the class president is and therefore decide to abstain. The election proceeds. Eleven people vote for Odelia and six for Marty. Since Odelia did not gain the necessary two-thirds majority (she captured only eleven of the seventeen votes), Marty remains president. Had you cast your vote for Odelia, she would have obtained the required two-thirds majority (twelve out of eighteen) and replaced Marty. By not participating in the election, you, in fact, voted for Marty. Maybe that's OK since you did not care either way. The point is, however, that you did influence the outcome.

MORE . . .

Let's say that you don't care about the candidate or the issue at hand, but you do care about the democratic process. Is the answer maybe to cast a blank ballot?

It may not be the answer to your dilemma because you may still influence the outcome. A blank ballot may make a difference in the outcome of an election or plebiscite if the blanks are counted as valid votes. For example, a decision may require an absolute majority, that is, more than 50 percent of *all* votes cast or of all *valid* votes cast. Let's say that 1,000 votes are cast, 499 for the building of a school, 498 against, and three blanks. If a majority of valid votes is required (50 percent of 997 votes), the school will be built. If an absolute majority of all votes cast is required, the school will not be built. In the latter case, the blanks determined the outcome.

So, "I don't participate in elections; I either do not vote at all, or I cast a blank ballot" is a logical contradiction.

59

PACKING AND CRACKING

Gerrymandering

In an election for parliament, each political party obtains more or less the number of representatives that reflects its share of the population. If, for example, Party A has the support of 60 percent of the population, then its delegation will make up about 60 percent of parliament.

Correct?

Sometimes. And sometimes not.

In the 2016 elections for the U.S. House of Representatives, 1,859,426 citizens in Virginia cast their votes for Democrats, 1,843,010 for Republicans. Nevertheless, the minority of Republicans sent seven representatives to Washington, DC, while the majority of Democrats sent only four. In Ohio, Republicans scored 58 percent of the vote but obtained 75 percent of the seats, while Democrats, with 42 percent of the vote, got 25 percent of the seats.

Overall in 2016, the Republican Party obtained 63,173,815 votes (50.56 percent), the Democratic Party 61,776,554 (49.44 percent). Had the 435 seats in the House been allocated strictly proportionately, the Republicans should have received 220 seats, the Democrats 215. In fact, the Republicans won 241 seats, twenty-one more than their fair share, while the Democrats won 194 seats, twenty-one less than their fair share.

It takes many kinds of people to make up a nation. Young and old, rich and poor, urban and suburban, religious and secular, conservative and liberal, hipsters and cranks, folks who love the sun and folks who enjoy the snow. To a certain degree, neighborhoods are homogeneous, and people living in proximity with one another usually have similar interests and worries. To represent the objectives and aspirations of these citizens, keeping in mind their characteristics and particularities, representative democracies divide the nation into geographic districts, each of which sends one representative to parliament.

The United States, for example, is divided into 435 congressional districts. These districts are each supposed to comprise approximately the same number of citizens—on average, about 710,000 people—and be compact and contiguous, unless there are natural obstacles, like broad rivers or mountain ranges, in the way. In that manner, the interests of groups of citizens, with all their idiosyncrasies and special interests, are represented by the 435 congresspeople.

Political parties try to push the maximum number of their party's representatives into parliament. And the party in power usually gets to design the congressional districts. Here's the rub: while the United States requires districts to comprise about 710,000 citizens, the ruling party in each state has some flexibility in designing the districts' contours. And it often uses this freedom to draw the contours to maximize the number of their party's candidates getting elected.

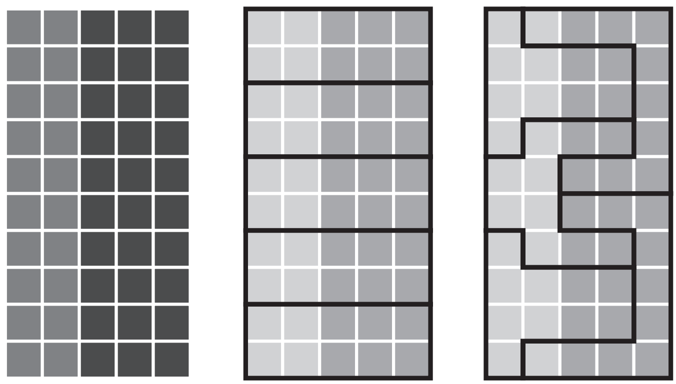
One early such manipulation occurred in 1812 when election districts were redrawn in Massachusetts. Leading the effort was Governor Elbridge Gerry. When a cartoonist depicted one of the strangely shaped districts as a salamander, a new term was born: Gerry's salamander became "gerrymander" and is used as both a noun and a verb.

Gerrymandered voting districts, as illustrated in a political cartoon from 1812.

Source: https://commons.wikimedia.org/wiki/File:The_Gerry-Mander_Edit.png.

DÉNOUEMENT

As we saw, in 2016 the Republican Party managed to gain twenty-one more seats in the House than what would have been proportionate to their share of the vote. How? By packing and cracking, that's how: packing the Democrats' adherents into a small number of districts and cracking its own electorate into many districts.



