THEOREMS AND CONJECTURES ON SOME RATIONAL GENERATING FUNCTIONS

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1. Introduction

This paper arose from my earlier paper [8]. (See also the follow-up by Speyer [4].) The prototypical result in [8] is the following. Define

$$S_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2^{i+1}}\right). \tag{1.1}$$

Set $S_n(x) = \sum_{k\geq 0} \left\langle {n \atop k} \right\rangle x^k$ (a finite sum), and define

$$u_2(n) = \sum_{k\geq 0} \left\langle {n \atop k} \right\rangle^2.$$

Then

$$\sum_{n\geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+x^2}.$$

Upon seeing this result and some similar ones, Doron Zeilberger asked what happens when 2^n is replaced by some other function satisfying a linear recurrence with constant coefficients, such as the Fibonacci numbers F_n (with initial conditions $F_1 = F_2 = 1$). We will prove some results of this nature, but the data suggests that much more is true. We give a number of conjectures in this direction.

This paper is written in somewhat casual style, with many proofs omitted or just outlined. My main motivation in posting it is to inspire someone to find a better approach, prove the conjectures, and develop a more general theory.

2. A Fibonacci product

In this section we consider the product

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right). \tag{2.1}$$

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In particular, $I_0(x) = 1$ (the empty product) and $I_1(x) = 1 + x$. Our main goal for this section is a proof of the following result.

Theorem 2.1. *Let $I_n(x) = \sum_{k\geq 0} c_n(k)x^k$, and set*

$$v_2(n) = \sum_{k\geq 0} c_n(k)^2,$$

so $v_2(0) = 1$, $v_2(1) = 2$, $v_2(2) = 4$, $v_2(3) = 10$, etc. Then

$$\sum_{n\geq 0} v_2(n)x^n = \frac{1-2x^2}{1-2x-2x^2+2x^3}.$$

The proof parallels the proofs in [8] of similar results by setting up a system of linear recurrences of order one. (In Section 4 we give another proof, the case $k = 2$ and $t = 1$ of Theorem 4.3). However, deriving these recurrences here is quite a bit more complicated. It simplifies somewhat the argument to replace $I_n(x)$ with another power series (with noninteger exponents). The justification for this replacement is provided by the following lemma. Part (b) is presumably known, though I couldn't find this anywhere.

Lemma 2.2. *Let $\phi = \frac{1}{2}(1+\sqrt{5})$.*

(a) *Suppose $\alpha = (a_0, a_1, \dots)$ and $\beta = (b_0, b_1, \dots)$ are sequences of 0's and 1's, with finitely many 1's, such that*

$$\sum_{i\geq 0} a_i\phi^i = \sum_{i\geq 0} b_i\phi^i.$$

Then α can be converted to β by a sequence of operations that replace three consecutive terms 001 with 110, and vice versa.

(b) *Suppose $\alpha = (a_0, a_1, \dots)$ and $\beta = (b_0, b_1, \dots)$ are sequences of 0's and 1's, with finitely many 1's, such that*

$$\sum_{i\geq 0} a_iF_{i+2} = \sum_{i\geq 0} b_iF_{i+2}.$$

Then α can be converted to β by a sequence of operations that replace three consecutive terms 001 with 110, and vice versa.

Proof. (a) This is a simple consequence of the fact that ϕ is a zero of the irreducible polynomial $x^2 - x - 1$.

(b) Simple proof by induction on the largest j for which $a_j = 1$ or $b_j = 1$. Details omitted. □

For a power series $P(x) = \sum_{i\geq 0} c_ix^{m_i}$ with real exponents $m_i \geq 0$, where each $c_i \neq 0$ and $m_0 < m_1 < \cdots$, we call the sequence $(c_0, c_1, \dots)$ the *sequence of coefficients* of $P(x)$. It's easy to see that Lemma 2.2 has the following consequence.

Corollary 2.3. *Let* $G_n(x) = \prod_{i=0}^{n-1}\left(1+x^{\phi^i}\right)$*, a power series whose exponents lie in the ring* $\mathbb{Z}[\phi]$*. Then the sequence of coefficients of* $G_n(x)$ *is equal to the sequence of coefficients of* $I_n(x)$*. Moreover, if the coefficient of* x^k *in* $I_n(x)$ *is 0, then* $k > \deg I_n(x)$*.*

To illustrate the next result, when we expand $G_5(x)$ we obtain the following expression, where the terms are listed in increasing order of their exponents:

$$G_5(x) = 1 + x + x^a + 2x^{a+1} + x^{a+2} + 2x^b + 2x^{b+1} + x^c + 3x^{c+1} + 2x^{c+2}$$
$$+2x^d+3x^{d+1}+x^{d+2}+2x^e+2x^{e+1}+x^f+2x^{f+1}+x^{f+2}+x^g+x^{g+1},$$

for certain numbers $a, b, \dots, g \in \mathbb{Z}[\phi]$. Note that the terms come in groups of length two or three, where within each group the exponents increase by one at each step.

