A Nash Equilibrium of Two-player Indian Poker

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Abstract

We discuss a Nash equilibrium of the simplified version of poker called *Indian Poker*. The background of game theory and linear programming model will be given.

Contents

1 Introduction

Poker is a popular strategy card game.

Figure 1: Scene from James Bond movie

Needless to say, it is a popular source for mathematician and computer scientist to analyze the game.

Simplified models of poker were first studied in a game theoretic frame work by Borel and von Neumann in the 1920s. Von Neumann's results were published later in the seminal book *Theory of Games and Economic Behavior* with Morgenstern [3]. (His version of game is similar to what we consider and will be explained in detail.)

Lately, after the rise of machine learning, people also adapted this method of *equilibrium* in game theory to it for solving the poker game (at least in a particular situation). The algorithm is called *GTO* which is short for *Game Theory Optimal.*

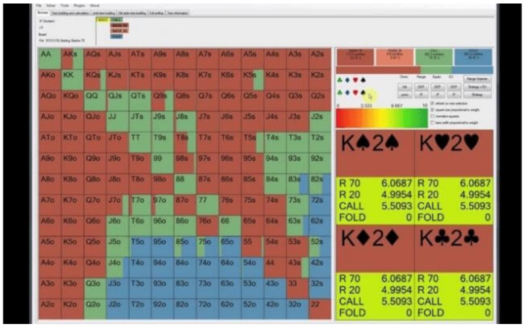


Figure 2: Screen shot of Poker solver which based on GTO algorithm

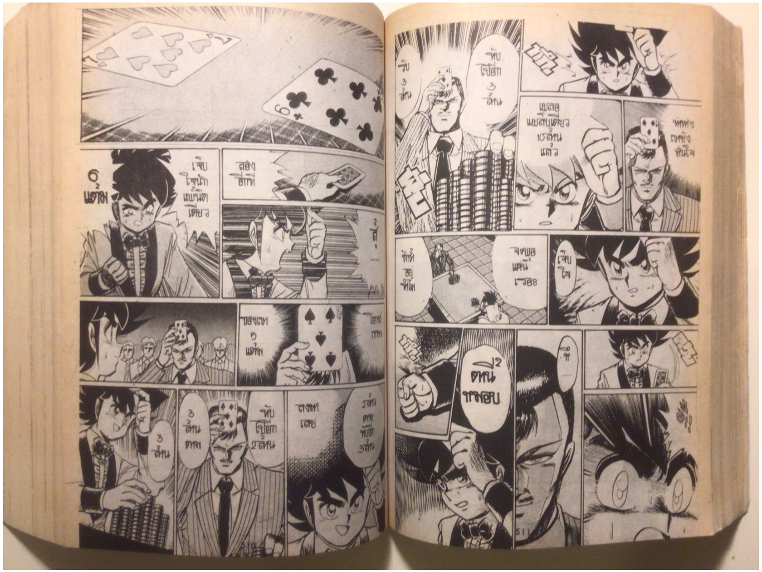
There is no universal definition of the best strategy. Some strategies are good in some situations and others are good in different situations. But many people consider the *equilibrium* to be standard that gives the reasonable outcome of the game. This is also what GTO strategy is based on.

GTO strategy explores the concepts of balance and indifference which minimizes exploitability. When you have minimal knowledge of the opponent's play style, it is a good defensive strategy to play close to GTO, which aims to optimize for the worst case by minimizing your own exploitability. GTO strategy assumes an opponent who also plays optimally, or knows how to exploit weaknesses in any strategy.

For me, the process of learning to play poker optimally is to study the mathematics behind it.

In this work, we start from scratch with minimal knowledge from existing literature. The main goal is to apply game theory along with its linear programming model to solve the simplified version of poker call *Indian Poker*.

Remark: The motivation of choosing this version of poker is from the anime that I read when I was in high school.



2 About Indian Poker

Figure 3: Picture from the website of Bicycle, a card selling company

Standard rule

Object of the Game: The goal of each player is to win the pot, which contains all the bets that the players have made in any one deal. A player makes a bet in hopes that they have the best hand, or to give the impression that they do.