Note: Light grey represents red and dark grey represents blue.

Source: https://commons.wikimedia.org/wiki/File:How_to_Steal_an_Election_-_Gerrymandering.svg.

The second figure illustrates what can happen when fifty citizens vote for five representatives. With sixty votes overall for Party Blue and forty overall for Party Red, the fair allocation would be three seats for Blue and two for Red. By judiciously packing and cracking, however, districts can be designed so that there will be five seats for Blue and none for Red or two for Blue and three for Red.

When the party in power designs congressional districts, each encompassing about 710,000 citizens, it can allow districts to snake through the nation, grabbing as many voters of the opposing party as possible along the way and wasting few of one's own, thus packing them into a single district. (The best strategy for a party is to let the opposing party win a few seats overwhelmingly while gaining as many seats as possible for itself, even if just barely.)

MORE . . .

As one may expect, whenever a party believes that it has lost seats because of gerrymandering, it files a lawsuit. The claim is usually that the "one person, one vote" maxim has been violated because the packed votes were wasted and the cracked votes carried more weight than they should have. Most suits have been unsuccessful, mainly because of the plaintiffs' inability to quantify the extent by which the district boundaries have been manipulated. After all, what does *compact* mean in this context? The most compact region mathematically is a circle, but there may be lakes, rivers, or mountain ranges that serve as natural boundaries to a district. Further, it is mathematically impossible to cover a nation with circular districts without leaving interstices between them. Judges in general are at a loss.

A few years ago, a law professor and a political scientist proposed a measure that can assess the degree of gerrymandering. The authors defined a so-called efficiency gap that takes account of wasted votes. There are two kinds of wasted votes: those for a losing candidate and those for a winning candidate that go beyond what is necessary for victory.

To illustrate, let's assume there are five districts with one hundred voters in each. Party A wins each of districts 1 to 4 by fifty-three to forty-seven votes. District 5 is won by Party B with eighty-five votes to fifteen. Thus, Party A garnered four seats against one seat for Party B, even though according to the proportion of votes (227 to 273), Party A should have obtained fewer seats than Party B.

The efficiency gap is computed thus: Party A had eight superfluous votes in its four winning districts (in each, it obtained fifty-three votes instead of the bare majority needed of fifty-one) and another fifteen in district 5. So, it wasted twenty-three votes. Party B, on the other hand, wasted thirty-four votes in district 5 (receiving eighty-five votes instead of the bare majority of fifty-one) and all its votes, 188, in the other four districts. Altogether, Party B wasted 222 votes. The efficiency gap is defined as the difference in wasted votes for each party as a proportion of total votes cast: (222 – 23)/500, or 40 percent.

The efficiency gap is a measure of the undeserved share of seats. In the example here, had the election been fair, Party A should have received only two seats: (227/500) × 5 = 2.27 (rounded to 2). In fact, it received two additional seats.

60

HOW CAN ONE BE A DEMOCRAT?

Wollheim's Paradox

Adrienne is an ardent environmentalist. She wants to live in a town with clean air and pure water. Accordingly, she initiates a bill to convert the nearby wood into a nature reserve. Unfortunately, developers want to install a fracking well in the very same wood. The townsfolk are split: a nature reserve will bring tourists; a fracking site will provide jobs. The question is put to a referendum, and the majority will decide.

On the day of the referendum, after the voting booths are closed, Adrienne waits with a pounding heart for the result. Of course, she hopes that her bill will pass. But apart from being an ardent environmentalist, Adrienne is also an ardent democrat; if the developers obtain the majority of votes, she will want the voters' verdict to be honored.[1]

So, Adrienne wants the wood to be converted into a nature reserve, but she also wants it to be converted into a fracking well in case it turns out, after the votes are counted, that her side lost.

Citizens who vote for a bill or to elect a candidate (let's call this state of affairs *B*) are faced with a dilemma as soon as the vote count

1. As much as democrats and Democrats may have been opposed to the election of Donald Trump as president in 2016, protests and slogans like "Donald Trump is not my president" were quite undemocratic.

Source: https://commons.wikimedia.org/wiki/File:Students_voting_in_Fall_Elections,_University_of_Texas_at_Arlington_(10005980).jpg.

indicates that the bill was voted down or the opposing candidate obtained more votes (let's call this state *notB*). As proponents of the bill, they want *B*, and as democrats they want *notB*. How can one simultaneously both want and not want the same thing?

A paradox!

In 1962, the British philosopher Richard Wollheim investigated the question in a book devoted to philosophical aspects of politics and society. He termed the internal conflict to which democratic voters are subject the "paradox of democracy."

At first, Wollheim did not consider the voters' dilemma a paradox. There is no inconsistency, he believed, in wanting *B* but thinking that *notB* ought to be implemented. One "may well have a desire and a moral belief that runs counter to that desire."

Principles and wants need not be inconsistent, even if they lead in different directions; belief in the democratic process supersedes the voters' personal wants because it represents a higher-order principle. Hence, if the voter wants *B* but simultaneously wants to adhere to the superseding democratic principles, she would accept *notB*, and there would be no paradox and no dilemma.