Theorem 2.4. *For* $n \geq 1$*, we can write* $G_n(x)$ *as a sum* $G_n(x) = T_1(x)+T_2(x)+\cdots+T_k(x)$*, where each* $T_i(x)$ *has the form* $x^a + x^{a+1}$ *or* $x^a + x^{a+1} + x^{a+2}$*. Moreover, the largest exponent of a term in* $T_i(x)$ *is less the smallest exponent of a term in* $T_{i+1}(x)$*. (As an aside, we have* $k = F_{n+1}$*.)*

Proof hint. The terms $T_i(x)$ with two summands are of the form

$$c_1 x^{\phi+\phi^2+a_3\phi^3+a_4\phi^4+\cdots} + c_2 x^{1+\phi+\phi^2+a_3\phi^3+a_4\phi^4+\cdots},$$

where $a_3, a_4, \dots$ is a sequence of 0's and 1's with finitely many 1's, and where c_1, c_2 are positive integers. Similarly, the terms $T_i(x)$ with three summands are of the form

$$c_1 x^{\phi+a_3\phi^3+a_4\phi^4+\cdots} + c_2 x^{\phi^2+a_3\phi^3+a_4\phi^4+\cdots} + c_3 x^{1+\phi^2+a_3\phi^3+a_4\phi^4+\cdots}. \quad \square$$

Note. Though we have no need of this result, let us mention that if $d(i)$ denotes the number of terms (either two or three) of $T_i(x)$, then for i fixed and n sufficiently large we have

$$d(i) = 1 + \lfloor i\phi \rfloor - \lfloor (i-1)\phi \rfloor.$$

The sequence $(d(1), d(2), \dots)$ is obtained from sequence A014675 in OEIS by prepending a 1 and adding 1 to every term.

We now define an array analogous to Pascal's triangle (or the arithmetic triangle) and Stern' triangle of [8]. We call the resulting array the *Fibonacci triangle* $\mathcal{F}$. (This definition is unrelated to some other definitions of Fibonacci triangle in the literature.)

Every row is a sequence of positive integers, together with a grouping of consecutive terms such that every group of the grouping has two or three terms. We will denote the grouping by a bullet (•) between groups. The first row is the sequence $1, 1$, which necessarily has a single element $1, 1$ in its grouping. Regard the first entry in each row as preceded by a 0 which is the last element of its group. (The length of this group ending with a 0 is irrelevant.) Similarly, the last entry in each row is followed by a 0 which is the first element of its group.

Row $i+1$ is obtained from row i by the following recursive procedure. If a term a_j of row i ends a group (so a_{j+1} begins a group), then below a_j, a_{j+1} write in row $i+1$ the 3-element group $a_j, a_j + a_{j+1}, a_{j+1}$. If a_j in row i is the middle element of a 3-element group, then write in row $i+1$ below a_j the 2-element group a_j, a_j. The first five rows of $\mathcal{F}$ look as follows:

											1						1											
					1						1			•			1						1					
			1		1			•			1			2			1			•			1		1			
	1		1	•	1			2			1	•	2		2	•	1			2			1	•	1		1	
1	1	•	1	2	1	•	2		2	•	1	3	2	•	2	3	1	•	2		2	•	1	2	1	•	1	1

Let $\left[{n \atop k}\right]$ be the kth entry (beginning with $k = 0$) in row n (beginning with $n = 1$) of $\mathcal{F}$. Set $H_n(x) = \sum_{k \geq 0} \left[{n \atop k}\right] x^k$. For instance,

$$H_3(x) = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6.$$

The following result can be proved by induction.

Theorem 2.5. *We have* $H_n(x) = I_n(x)$.

We now have all the ingredients for proving Theorem 2.1. Define

$$\left[{n \atop k}\right]_1 = \begin{cases} \left[{n \atop k}\right], & \text{if the } k\text{th entry in row } n \text{ of } \mathcal{F} \text{ is the first entry of its group} \\ 0, & \text{otherwise} \end{cases}$$

$$\left[{n \atop k}\right]_2 = \begin{cases} \left[{n \atop k}\right], & \text{if the } k\text{th entry in row } n \text{ of } \mathcal{F} \text{ is the middle entry of its group} \\ 0, & \text{otherwise} \end{cases}$$

$$\left[{n \atop k}\right]_3 = \begin{cases} \left[{n \atop k}\right], & \text{if the } k\text{th entry in row } n \text{ of } \mathcal{F} \text{ is the last entry of its group} \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$\begin{aligned}
A_1(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_1^2 \\
A_2(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_2^2 \\
A_3(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_3^2 \\
A_{3,1}(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_3 \begin{bmatrix} n \\ k+1 \end{bmatrix}_1 \\
A_{1,2}(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_1 \begin{bmatrix} n \\ k+1 \end{bmatrix}_2 \\
A_{1,3}(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_1 \begin{bmatrix} n \\ k+1 \end{bmatrix}_3 \\
A_{2,3}(n) &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix}_2 \begin{bmatrix} n \\ k+1 \end{bmatrix}_3 .
\end{aligned}$$

Using the definition of $\mathcal{F}$ one checks the following (all sums are over $k \geq 0$):

$$\begin{aligned}
A_1(n+1) &= \sum \left(\begin{bmatrix} n \\ k \end{bmatrix}_3^2 + \begin{bmatrix} n \\ k \end{bmatrix}_2^2 \right) \\
&= A_2(n) + A_3(n) \\
A_2(n+1) &= \sum \left(\begin{bmatrix} n \\ k \end{bmatrix}_3 + \begin{bmatrix} n \\ k+1 \end{bmatrix}_1 \right)^2 \\
&= A_1(n) + A_3(n) + 2A_{3,1}(n) \\
A_3(n+1) &= \sum \left(\begin{bmatrix} n \\ k \end{bmatrix}_1^2 + \begin{bmatrix} n \\ k \end{bmatrix}_2^2 \right) \\
&= A_1(n) + A_2(n) \\
A_{3,1}(n+1) &= \sum \left(\begin{bmatrix} n \\ k \end{bmatrix}_1 \begin{bmatrix} n \\ k+1 \end{bmatrix}_2 + \begin{bmatrix} n \\ k \end{bmatrix}_1 \begin{bmatrix} n \\ k+1 \end{bmatrix}_3 + \begin{bmatrix} n \\ k \end{bmatrix}_2 \begin{bmatrix} n \\ k+1 \end{bmatrix}_3 \right) \\
&= A_{1,2}(n) + A_{1,3}(n) + A_{2,3}(n)
\end{aligned}$$