The Deal: A single card is dealt face down to each player.

The Play: On a signal from the dealer, each player simultaneously lifts their card, placing it on their forehead so that all of the other players can see it, but the player cannot see his own. In standard play, there is only one round of betting. In this version, we allow multiple rounds of betting (to add more complexity) and then a show down.

In some games the suits have rank - Spades (high), Hearts, Diamonds, Clubs - so that the Ace of Spades would be the highest card, the Ace of Hearts the next highest, and so on. Here we rank the card from 1 to n for some integer n.

Rules for this work

- Two players
- Cards from 1 to n are randomly dealt to players. Player 1 sees card j of player 2 and player 2 sees card i of player 1. But each of them does not know their own cards.
- Blind is 1 for first player and 2 for second player.
- Each player has a choice of fold, call (check) or raise. The raise has to be at least double the size of the last raise, i.e. first raise must be at least 4, and so on.
- Player 1 starts the play. Players alternate their plays. Game stops when there is a call or fold. In case of call, player with the higher card wins the pot.

3 Game Theory Background

3.1 Zero-sum Game

Poker is a **zero-sum game** i.e. the game where the entries in each cell add up to 0. There is no external banker and that P1's gain is covered by P2's loss and vice versa.

Example 1.

	B plays 1	*B plays 2*	*B plays 3*
A plays 1	*(3, -3)*	*(-4, 4)*	*(2, -2)*
A plays 2	*(-1, 1)*	*(4, -4)*	*(-2, 2)*
A plays 3	*(-3, 3)*	*(1, -1)*	*(4, -4)*
A plays 4	*(1, -1)*	*(-1, 1)*	*(1, -1)*

A pay-off matrix of the zero-sum game is then written from A's point of view only, i.e. $\begin{bmatrix} 3 & -4 & 2 \\ -1 & 4 & -2 \\ -3 & 1 & 4 \\ 1 & -1 & 1 \end{bmatrix}$

3.2 Play safe strategy

Play safe strategy:

For player A (rows): the row maximin (maximum of the minimum values from each row).

For player B (columns): the column minimax (minimum of the maximum values from each column).

Determine the play equilibrium for players A and B for the game below.

$$\begin{vmatrix} 3 & -4 & 2 \\ -1 & 4 & -2 \\ -3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

Solution:

The worst outcomes for A in each row are $-4, -2, -3, -1$. The maximum of these numbers is -1. Hence the play safe strategy for A is to play 4 (which guarantee to get him at least -1 points.)

The worst outcomes for B (maximum) in each column are 3, 4, 4. The minimum of these numbers is 3. Hence the play safe strategy for B is to play 1 (which guarantee to get him at least -3 points.)

3.3 Equilibrium

The solution is called **equilibrium** if neither player has any incentive to change their play safe strategy.

An equilibrium point is the most important thing in game theory. It is considered to be the solution of the game.

Example 2. *Consider the following pay off matrix*

$$\begin{vmatrix} 4 & -1 & 2 & 3 \\ 4 & 6 & 3 & 7 \\ 1 & 2 & -2 & 4 \end{vmatrix}$$

The row minimums are $-1, 3, -2$. The row maximum is 3, so A should play choice 2.

The column maximums are 4,6,3,7. The column minimum is 3, so B should play choice 3.

This results in a win 3 points for A (and a lose of 3 points for B).

We see that there is no incentive for A to change her/his strategies as 3 points is the highest outcome in column 3. There is no incentive for B to change her/his strategies neither as 3 points is the lowest outcome in row 2.

So we conclude that the entry (2,3) is an equilibrium point.

So example 1 has no equilibrium point. But example 2 does have an equilibrium point.

Important theorem!

In a zero-sum game, there will be a equilibrium point if and only if

the row maximin =the column minimax.

3.4 A mixed strategy to guarantee equilibrium point

When the game does not have an equilibrium point, it makes sense to not stick to a particular choice, as your opponent could take an advantage of it. Here the idea of mixed strategy (play each choice randomly with particular probability) comes into play.

