AN INFINITE NESTED RECURRENCE RELATION THAT TAKES ON FIBONACCI VALUES AT FIBONACCI INDICES

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Abstract. Let $a(0) = 0$ and $a(n) = n - 1 - \sum_{k \geq 1} a^k(n-k)$ for $n > 1$, where the exponent k indicates k-fold composition. We prove that $a(F_{n+2}) = F_n$ using a certain infinite morphism. The recursive structure of the underlying morphic word is elucidated.

1. Introduction

There is a close connection between several nested recurrence relations and Fibonacci numbers. A nested recurrence relation, such as Hofstadter's G recurrence, $G(n) = n - G(G(n-1))$, has the property that the function name (G in his case) occurs somewhere in the parameter for another instance of the function name. With the intial condition $G(0) = 0$, this recurrence relation has the solution $G(n) = \lfloor (n+1)/\phi \rfloor$ where ϕ is the golden ratio; see Downey and Griswold [4] (and [6],[7]). As a consequence $G(F_{n+1}) = F_n$, where as usual F_n is the n-th Fibonacci number. Similar relationships for another nested recurrence *equation* are noted in [1, 2].

More recently, it was shown in [8] that, with carefully selected initial conditions, $Q(3m + 2) = F_{m+5}$, where $Q(n)$ is Hoftadter's famous "meta-Fibonacci" recurrence relation $Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2))$ [5].

Our purpose in this paper is to prove an equation similar to $G(F_{n+1}) = F_n$ for an infinite nested recurrence relation that occurs in the paper [3], namely

$$a(n) = \begin{cases} 0 & \text{if } n \leq 0 \\ n-1-a(n-1)-a(a(n-2))-a(a(a(n-3)))-a(a(a(a(n-4))))-\cdots & \text{otherwise.} \end{cases} \tag{1.1}$$

There they noted experimentally that

$$a(F_{n+2}) = F_n, \tag{1.2}$$

and proved it when n is even, leaving open the odd case. The objective of this paper is to prove (1.2) and elucidate some structural properties of the sequence $a(1), a(2), \ldots$.

2. Results

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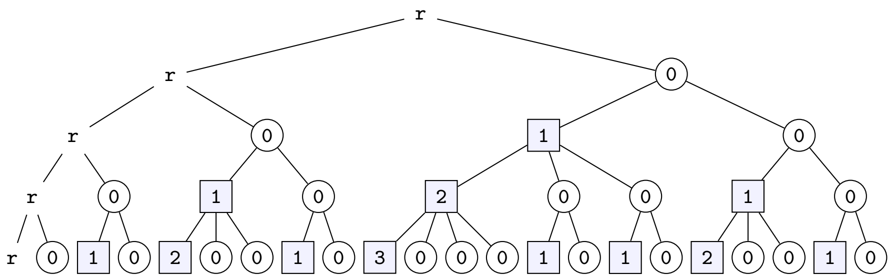

FIGURE 1. The tree of the morphism.

As proved by Meek and van Rees [6], the value of $G(n)$ may be obtained by forming the Zeckendorf representation of n and the deleting the least significant bit. It might be hoped that the solution to $a(n)$ could be obtained from the Zeckendorf representation by deleting the two least significant bits of n; however, that is not the case, something more subtle is happening with $a(n)$.

The starting point for our approach to the problem is the paper [3]. There they establish a correspondence between the solution to a certain class of nested recurrence relation and a counting problem on words arising from a certain class of morphisms. See Section 5.2 of [3] where (1.1) is introduced.

The relevant morphism, σ, is defined as follows.

$$r \to r0, 0 \to 10, 1 \to 200, \dots, k \to (k+1)0^{k+1}, \dots.$$

The starting symbol is r, and the initial levels of the morphic tree of σ are shown in Figure 1. Define the infinite morphic word

$$w = u_1 u_2 u_3 \cdots = r^{-1}\sigma^{\infty}(r)$$

and let

$$w_k = u_1 u_2 \cdots u_k$$

be the length k prefix of w. E.g.,

$$w_{47} = 01020010300010102001040000101010200102001030001.$$

The results of [3] then imply that $a(n)$ is the number of non-zero characters in w_n. Define $s(n)$ to be the sum of the numbers in w_n and $b(n)$ to be the length of $r^{-1}\sigma(rw_n)$. It is worth noting that $r^{-1}\sigma(rw_n) = 0\sigma(w_n)$ for $n \geq 1$. A table of the relevant numbers may be found below. The $a(n)$ numbers are in the OEIS as sequence A227145 [9].