$$\begin{aligned} A_{1,2}(n+1) &= \sum \begin{bmatrix} n \\ k \end{bmatrix}_3 \left(\begin{bmatrix} n \\ k \end{bmatrix}_3 + \begin{bmatrix} n \\ k+1 \end{bmatrix}_1 \right) \\ &= A_3(n) + A_{3,1}(n) \\ A_{1,3}(n+1) &= \sum \begin{bmatrix} n \\ k \end{bmatrix}_2^2 \\ &= A_2(n) \\ A_{2,3}(n+1) &= \sum \left(\begin{bmatrix} n \\ k \end{bmatrix}_3 + \begin{bmatrix} n \\ k+1 \end{bmatrix}_1 \right) \begin{bmatrix} n \\ k+1 \end{bmatrix}_1 \\ &= A_1(n) + A_{3,1}(n). \end{aligned}$$

Let M denote the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and let $v(n)$ denote the column vector

$$v(n) = [A_1(n), A_2(n), A_3(n), A_{3,1}(n), A_{1,2}(n), A_{1,3}(n), A_{2,3}(n)]^t$$

(where ^{t} denotes transpose). The recurrences above take the form $v(n+1) = Mv(n)$. Hence, as in [8, §2], the seven functions $A_\alpha(n)$ all satisfy a linear recurrence relation whose characteristic polynomial $Q_2(x)$ is the characteristic polynomial $\det(xI - M)$ of M. Then $\sum_{n\geq 0} A_\alpha(n)x^n$ is a rational function with denominator $x^{\deg Q_2(x)}Q_2(1/x)$. One computes $Q_2(x) = x^2(x+1)^2(x^3 - 2x^2 - 2x + 2)$. Taking into account the initial conditions for the case $A_1(n) + A_2(n) + A_3(n) = v_2(n)$ yields Theorem 2.1.

Note that the factors $x^2(x+1)^2$ of $Q_2(x)$ were spurious. This suggests that there should be a simpler argument involving a 3×3 matrix rather than a 7×7 matrix.

3. Some generalizations

There are several ways we can try to generalize Theorem 2.1. In this section we will consider generalizing the product $I_n(x)$ and the function $v_2(n)$. However, we continue to deal with Fibonacci numbers. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{m-1}) \in \mathbb{N}^m$ (where $\mathbb{N} = \{0, 1, \dots\}$), and define

$$v_\alpha(n) = \sum_{k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}^{\alpha_0} \begin{bmatrix} n \\ k+1 \end{bmatrix}^{\alpha_1} \cdots \begin{bmatrix} n \\ k+m-1 \end{bmatrix}^{\alpha_{m-1}}.$$

This definition is completely analogous to the definition of $u_\alpha(n)$ in [8]. As in [8], we write $v_{\alpha_0,\dots,\alpha_{m-1}}$ as short for $v_{(\alpha_0,\dots,\alpha_{m-1})}$.

Our proof of Theorem 2.1 carries over to the following result. The argument is analogous. We just have to ascertain that we don't end up with a system of infinitely many equations. This is proved in the same way as in [8, Thm. 2].

Theorem 3.1. *For any $\alpha \in \mathbb{N}^m$, the generating function*

$$J_\alpha(x) = \sum_{n\geq 0} v_\alpha(n) x^n$$

is rational.

We used the Maple package gfun to "guess" the rational function $J_\alpha(x)$ for some small α. Gfun finds the "simplest" rational function fitting the data, which consists of values of $v_\alpha(n)$ for small n (typically around $0 \leq n \leq 36$). Thus the examples below have not been rigorously proved. First we give some examples where $\alpha = (r)$:

$$\begin{aligned} J_3(x) &= \frac{1-4x^2}{1-2x-4x^2+2x^3} \\ J_4(x) &= \frac{1-7x^2-2x^4}{1-2x-7x^2-2x^4+2x^5} \\ J_5(x) &= \frac{1-11x^2-20x^4}{1-2x-11x^2-8x^3-20x^4+10x^5} \qquad (3.1) \\ J_6(x) &= \frac{1-17x^2-88x^4-4x^6}{1-2x-17x^2-28x^3-88x^4+26x^5-4x^6+4x^7} \\ J_7(x) &= \frac{1-26x^2-311x^4-84x^6}{1-2x-26x^2-74x^3-311x^4+34x^5-84x^6+42x^7}. \end{aligned}$$

Note that the denominators all have odd degree, and the numerator is the even part of the denominator. This behavior has been verified empirically (not rigorously) for $n \leq 17$. For $8 \leq n \leq 17$, the denominator degrees are $9, 7, 9, 9, 13, 11, 13, 11, 13, 13$, respectively. See Conjecture 4.6 for a generalization.