Example 3 (Rock-Paper-Scissors)**.** *What is your strategy to play this game?*

	Rock	*Paper*	*Scissors*
Rock	*0*	-1	*1*
Paper	1	*0*	-1
Scissors	-1	*1*	0

Example 4 (Matching pennies)**.** *We consider a matching pennies. The pay-off matrix gives a big pay off for P1 if he can match Head with Head.*

	H	T
H	*100*	-1
T	-1	*1*

What is your recommended strategy to play this game now?

The standard way to solve the mixed strategy problem is to apply **linear programming**.

Solution:

Let p_1 be the probability that player 1 plays H
and p_2 be the probability that player 1 plays T.

Maximize: V

under the constraints:

$$\begin{aligned} p_1 \cdot 100 + p_2 \cdot (-1) &\geq V \\ p_1 \cdot (-1) + p_2 \cdot (1) &\geq V \\ p_1 + p_2 &= 1 \\ p_i &\geq 0, \quad i = 1, 2. \end{aligned}$$

(this is the maximin version of mixed strategy)

Answers: $V = \frac{99}{103}, p_1 = \frac{2}{103}$ and $p_2 = \frac{101}{103}$.

Solution of player 2 is the dual problem of player 1.

Minimize: V

under the constraints:

$$\begin{aligned} q_1 \cdot 100 + q_2 \cdot (-1) &\leq V \\ q_1 \cdot (-1) + q_2 \cdot (1) &\leq V \\ q_1 + q_2 &= 1 \\ q_i &\geq 0, \quad i = 1, 2. \end{aligned}$$

(this is the minimax version of mixed strategy)

Answers: $V = \dfrac{99}{103}, q_1 = \dfrac{2}{103}$ and $q_2 = \dfrac{101}{103}$.

4 Back to Indian Poker

4.1 Trivial Model

Recall that the deck has n cards with the blind to be 1 (first player) and 2 (second player). Player 1 sees card of player 2 (say j) and player 2 sees card of player 1. But neither see the cards of themselves.

For this model, we assume **player 1 can only call or fold**. Very simple model!

Again if he fold, he loses 1. If he calls, he could either win 2 or lose 2.

What is the solution for this model?

Player 1 should call if he sees the low number card of player 2 and fold if he sees the high number card of player 2.

But at which point he should switch from call to fold?

Payoff call is $p_{win} \cdot 2 + p_{lose} \cdot (-2)$
Payoff fold is -1

Set equal:

$$\frac{n-j}{n-1}2 + \frac{j-1}{n-1}(-2) = -1.$$

Answer: $j^* = \left\lfloor \frac{3n+1}{4} \right\rfloor$.

The average pay off is

$$\frac{1}{n}\left[\sum_{j=1}^{j^*} \frac{n-2j+1}{n-1}(2) + \sum_{j=j^*+1}^{n} -1\right] = \frac{1}{8} - \frac{5}{8n}.$$

Summarize (from player 1 point of view): Call if see $1, 2, \ldots, \left\lfloor \frac{3n+1}{4} \right\rfloor$, and fold if see $\left\lfloor \frac{3n+1}{4} \right\rfloor + 1, \ldots, n$.

The continuous version of n cards can also be considered. That is the number on the card is chosen randomly from the interval $(0, 1)$.

Solution: Player 1 call if see card from $(0, 3/4)$ and fold if see card from $(3/4, 1)$. The average payoff becomes