It is when voters do not simply express a want but an evaluation of the candidates that "a paradox at the very heart of democracy" arises. If the voter evaluates *B* as being the preferable candidate, but the electorate expresses preference for *notB*, democratically minded voters are obliged to think that *notB* ought to be enacted. So, which of the two incompatible desires should the voter ignore, her own preference or her wish to honor the democratic choice?

DÉNOUEMENT

Wollheim asserted that if voters want to remain true to themselves and faithful to democratic values, neither avenue is acceptable.

After all, a voter who is prepared to drop his preference for *B* if the democratic machine indicates *notB*, says, in effect, "I think that *B* ought to be enacted, provided that enough other people are of the same opinion." Such a personal choice expresses a lack of conviction in his own preference. By the same token, the voter could have cast his ballot for *notB* or, indeed, abstained from voting altogether. In this situation, he would have remained faithful to democracy but untrue to himself.

And a voter who remains steadfast in her support for *B* may accept *notB* if the democratic machine says so out of pragmatic considerations. This voter expresses a lack of genuine conviction in democracy, however. True, she accepts the democratic choice but not because it ought to be enacted as a moral value but—rather hypocritically—only because it seems wise and practical. She remains true to herself but unfaithful to democracy.

Is there a way out of the paradox of democracy? According to Wollheim, there is, though his suggestion does not make things easier for the bewildered voter.

Wollheim distinguished between direct principles (policies such as "murder is wrong") and oblique principles (decision procedures, such as "what is willed by the people is right"). *B* and *notB* are not contradictory if they are used in different senses; they operate on different levels. In our example, *B* is the direct principle, and *notB* is the oblique principle. In effect, voters answer two questions with different factors to consider for each.

Wollheim's point is that even though *B* and *notB* cannot be realized simultaneously, they are not incompatible since they carry different meanings. Hence, from a logical perspective, they do not contradict each other. It is acceptable for voters to remain true to a direct principle while remaining faithful to an oblique principle.

MORE . . .

So, where does that leave democratic voters? What are they to do? Unfortunately, on this point Wollheim remained silent. Once he had shown that *B* and *notB* are not contradictory in a logical sense, the philosopher considered his work done. Frustrated, democratically minded voters are left to fend for themselves.

The paradox is not unique to democracies. Monarchists may evaluate a proposal as superior to the king's suggestion but nevertheless defer to the king's wishes; communists may comply with the politburo's decision even though they consider their own policy better.

EPILOGUE

As the pages of this book come to a close, it is clear that the study of paradoxes is a never-ending journey. From the seemingly simple paradoxes of daily life to the complex and nuanced paradoxes found in law, economics, philosophy, mathematics, and other disciplines, they challenge our understanding of the world around us and force us to question our assumptions.

Some of the paradoxes discussed in this book have been resolved through further reflection and analysis, providing us with a deeper understanding of the intricacies of human behavior and the workings of the universe. Others, however, remain stubbornly paradoxical, defying easy explanation and continuing to perplex us.

Despite this, the exploration of paradoxes is a valuable endeavor for it pushes us to think critically and question our understanding of the world. It reminds us that the world is not always black and white and that there is often more to a situation than meets the eye. It is important to remember that paradoxes are a natural part of human understanding and experience. They challenge our assumptions and force us to think critically about the world around us.

In the end, it is important to remember that paradoxes are not simply frustrating obstacles to be overcome but rather opportunities to deepen our understanding and appreciation of the complexities of the world. As we continue to explore and contemplate these paradoxes, we may find that they lead us to new insights and perspectives that enrich our lives. We are reminded that the study of paradoxes is not only a fascinating intellectual pursuit but also a reminder to remain open-minded and curious in our quest for understanding. We may never fully resolve all the paradoxes we encounter, but the journey is well worth the effort. They remind us that the world is not always as simple as we may think and that there is always more to explore and understand.

It is my hope that you have been inspired to think more critically about the paradoxes in your own life and to approach them with a sense of curiosity and wonder. Thank you for joining me on this journey of exploration and discovery.

POSTSCRIPT

When I was about to write the epilogue to this book in January 2023, a fascinating software application had just hit the internet, an artificial intelligence chatbot called ChatGPT. Within a few weeks, millions of intrigued users all over the world tried their hand at it. I did, too. So, please forgive me, dear reader . . . the preceding epilogue to this book was written entirely by ChatGPT! It is an amalgamation (with only light editing by me) of three responses to the request "Write an epilogue to a book about paradoxes in daily life, law, economics, philosophy, math, etc."

Of course, this immediately prompts the question: Is the epilogue a lucid comment or just a jumble of confusing sentences? If you know about paradoxes—maybe by having read this book—you will realize that it is, in fact, a sensible closure to this book—but you will not have learned anything new. On the other hand, if you don't know anything about paradoxes—for example, because you began this book by reading the epilogue—you will not know whether it is real or humbug; reading the epilogue will not enlighten you. So, you either know and thus had no need to read the epilogue, or you don't and thus won't be able to determine whether the epilogue is informative or nonsensical. So, there you are: it's Meno's paradox (chapter 38) all over again.

QED!

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