Here are some examples where α has at least two terms:

$$
\begin{aligned}
J_{1,1}(x) &= \frac{x+x^2}{1-2x-2x^2+2x^3}\\
J_{1,0,1}(x) &= \frac{2x^2+x^3-x^4}{(1-x)(1-2x-2x^2+2x^3)}\\
J_{2,1}(x) &= \frac{x+x^2}{1-2x-4x^2+2x^3}\\
J_{1,3}(x) &= \frac{x+x^2+x^3+x^4}{1-2x-7x^2-2x^4+2x^5}\\
J_{2,2}(x) &= \frac{x+x^2-x^3-x^4}{1-2x-7x^2-2x^4+2x^5}\\
J_{2,3}(x) &= \frac{x+x^2-x^3-x^4}{1-2x-11x^2-8x^3-20x^4+10x^5}\\
J_{1,1,1}(x) &= \frac{2x^2+2x^3-2x^4}{(1-x)(1-2x-4x^2+2x^3)}\\
J_{1,0,2}(x) &= \frac{2x^2+x^3-2x^4+x^5}{(1-x)^2(1-2x-4x^2+2x^3)}\\
J_{2,1,1}(x) &= \frac{2x^2+2x^3-4x^4+4x^5}{(1-x)^2(1-2x-7x^2-2x^4+2x^5)}\\
J_{1,2,1}(x) &= \frac{2x^2+4x^3-2x^4}{(1-x)(1-2x-7x^2-2x^4+2x^5)}
\end{aligned}
$$

It appears that $J_\alpha(x)$ has a denominator of the form $(1-x)^{c_\alpha}D_r(x)$, where $c_\alpha \geq 0$, $r = \sum \alpha_i$, and $D_r(x)$ is the denominator of $J_r(x)$. This heuristic observation is in complete analogy to [8, Thm. 3] and presumably has a similar proof.

We can also generalize the definition of $I_n(x)$. In analogy to [8, Thm. 4] we have the following conjecture.

Conjecture 3.2. *Let $h \geq 1$, $(a_1, \ldots, a_h) \in \mathbb{C}^h$, and $P(x) \in \mathbb{C}[x]$. Set*

$$I_{h,P,n}(x) = P(x) \prod_{i=1}^{n} \left(1 + a_1 x^{F_i} + a_2 x^{F_{i+1}} + \cdots + a_h x^{F_{i+h-1}}\right).$$

Regarding h, P *as fixed, let* $c_n(p)$ *denote the coefficient of* x^p *in* $I_{h,P,n}(x)$*. For* $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$ *define*

$$v_{h,P,\alpha}(n) = \sum_{p\geq 0} c_n(p)^{\alpha_0} c_n(p+1)^{\alpha_1} \cdots c_n(p+m-1)^{\alpha_{m-1}}.$$

Then the generating function $\sum_{n\geq 0} v_{h,P,\alpha}(n)x^n$ *is rational.*

Let us consider one simple special case of this conjecture. Let t be any complex number (or an indeterminate), and define

$$I_{n,t}(x) = \prod_{i=1}^{n} \left(1 + tx^{F_{i+1}}\right).$$

We now get a triangle $\mathcal{F}(t)$ with the same grouping into two or three terms as in $\mathcal{F}$, but the first row is $1, t$. Row $i+1$ is obtained from row i by the following recursive procedure. If a term a_j of row i ends a group (so a_{j+1} begins a group), then below a_j, a_{j+1} write in row $i+1$ the 3-element group $a_j, a_j + ta_{j+1}, ta_{j+1}$. If a_j in row i is the middle element of a 3-element group, then write in row $i+1$ below a_j the 2-element group a_j, ta_j.

The following result now is proved in complete analogy with the proof of Theorem 2.1.

Theorem 3.3. *Let* $v_{2,t}(n)$ *denote the sum of the squares of the coefficients of* $I_{n,t}(x)$*. Then*

$$\sum_{n\geq 0} v_{2,t}(n)x^n = \frac{1-(t^3+t)x^2}{1-(t^2+1)x - t(t^2+1)x^2 + t(t^4+1)x^3}.$$

The polynomial $I_{n,-1}(x) = \prod_{i=1}^{n} \left(1 - x^{F_{i+1}}\right)$ has been considered before. It was shown by Yufei Zhao [10] that all its nonzero coefficients are equal to ± 1. Thus $v_{2,-1}(n)$ is equal to the number of nonzero coefficents of $I_{n,-1}(x)$, with generating function

$$\sum_{n\geq 0} v_{2,-1}(n)x^n = \frac{1+2x^2}{1-2x+2x^2-2x^3}.$$

This fact is stated (in equivalent form) in the OEIS [2]. Note that we can also directly compute, using the technique in the proof of Theorem 2.1, that $v_{4,-1} = v_{2,-1}$. This gives a new proof (albeit involving a cumbersome computation) of Zhao's result.