$$\int_{x=0}^{3/4} (1-x)2 + x(-2)\, dx + \int_{x=3/4}^{1} (-1)\, dx = 1/8.$$

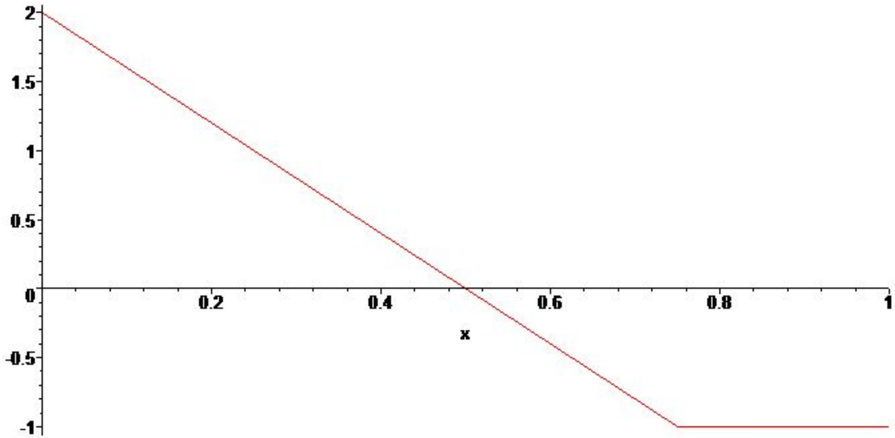


Figure 4: Payoff for player 1, where x-axis is the value of the card he sees ranging from 0 to 1.

4.2 Simple Model: Only player 1 can raise

Once again, we recall that the deck has n cards with the blind to be 1 (first player) and 2 (second player). Player 1 sees card of player 2 (say j) and player 2 sees card of player 1 (say i). But neither see the cards of themselves.

For this model, we assume **player 1 can only call, fold or raise**. In case of player 1 raises, **player 2 can call or fold**.

For simplicity, we set the raise to be $R = 4$ which is the minimum raise allowed.

We need to some game theory/linear programming to solve for numeric result for a fixed n cards.

Exponential number of Constraints Model

For $n = 2$, this is the payoff table,

	[c,c]	[c,f]	[f,c]	[f,f]
[r,r]	0	-1	3	2
[r,c]	1	0	1	0
[r,f]	3/2	1/2	3/2	1/2
[c,r]	-1	-1	2	2
[c,c]	0	0	0	0
[c,f]	1/2	1/2	1/2	1/2
[f,r]	$-5/2$	$-5/2$	1/2	1/2
[f,c]	$-3/2$	$-3/2$	$-3/2$	$-3/2$
[f,f]	-1	-1	-1	-1

Do you notice an equilibrium point?

By our LP solution, we have the solution that $V = 1/2$ which the equilibrium point (not necessary unique) is $[r, f]$ for player 1 and $[c, f]$ for player 2.

To give you more idea. Here is the result for $n = 6$, i.e. 6 cards:

$V = 3/10$
Player 1 plays mixed strategy: $[r, c, c, c, r, f]$ with probability $1/3$ and $[r, c, c, c, f, f]$ with probability $2/3$.
Meanwhile player 2 also plays mixed strategy: $[c, c, c, f, f, f]$ with probability $1/2$, and $[c, c, f, f, f, f]$ with probability $1/2$.

The main drawback of doing it this way is there are **exponential number** of constraints.

Linear number of Constraints Model

New Model:

Maximize $V_1 + V_2 + \cdots + V_n$

Constraints:

$$\frac{1}{n-1} \cdot \sum_{j, j \neq i} [\text{Paid}_{call}(i,j) \cdot p_2(j) + \text{Paid}_{fold}(i,j) \cdot p_1(j) + \text{Paid}_{raise}(i,j) \cdot p_3(j)] \geq V_i,$$
$i = 1, \ldots, n.$

$$\frac{1}{n-1} \cdot \sum_{j, j \neq i} [\text{Paid}_{call}(i,j) \cdot p_2(j) + \text{Paid}_{fold}(i,j) \cdot p_1(j) + \text{Paid}_{fold}(i,j) \cdot p_3(j)] \geq V_i,$$
$i = 1, \ldots, n.$

$p_1(j), \;\; p_2(j), \;\; p_3(j) \geq 0$ and $p_1(j) + p_2(j) + p_3(j) = 1, \qquad j = 1, 2, \ldots, n.$

The program runs much faster with this new constraints.

Here is the result for $n = 100$, i.e. 100 cards:

$V = 25/99$
Player 1 plays mixed strategy:

$$[r, r, \ldots, r, c, c, \ldots, c, f, f, \ldots, f, r, \ldots, r]$$

Raise for the first 25 cards, call the next 50, fold the next 16+2/3 and raise for the rest.