n	1	2	3	4	5	6	7	8	9	10	11	12	13
w	0	1	0	2	0	0	1	0	3	0	0	0	1
$a(n)$	0	1	1	2	2	2	3	3	4	4	4	4	5
$s(n)$	0	1	1	3	3	3	4	4	7	7	7	7	8
$b(n)$	3	6	8	12	14	16	19	21	26	28	30	32	35

Sufficiently far down in the morphism tree, w_n is the prefix of the string of characters at a given level, ignoring the initial r. For example 0102 occurs at level 3 (with the root at level 0) and greater. In terms of the tree (again, ignoring the leading r), $a(n)$ is the number of parents of w_n and $b(n)$ is the number of children of w_n.

The following identities follow fairly directly from the definitions. For example, $a(b(k)) = k$ says that the number of parents of the number of children of the first k nodes is k. And (2.2) is true because if one of the first k nodes is a j then it has $j+2$ children (the plus 1 comes from the initial 0).

$$a(b(k)) = k \text{ and } a(b(k)-1) = k \text{ for all } k \geq 0, \tag{2.1}$$

$$b(k) = 1 + 2k + s(k), \text{ for all } k \geq 1, \text{ and} \tag{2.2}$$

$$k + s(k) = s(b(k)) = s(b(k)-1) \text{ for all } k \geq 1. \tag{2.3}$$

Claim 2.1. *For all $m \geq 1$ we have $b(F_{2m-1}) = 1 + F_{2m+1}$ and $b(F_{2m}) = F_{2m+2}$.*

Proof: We break this down into the following two enhanced statements.

(a) For all $m \geq 1$ we have $b(F_{2m-1}) = 1 + F_{2m+1}$ and $s(F_{2m+1}) = F_{2m}$.
(b) For all $m \geq 1$ we have $b(F_{2m-2}) = F_{2m}$ and $s(F_{2m}) = -1 + F_{2m-1}$.

Each of (a) and (b) is proven independently by induction on m. The initial cases can be checked in the table. We first prove (a).

$$\begin{aligned} b(F_{2m-1}) &= 1 + 2F_{2m-1} + s(F_{2m-1}) \\ &= 1 + F_{2m-1} + F_{2m-1} + F_{2m-2} \\ &= 1 + F_{2m-1} + F_{2m} \\ &= 1 + F_{2m+1}. \end{aligned}$$

Using (2.3) we get

$$\begin{aligned} s(F_{2m+1}) &= s(b(F_{2m-1}) - 1) \\ &= s(b(F_{2m-1})) \\ &= F_{2m-1} + s(F_{2m-1}) \\ &= F_{2m-1} + F_{2m-2} \\ &= F_{2m}. \end{aligned}$$

And now we prove (b), using (2.2).

$$\begin{aligned} b(F_{2m-2}) &= 1 + 2F_{2m-2} + s(F_{2m-2}) \\ &= 1 + F_{2m-2} + F_{2m-2} + F_{2m-3} - 1 \\ &= F_{2m-2} + F_{2m-1} \\ &= F_{2m}. \end{aligned}$$

Again using (2.3),

$$\begin{aligned} s(F_{2m}) &= s(b(F_{2m-2})) \\ &= F_{2m-2} + s(F_{2m-2}) \\ &= F_{2m-2} + F_{2m-3} - 1 \\ &= -1 + F_{2m-1}. \end{aligned}$$

□

Corollary 2.2. *For all $n \geq 3$, $a(F_n) = F_{n-2}$.*

Proof: The proof also uses (2.1). For the even indexed case, note that

$$a(b(F_{2m})) = F_{2m} \text{ and } a(b(F_{2m})) = a(F_{2m+2}).$$

For the odd indexed case, note that

$$a(b(F_{2m-1} - 1)) = F_{2m-1} \text{ and } a(b(F_{2m-1}) - 1) = a(F_{2m+1}).$$

□

We will now give a recursive structural characterization of the w_{F_n}. Somewhat similar characterizations for $G(n)$ were undertaken in Rahman [7]. However, here the situation is considerably more complicated. In the claim and proof below, we use 0^{-1} to mean: delete the trailing or leading 0 from the string that precedes or follows it.