Example 3.4. As a somewhat random special case of Conjecture 3.2, let $h = 3$, $(a_1, a_2, a_3) = (0, 1, 1)$, $P(x) = 1$, and $\alpha = (2)$. Thus we are considering the sum $w(n)$ of the squares of the coefficients of the product $\prod_{i=1}^{n}\left(1+x^{F_{i+1}}+x^{F_{i+2}}\right)$. Then gfun suggests that

$$\sum_{n\geq 0} w(n)x^n = \frac{1-4x-5x^2+24x^3+4x^4-34x^5+2x^6+10x^7-4x^8}{1-7x+x^2+47x^3-32x^4-84x^5+50x^6+34x^7-18x^8}.$$

Note. There is an alternative way of describing the nonzero coefficients of the polynomial $I_n(x)=\prod_{i=1}^{n}\left(1+x^{F_{i+1}}\right)$. Let $\mathcal{A}_n$ denote the set of all words of length n in the letters a, b, so $\#\mathcal{A}_n = 2^n$. Define $\pi, \sigma \in \mathcal{A}_n$ to be *equivalent* if σ can be obtained from π by a sequence of substitutions (on three consecutive terms) $baa \to abb$ and $abb \to baa$, an obvious equivalence relation $\sim$. For instance, when $n = 5$ one of the equivalence classes is $\{baaaa, abbaa, ababb\}$. The quotient monoid of the free monoid generated by a, b modulo $\sim$ is called the *Fibonacci monoid* in [9], though other monoids are also called the Fibonacci monoid. Here we are interested not in the monoid itself, but rather the sizes of its equivalence classes. It follows easily from Lemma 2.2(b) that the multiset M_n of equivalence class sizes of $\sim$ on $\mathcal{A}_n$ coincides with the multiset of (nonzero) coefficients of $I_n(x)$. Thus if $u_n^*(r)=\sum_{j\in M_n} j^r$ $(r\in\mathbb{N})$, then the generating function $\sum_{n\geq 0} u_n^*(r)x^n$ is rational. What other equivalence relations on $\mathcal{A}_n$ obtained by substitutions of words of equal length are rational? For instance, the substitutions $ab \leftrightarrow ba$ do not give rational generating functions for $r \geq 2$. For $r = 2$ the generating function is algebraic but not rational, while for $r \geq 3$ it is D-finite but not algebraic [5, Exer. 6.3, 6.54]. Thus we can also ask in general when we get algebraic and D-finite generating functions.

Note. It is a nice exercise to show that if $f_1, f_2, \dots$ is a sequence of positive integers satisfying $f_1 \neq f_2$ and $f_{i+1} = f_i + f_{i-1}$ for all $i \geq 2$, then for all $n \geq 1$ the sequence of nonzero coefficients of the polynomial $\prod_{i=1}^{n}\left(1+x^{f_i}\right)$ depends only on n.

4. Generalizing the Fibonacci numbers

What happens if we replace F_{i+1} in the definition (2.1) and its generalizations with some other sequence? We consider only sequences $f_1, f_2, \dots$ satisfying linear recurrences with constant integer coefficients. Note that if $f_{i+1} \geq 2f_i$ for all i, then the nonzero coefficients of $\prod_{i=1}^{n}\left(1+x^{f_i}\right)$ are all equal to 1, which is not so interesting. One class of sequences that have more interesting behavior is given for fixed $k \geq 1$ by

$$F_{i+1}^{(k)} = F_i^{(k)} + F_{i-1}^{(k)} + \cdots + F_{i-k+1}^{(k)},$$

say with initial conditions $F_1^{(k)} = F_2^{(k)} = \cdots = F_k^{(k)} = 1$. Thus $F_i^{(2)} = F_i$.

We conjecture that Conjecture 3.2 has a direct $F^{(k)}$-analogue.

Conjecture 4.1. *Let $k \geq 2$, $h \geq 1$, $(a_1, \dots, a_h) \in \mathbb{C}^h$, and $P(x) \in \mathbb{C}[x]$. Set*

$$I^{(k)}_{h,P,n}(x) = P(x) \prod_{i=1}^{n} \left(1 + a_1 x^{F_i^{(k)}} + a_2 x^{F_{i+1}^{(k)}} + \cdots + a_h x^{F_{i+h-1}^{(k)}}\right).$$

Regarding h, P, k as fixed, let $c_n(p)$ denote the coefficient of x^p in $I^{(k)}_{h,P,n}(x)$. For $\alpha = (\alpha_0, \dots, \alpha_{m-1}) \in \mathbb{N}^m$ define

$$v^{(k)}_{h,P,\alpha}(n) = \sum_{p \geq 0} c_n(p)^{\alpha_0} c_n(p+1)^{\alpha_1} \cdots c_n(p+m-1)^{\alpha_{m-1}}.$$

Then the generating function $\sum_{n \geq 0} v^{(k)}_{h,P,\alpha}(n) x^n$ is rational.

For the special case

$$I^{(k)}_{H,P,n}(x) = \prod_{i=1}^{n} \left(1 + t x^{F_{i+k-1}}\right),$$

we can prove this conjecture by a combinatorial technique. When $k = 2$ this gives a new proof of Theorem 2.1.

To give this proof, for $k \geq 2$ define $\mathcal{M}^{(k)}$ to be the set of all pairs π of finite binary sequences of the same length, say n, denoted

$$\pi = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}, \tag{4.1}$$

such that

$$\sum_{i=1}^{n} a_i F^{(k)}_{i+k-1} = \sum_{i=1}^{n} b_i F^{(k)}_{i+k-1}.$$

It is easily seen that if $\pi \in \mathcal{M}^{(k)}$ (where π is given by equation (4.1)) and if

$$\sigma = \begin{pmatrix} c_1 & c_2 & \cdots & c_p \\ d_1 & d_2 & \cdots & d_p \end{pmatrix} \in \mathcal{M}^{(k)},$$

then the concatenation

$$\pi\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_n & c_1 & c_2 & \cdots & c_p \\ b_1 & b_2 & \cdots & b_n & d_1 & d_2 & \cdots & d_p \end{pmatrix}$$

also belongs to $\mathcal{M}^{(k)}$. Thus $\mathcal{M}^{(k)}$ is a monoid under concatenation. (The empty array is the identity element.)