Meanwhile player 2 also plays mixed strategy:

$$[c, c, c, \ldots, c, f, f, f, \ldots f]$$

Call 49.5 cards and fold 50.5 cards.

Continuous Version

So we see the clear pattern from running the linear programming. We turns the problem to continuous version once again. That is the number on the card is chosen randomly from the interval $(0, 1)$.

Solution:

The strategy of player 1 can be written as

$$S_1 = \begin{cases} raise & \text{if } x \in (0, A) \text{ or } x > A + B + C, \\ call & \text{if } x \in (A, A + B), \\ fold & \text{if } x \in (A + B, A + B + C). \end{cases}$$

The strategy of player 2 can be written as

$$S_2 = \begin{cases} call & \text{if } y \in (0, D), \\ fold & \text{if } y > D. \end{cases}$$

From the numerical results, it is safe to assume that $A < D < A + B$.

We solve for A, B, C, D with the *method of indifference.*

If player 1 sees card A:

Payoff if player 1 raises $= -4A + 4(D - A) + 2(1 - D)$

Payoff if player 1 calls $= -2A + 2(1 - A)$

Combine to get $D = 2A$.

If player 1 sees card $A + B$:

Payoff if player 1 calls $= -2(A + B) + 2(1 - A - B)$

Payoff if player 1 folds $= -1$

Combine to get $A + B = 3/4$. (we did this earlier)

If player 1 sees card $A + B + C$:

Payoff if player 1 folds $= -1$

Payoff if player 1 raises $= -4D + 2(1 - D)$

Combine to get $D = 1/2$.

So so far, we know $D = 1/2, A = 1/4$ and $B = 1/2$.

If player 2 sees card D:

Payoff if player 2 calls $= 4\dfrac{A}{A + (1 - A - B - C)} - 4\dfrac{1 - A - B - C}{A + (1 - A - B - C)}$

Payoff if player 2 folds $= 2$

Combine all these to get $C = 1/6$.

The average payoff becomes

$$\int_{x=0}^{1/4} -4x + 4(1/2 - x) + 2(1/2)\,dx + \int_{x=1/4}^{3/4} (1-x)2 + x(-2)\,dx$$
$$+ \int_{x=3/4}^{11/12} (-1)\,dx + \int_{x=11/12}^{1} -4(1/2) + 2(1/2)\,dx = 1/4.$$

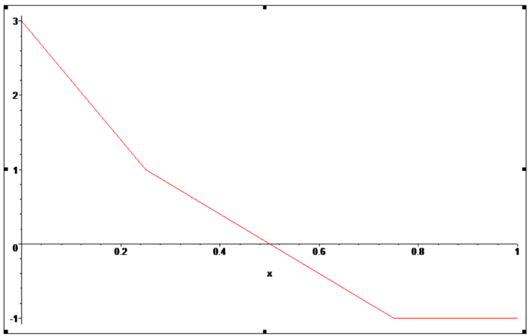


Figure 5: Payoff for player 1. Raise option helps him to improve his payoff

More Analysis: other values of raise

Now let's consider raise $= R$.

With similar calculations, we have

$$D = 2A, \quad B = 3/4, \quad D = \frac{3}{R+2}, \quad C = \frac{1}{4} - \frac{R-2}{R+2}A.$$

The average payoff becomes

$$\int_{x=0}^{\frac{3}{2(R+2)}} -Rx + R\left(\frac{3}{R+2} - x\right) + 2\left(1 - \frac{3}{R+2}\right) dx$$
$$+ \int_{\frac{3}{2(R+2)}}^{3/4} (1-x)2 + x(-2)\, dx$$
$$+ \int_{3/4}^{1-\frac{3}{2}\frac{R-2}{(R+2)^2}} (-1)\, dx + \int_{x=1-\frac{3}{2}\frac{R-2}{(R+2)^2}}^{1} -R\left(\frac{3}{R+2}\right) + 2\left(1 - \frac{3}{R+2}\right) dx$$
$$= \frac{R^2 + 22R - 32}{8(R+2)^2}.$$

The optimal payoff is $\frac{17}{64}$ when $R = 6$.