Claim 2.3. *If $m \geq 3$, then*

$$\begin{aligned} w_{F_{2m}} &= w_{F_{2m-1}} w_{F_{2m-2}} && (2.4)\\ &= 0\sigma(w_{F_{2m-2}}) && (2.5)\\ w_{F_{2m-1}} &= w_{F_{2m-2}}(m-1)w_{F_1}^{m-2} w_{F_3}^{m-3} \cdots w_{F_{2m-7}}^{2} w_{F_{2m-5}}^{1} && (2.6)\\ &= 0\sigma(w_{F_{2m-3}})0^{-1} && (2.7) \end{aligned}$$

Proof: Our proof is by induction on m, where the base cases may be checked using the table. We first prove that the right hand sides of (2.6) and (2.7) are equal, the most interesting of the four equations.

$$\begin{aligned} 0\sigma(w_{F_{2m-1}}) &= 0\sigma(w_{F_{2m-2}}(m-1)w_{F_1}^{m-2} w_{F_3}^{m-3} \cdots w_{F_{2m-7}}^{2} w_{F_{2m-5}}^{1}) \\ &= 0\sigma(w_{F_{2m-2}})\sigma((m-1))\sigma(w_{F_1}^{m-2})\sigma(w_{F_3}^{m-3}) \cdots \sigma(w_{F_{2m-7}}^{2})\sigma(w_{F_{2m-5}}^{1})\sigma(w_{F_{2m-2}}) \\ &= w_{F_{2m}} m 0^{m-1} 0 (0^{-1} w_{F_3} 0)^{m-2} (0^{-1} w_{F_5} 0)^{m-3} \cdots (0^{-1} w_{F_{2m-5}} 0)^2 (0^{-1} w_{F_{2m-3}} 0)^1 \\ &= w_{F_{2m}} m w_{F_1}^{m-1} (w_{F_3})^{m-2} (w_{F_5})^{m-3} \cdots (w_{F_{2m-5}})^2 (w_{F_{2m-3}})^1 0 \qquad (2.8) \end{aligned}$$

This proves (2.7).

To prove (2.6) we will show that the string in (2.8) has the correct length. We will use the following identity. This should be a well-known identity but we have no reference; in any event, it is easily proven by induction.

$$\sum_{j=1}^{m-1} (m-j) F_{2j-1} = F_{2m-1} - 1. \qquad (2.9)$$

It then follows that

$$\begin{aligned} |0\sigma(w_{F_{2m-1}})0^{-1}| &= F_{2m} + 1 + (m-1)F_1 + (m-2)F_3 + \cdots + 2F_{2m-5} + F_{2m-3} \\ &= F_{2m} + 1 + F_{2m-1} - 1 \ = \ F_{2m+1}. \end{aligned}$$

Below we show that (2.5) is true.

$$\begin{aligned} 0\sigma(w_{F_{2m}}) &= 0\sigma(w_{F_{2m-1}})\sigma(w_{F_{2m-2}}) \\ &= w_{F_{2m+1}} 0\sigma(w_{F_{2m-2}}) \\ &= w_{F_{2m+1}} w_{F_{2m}}. \end{aligned}$$

And (2.4) follows from the basic recurrence for Fibonacci numbers. □

3. Algorithmic Implications

It is natural to wonder whether these sorts of decompositions and recurrences might lead to an efficient way to compute $a(n)$. Let us assume that the Fibonacci numbers of size at most n have be pre-computed in a table; this takes $O(\log n)$ arithmetic operations.

The main thing to note is that the decomposition of (2.6) is into $O(\log n)$ ranges whose sizes are constant-sized linear expressions of m and Fibonacci numbers, and thus their a numbers are easily computed. For example,

$$(m-2)F_1 + (m-3)F_3 + \cdots + (m-7)F_{11} = (m-7)F_{12} + F_9 - 1$$

is the cumulative size of the first 6 ranges to the right of the $(m-1)$ in (2.6). We can use binary search to locate the correct range and then associated $a()$ value; for example, again in the first 6 ranges we would have the $a()$ value

$$(m-2)a(F_1)+(m-3)a(F_3)+\cdots+(m-7)a(F_{11}) = (m-7)a(F_{12})+a(F_9)-1 = (m-7)F_{10}+F_7-1.$$

The range in which n lies is at largest a positive fraction (related to golden ratio) of n. And since we are doing a binary search for the right range among $O(m) = O(\log n)$ of them, the search will take $O(\log\log n)$ steps. Once we are in the right range, $O(1)$ arithmetic operations, together with a recursive call, will give us the result. Thus the total number of arithmetic operations, $T(n)$, satisfies the recurrence relation

$$T(n) \leq O(\log\log n) + T(n/\phi),$$

whose solution is $O(\log\log n \log n)$. This gives us a fairly efficient way of computing $a(n)$; certainly better than using the original recurrence relation (1.1) from the problem definition!

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