For a binary letter $a = 0, 1$ let a^j denote a sequence of j a's. For instance, $0^4 = 0000$. Given $k \geq 2$, let $\mathcal{G}^{(k)}$ be the set of all pairs of binary sequences equal to

$$\left(\begin{array}{c}0\\0\end{array}\right) \text{ or } \left(\begin{array}{c}1\\1\end{array}\right), \tag{4.2}$$

or equal to one of the two forms (which differ by interchanging the rows)

(4.3)

$$\pi = \left(\begin{array}{cccccccccccc} 1^k & * & 1^{k-1} & * & 1^{k-1} & * & 1^{k-1} & * & \cdots & * & 1^{k-1} & 0 \\ 0^k & * & 0^{k-1} & * & 0^{k-1} & * & 0^{k-1} & * & \cdots & * & 0^{k-1} & 1 \end{array}\right), \text{ or}$$

$$\sigma = \left(\begin{array}{cccccccccccc} 0^k & * & 0^{k-1} & * & 0^{k-1} & * & 0^{k-1} & * & \cdots & * & 0^{k-1} & 1 \\ 1^k & * & 1^{k-1} & * & 1^{k-1} & * & 1^{k-1} & * & \cdots & * & 1^{k-1} & 0 \end{array}\right),$$

where * can be 0 or 1, but two *'s in the same column must be equal. It's easy to see that $\mathcal{G}^{(k)} \subset \mathcal{M}^{(k)}$. Then the following key lemma is fairly straightforward to prove.

Lemma 4.2. *The set $\mathcal{G}^{(k)}$ freely generates $\mathcal{M}^{(k)}$. That is, every element π of $\mathcal{M}^{(k)}$ can be written uniquely as a product of words in $\mathcal{G}^{(k)}$.*

We can now state the main (nonconjectural)) result of this section.

Theorem 4.3. *Let $v_2^{(k)}(n,t)$ denote the sum of the squares of the coefficients of the polynomial $\prod_{i=1}^n \left(1+tx^{F^{(k)}_{i+k-1}}\right)$. Then*

$$\sum_{n\geq 0} v_2^{(k)}(n,t)x^n = \frac{1-t^{k-1}(1+t^2)x^k}{1-(1+t^2)x-t^{k-1}(1+t^2)x^k+t^{k-1}(1+t^4)x^{k+1}}.$$

Proof. Write $\ell(\pi)$ for the length of $\pi \in \mathcal{M}^{(k)}$, and write $N(\pi)$ for the total number of 1's in π. Note that $\ell(\pi\sigma) = \ell(\pi) + \ell(\sigma)$ and $N(\pi\sigma) = N(\pi) + N(\sigma)$. Define

$$G^{(k)}(x) = \sum_{\pi \in \mathcal{G}^{(k)}} t^{N(\pi)} x^{\ell(\pi)}.$$

By a standard simple argument (see [6, §4.7.4]),

$$\sum_{n\geq 0} v_2^{(k)}(n,t)x^n = \frac{1}{1-G^{(k)}(x)}.$$

We can use Lemma 4.2 to compute $G^{(k)}(x)$. The two generators in equation (4.2) contribute $(1+t^2)x$ to $G^{(k)}(x)$. Now consider the generators π and σ of equation (4.3). The two generators differ only by switching rows, so consider just π. Suppose there are $j \geq 0$ columns of $*$'s. The number of 1's in the remaining columns is $k + j(k-1) + 1$. The length of π is $(j+1)k+1$. Since each of the j columns of $*$'s has zero or two 1's, the contribution to $G^{(k)}$ of all generators (4.3) is $t^{j(k-1)+k+1}(1+t^2)^j x^{jk+k+1}$. The same is true of the second generator σ. Hence

$$\begin{aligned} G^{(k)}(x) &= (1+t^2)x + 2\sum_{j\geq 0} t^{j(k-1)+k+1}(1+t^2)^j x^{jk+k+1} \\ &= \frac{2t^{k+1}x^{k+1}}{1-t^{k-1}(1+t^2)x^k}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n\geq 0} v_2^{(k)}(n,t)x^n &= \frac{1}{1-(1+t^2)x - \frac{2t^{k+1}x^{k+1}}{1-t^{k-1}(1+t^2)x^k}} \\ &= \frac{1-t^{k-1}(1+t^2)x^k}{1-(1+t^2)x - t^{k-1}(1+t^2)x^k + t^{k-1}(1+t^4)x^{k+1}}. \end{aligned}$$

□

Naturally we can ask how the statement and proof of Theorem 4.3 can be extended. For any $r \geq 2$ let $v_r^{(k)}(n,t)$ denote the sum of the rth powers of the coefficients of the polynomial $\prod_{i=1}^n \left(1 + tx^{F_{i+k-1}}\right)$. Define the monoid $\mathcal{M}^{(k)}(r)$ analogously to $\mathcal{M}^{(k)}$ by letting the elements of $\mathcal{M}^{(k)}(r)$ be r-tuples of binary words of the same length such that $\sum_{i=1}^n a_i F_{i+k-1}^{(k)}$ is the same for all the r words $a_1a_2\cdots a_n$. It's easy to see that $\mathcal{M}^{(k)}(r)$ is a free monoid, basically because if π and σ are r-tuples of binary words such that $\pi \in \mathcal{M}^{(k)}(r)$ and $\pi\sigma \in \mathcal{M}^{(k)}(r)$, then $\sigma \in \mathcal{M}^{(k)}(r)$. However, finding the free generators of $\mathcal{M}^{(k)}(r)$ seems complicated for $r \geq 3$, and we have not tried to do so. For $r = 3$ we have the following conjecture. We use the notation