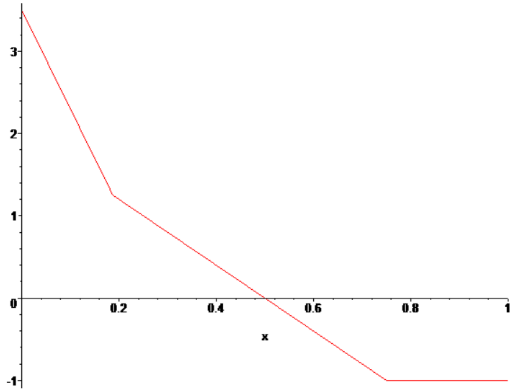


Figure 6: Payoff for player 1 with option of Raise=6

Multiple values of raises

Now we allow the first player 1 to raise to any amount (≥ 4).

Example

Consider the option of raise to be raise $R = 6$ or 4. We will find the "best strategies" for both player 1 and player 2 and the payoff.

Strategy of player 1 upon seeing a card $x \in (0, 1)$:

[raise R, raise 4, call, fold, raise 4, raise R] that is raise R if $x \in (0, A)$, raise 4 if $x \in (A, B)$, call if $x \in (B, C)$, fold if $x \in (C, D)$, raise 4 if $x \in (D, E)$, raise R if $x \in (E, 1)$.

Strategy for player 2:

Respond to raise=R: Player 2 should call if he sees card in $(0, F_1)$ and fold otherwise.
Respond to raise=4: Player 2 should call if he sees card in $(0, F_2)$ and fold otherwise.

The payoff function is

$$P = L_1 + L_2 + L_3 + L_4 + L_5 + L_6,$$

where

$$L_1 = \int_0^A -Rx + R(F_1 - x) + 2(1 - F_1)\,dx,$$
$$L_2 = \int_A^B -4x + 4(F_2 - x) + 2(1 - F_2)\,dx,$$
$$L_3 = \int_B^C -2x + 2(1 - x)\,dx,$$
$$L_4 = \int_C^D -1\,dx,$$
$$L_5 = \int_D^E -4F_2 + 2(1 - F_2)\,dx,$$
$$L_6 = \int_E^1 -RF_1 + 2(1 - F_1)\,dx.$$

Use calculus to optimize P with respect to A, B, C, D, E, F_1, F_2 to get

$$(A, B, C, D, E) = (1/8, 1/4, 3/4, 43/48, 15/16)$$

and $F_1 = 3/8,\ F_2 = 1/2$.

The optimal value of payoff P is $\frac{9}{32}$.

Conclusion: Player 1 raises with 6 when he sees card in (0,1/8), raises with 4 when sees card in (1/8,1/4), call when sees card in (1/4,3/4) and so on.

Respond to raise = 6: Player 2 should call if he sees card in $(0, 3/8)$ and fold otherwise. Respond to raise = 4: Player 2 should call if he sees card in $(0, 1/2)$ and fold otherwise.

The payoff of player 1 improves from $\frac{17}{64}$ to $\frac{9}{32} = 0.28125$.

Note: We wrote the program on Maple to calculate the best strategies and payoff for each options of raises.

If we limit to **two** options of raises as in the previous example, we can show that the best strategies of player 1 is to raise to 10 or 4. The payoff for him will be $\frac{7}{24} = 0.291667$.

If we limit to **three** options of raises (each raise ≥ 4), we can show that the best strategies of player 1 is to raise to 16, 7 or 4. The payoff for him will be $\frac{97}{324} = 0.299382$.

If we limit to **four** options of raises (each raise ≥ 4), we can show that the best strategies of player 1 is to raise to 22, 10, 6 or 4. The payoff for him will be $\frac{29}{96} = 0.302083$.

. . .

Payoff of the Optimal strategy

We will not stop for the fixed number of option for raising. We will answer the question of the payoff if Player 1 can raise freely with no limited number of options.