$$J_r^{(k)}(t,x) = \sum_{n\geq 0} v_r^{(k)}(n,t)x^n.$$

Conjecture 4.4. *We have*

$$J_3^{(k)}(t,x) = \frac{1-t^3(1+t^3)^2x^k + t^9(t^3-1)^2x^{2k}}{D_3^{(k)}(x)},$$

where

$$\begin{aligned} D_3^{(k)}(x) &= 1-(1+t^3)x - t^3(1+t^3)^2x^k + t^3(1+t^9)x^{k+1} \\ &\quad + t^9(t^3-1)^2x^{2k} - t^9(t^3-1)^2(t^3+1)x^{2k+1}. \end{aligned}$$

Note that the numerator in the above conjecture consists of the terms in the denominator with x-exponents $0, k, 2k$. For higher values of r the coefficients seem to be more complicated. For instance, it seems that the coefficient of x^4 in the denominator of $J_4^{(k)}(t,x)$ is $t^2(t^{12}+t^{10}-$

$t^8 - 4t^6 - t^4 + t^2 + 1)$. The factor $t^{12} + t^{10} - t^8 - 4t^6 - t^4 + t^2 + 1$ is irreducible over $\mathbb{Q}$. We give below some conjectures when $t = 1$.

Conjecture 4.5. *We have*

$$\begin{aligned}
J_4^{(k)}(1,x) &= \frac{1 - 7x^k - 2x^{2k}}{1 - 2x - 7x^k - 2x^{2k} + 2x^{2k+1}} \\
J_5^{(k)}(1,x) &= \frac{1 - 11x^k - 20x^{2k}}{1 - 2x - 11x^k - 8x^{k+1} - 20x^{2k} + 10x^{2k+1}} \\
J_6^{(k)}(1,x) &= \frac{1 - 17x^k - 88x^{2k} - 4x^{3k}}{D_6^{(k)}(1,x)} \\
J_7^{(k)}(1,x) &= \frac{1 - 26x^k - 311x^{2k} - 84x^{3k}}{D_7^{(k)}(1,x)},
\end{aligned}$$

where

$$D_6^{(k)}(1,x) = 1 - 2x - 17x^k - 28x^{k+1} - 88x^{2k} + 26x^{2k+1} - 4x^{3k} + 4x^{3k+1}$$

and

$$D_7^{(k)}(1,x) = 1 - 2x - 26x^k - 74x^{k+1} - 311x^{2k} + 34x^{2k+1} - 84x^{3k} + 42x^{3k+1}.$$

Theorem 3.3 and Conjectures 4.4 and 4.5 suggest the following conjecture.

Conjecture 4.6. *For $r \geq 2$ there is an integer $m \geq 1$ (depending on r) for which $J_r^{(k)}(t,x)$ has the form*

$$J_r^{(k)}(t,x) = \frac{1 + a_2(t)x^k + a_4(t)x^{2k} + \cdots + a_{2m}(t)x^{mk}}{D_r^{(k)}(t,x)},$$

where

$$\begin{aligned}
D_r^{(k)}(t,x) = 1 &+ a_1(t)x + a_2(t)x^k + a_3(t)x^{k+1} + a_4(t)x^{2k} + a_5(t)x^{2k+1} \\
&+ \cdots + a_{2m}(t)x^{mk} + a_{2m+1}(t)x^{mk+1},
\end{aligned}$$

and where each $a_i(t)$ is a polynomial in t (depending on r but independent from k) such that $a_{2m+1}(t) \neq 0$, and possibly even $a_i(t) \neq 0$ for all $0 \leq i \leq 2m+1$. Moreover (in order to account for the odd denominator degrees in equation (3.1)), the largest index j for which $a_j(1) \neq 0$ is odd.

Note that this conjecture has the suprising consequence that once we know $J_r^{(2)}(t,x)$ (or even just its denominator), then we can immediately determine $J_r^{(k)}(t,x)$ for all k.

What other sequences satisfying linear recurrences with constant coefficients have interesting behaviour related to this paper? We haven't found any further recurrences with "nice" behavior. For instance, gfun fails to find rational generating functions (using the values for $0 \le n \le 40$) for the sum of the squares of the coefficients of $\prod_{i=1}^{n}\left(1+x^{f_{i+2}}\right)$, when either $f_{i+1}=f_i+f_{i-2}$ or $f_{i+1}=f_{i-1}+f_{i-2}$, with initial conditions $f_1=f_2=f_3=1$.

5. Congruence properties

For $0 \le a < m$, let $g_{m,a}(n)$ denote the number of coefficients of $S_n(x)$ (defined by equation (1.1)) that are congruent to a modulo m. Reznick [3] showed that the generating function

$$G_{m,a}(x)=\sum_{n\ge 0} g_{m,a}(n)x^n$$

is rational. See also [7, pp. 28–37], where some open questions are on page 32. In particular, the denominator of $G_{m,a}(x)$ has quite a bit of factorization that remains unexplained. (For some small progress related to the denominator factorization, see Bogdanov [1].) For the proof that $G_{m,a}(x)$ is rational, it is necessary to introduce auxiliary generating functions $G_{m,a,b}(x)=\sum_{n\ge 0} g_{m,a,b}(n)x^n$, where $g_{m,a,b}(n)$ is equal to the number of integers $0 \le k \le \deg S_n(x)$ for which $\left\langle {n \atop k} \right\rangle \equiv a \pmod{m}$ and $\left\langle {n \atop k+1} \right\rangle \equiv b \pmod{m}$.

We can do something analogous for the Fibonacci triangle. For $0 \le a < m$, let $h_{m,a}(n)$ denote the number of coefficients of $I_n(x)$ (defined by equation (2.1)) that are congruent to a modulo m. Define

$$H_{m,a}(x)=\sum_{n\ge 0} h_{m,a}(n)x^n.$$

The proof sketched in [7] that $G_{m,a}(x)$ is rational carries over, *mutatis mutandis*, to $H_{m,a}(x)$. As in the proof for $G_{m,a}(x)$, we need to introduce some auxiliary generating functions that take into account consecutive coefficients of $I_n(x)$. However, we need also specify whether these coefficients are the beginning, middle, or end of a group (as defined in Section 2). Thus we will have numbers like $g^{(3,1)}_{m,a,b}(n)$ which count the number of integers $0 \le k \le \deg I_n(x)$ for which $\left[{n \atop k} \right]$ ends a group and satisfies $\left[{n \atop k} \right] \equiv a \pmod{m}$, while $\left[{n \atop k+1} \right]$ begins a group and satisfies $\left[{n \atop k+1} \right] \equiv b \pmod{m}$. When these procedures are carried out we obtain the following result.

Theorem 5.1. *The generating function $H_{m,a}(x)$ is rational.*

Naturally we would like to say more about $H_{m,a}(x)$ than just its rationality. Here are some values suggested by gfun. None have been proved.

$$\begin{aligned}
H_{2,0}(x) &= \frac{x^3(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-2x^3)} \\
H_{2,1}(x) &= \frac{1+2x^2}{1-2x+2x^2-2x^3} \\
H_{3,0}(x) &= \frac{2x^5(1-2x^2)}{(1-x)(1-x-x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{3,1}(x) &= \frac{1-2x+4x^2-6x^3+8x^4-10x^5+8x^6-6x^7}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{3,2}(x) &= \frac{x^3(1+2x^4)}{(1-x)(1-x+x^2)(1-2x+2x^2-3x^3+4x^4-4x^5)} \\
H_{4,0}(x) &= \frac{x^6(1-2x^2)(1-3x^2+4x^3-4x^4)}{(1-x)(1-x-x^2)(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,1}(x) &= \frac{1-2x+5x^2-8x^3+10x^4-12x^5+8x^6-6x^7}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)} \\
H_{4,2}(x) &= \frac{x^3(1+x^2)(1-2x^2)}{(1-x^2+2x^4)(1-2x+2x^2-2x^3)^2} \\
H_{4,3}(x) &= \frac{2x^5(1+x^2)}{(1-x)(1-2x+2x^2-2x^3)(1-x+2x^2-2x^3+2x^4)}
\end{aligned}$$

Note that just as for $G_{m,a}(x)$, there is a lot of denominator factorization. Moreover, some of the numerators of $H_{m,a}(x)$ have only two terms, in analogy to some numerators of $G_{m,a}(x)$ having just one term.

We can try to extend Theorem 5.1 to

$$I_n^{(k)}(x) = \prod_{i=1}^{n} \left(1 + x^{F^{(k)}_{i+k-1}}\right).$$

For $0 \le a < m$, let $h^{(k)}_{m,a}(n)$ denote the number of coefficients of $I_n^{(k)}(x)$ that are congruent to a modulo m. Define

$$H^{(k)}_{m,a}(x) = \sum_{n \ge 0} h^{(k)}_{m,a}(n) x^n.$$

Conjecture 5.2. *The generating function $H^{(k)}_{m,a}(x)$ is rational.*

We have some scanty evidence for a "congruence analogue" of Conjecture 4.6. For $(m, a) = (2, 1)$ we found enough evidence to conjecture the following.

Conjecture 5.3. *We have*

$$H_{2,1}^{(k)}(x) = \frac{1 + 2x^k}{1 - 2x + 2x^k - 2x^{k+1}}.$$

For $(m, a) = (3, 1)$ gfun suggests the following:

$$H_{3,1}^{(2)}(x) = \frac{1 - 2x + 4x^2 - 6x^3 + 8x^4 - 10x^5 + 8x^6 - 6x^7}{1 - 4x + 8x^2 - 12x^3 + 16x^4 - 20x^5 + 19x^6 - 12x^7 + 4x^8}$$

$$H_{3,1}^{(3)}(x) = \frac{1 - 2x + 4x^3 - 6x^4 + 8x^6 - 10x^7 + 9x^9 - 6x^{10}}{D(x)},$$

where

$$D(x) = 1 - 4x + 4x^2 + 4x^3 - 12x^4 + 8x^5 + 8x^6 - 20x^7 + 11x^8 + 8x^9 - 12x^{10} + 4x^{11}.$$

The connection between the two numerators is obvious. Note the denominator coefficients of $H_{3,1}^{(2)}(x)$ are obtained from those of $H_{3,1}^{(3)}(x)$ by adding the coefficients of the pairs (x^2, x^3), (x^5, x^6) and (x^8, x^9), keeping the other coefficients unchanged. Someone with better computer skills and/or power should be able to come up with more general conjectures. Even better, of course, would be some theorems!

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