Strategy of player 1 upon seeing a card $x \in (0, 1)$:

[raise R_i, call, fold, raise R_i]

that is raise R_i if $x \in (0, A)$, call if $x \in (A, B)$, fold if $x \in (B, C)$ and raise R_i if $x \in (C, 1)$,

Strategy for player 2:

For each respond to raise $= R_i$: Player 2 should call if he sees card in $(0, D_i)$ and fold otherwise.

Note: By the rule of betting, $R_i \geq 4$ for each i.

The payoff function is

$$P = \sum_{i=1}^{k} \int_{A_{i-1}}^{A_i} [-R_i x + R_i(D_i - x) + 2(1 - D_i)]\, dx + \int_{A_k}^{B} 2(1-x) - 2x\, dx$$
$$+ \int_{B}^{C_k} (-1)\, dx + \sum_{i=1}^{k} \int_{C_i}^{C_{i-1}} [-R_i D_i + 2(1 - D_i)]\, dx.$$

We use calculus to find the relation of A_i, B, C_i, D_i. Some of the values and relations can be solved right away.

$$\frac{\partial P}{\partial B} = 0 \implies 2(1-B) - 2B = -1 \implies B = \frac{3}{4}.$$
$$\frac{\partial P}{\partial C_k} = 0 \implies -1 = -R_k D_k + 2(1 - D_k) \implies D_k = \frac{1}{2},$$

Note $R_k = 4$.

$$\frac{\partial P}{\partial A_k} = 0 \implies -R_k A_k + R_k(D_k - A_k) + 2(1 - D_k) = 2 - 4A_k$$
$$\implies D_k = 2A_k \implies A_k = \frac{1}{4}.$$

For each i, $1 \leq i \leq k - 1$,

$$\frac{\partial P}{\partial C_i} = 0 \implies -R_i D_i + 2(1 - D_i) = -R_{i+1} D_{i+1} + 2(1 - D_{i+1})$$
$$\implies (R_i + 2)D_i = (R_{i+1} + 2)D_{i+1}.$$

Solving this successively to get

$$(R_i + 2)D_i = (R_k + 2)D_k = 3.$$

Remark: These D_i turns out to be the same as those from one fixed raise previously:

$$D_i = \frac{3}{R_i + 2}.$$

With this fact, the payoff for the **first part** at each $x \in (0, A)$ becomes

$$-R \cdot x + R \cdot \left(\frac{3}{R+2} - x \right) + 2 \cdot \left(1 - \frac{3}{R+2} \right).$$

Hence, for a given x, the optimal value of raise satisfies the relation:

$$x = \frac{6}{(R+2)^2}.$$

Then the whole payoff for the interval $x \in (0, A)$ becomes (recall that $R \geq 4$)

$$
\begin{aligned}
P_1 &= \int_0^{1/6} -Rx + R\left(\frac{3}{R+2} - x\right) + 2\left(1 - \frac{3}{R+2}\right) dx \\
&+ \int_{1/6}^{1/4} -4x + 4\left(\frac{1}{2} - x\right) + 1\, dx \\
&= \left[\int_{\infty}^{4} \frac{-6R}{(R+2)^2} + R\left(\frac{3}{R+2} - \frac{6}{(R+2)^2}\right) + 2\left(1 - \frac{3}{R+2}\right)\right] \cdot \frac{(-12)}{(R+2)^3}\, dR \\
&+ \int_{1/6}^{1/4} 3 - 8x\, dx \\
&= 4/9 + 1/9 = 5/9.
\end{aligned}
$$

For the **second part** of $x \in (A, B)$, the payoff is

$$P_2 = \int_{1/4}^{3/4} (-2)x + 2(1-x)\, dx = 0.$$

For the **last part** of $x \in (B, 1)$, the payoff is

$$\begin{aligned} P_3 &= \int_{3/4}^{C} (-1)\,dx + \int_{C}^{1} -RD + 2(1-D)\,dx \\ &= \int_{3/4}^{C} (-1)\,dx + \int_{C}^{1} (-1)\,dx \\ &= \frac{3}{4} - 1 = -\frac{1}{4}. \end{aligned}$$

Hence the total payoff is $\frac{5}{9} + 0 - \frac{1}{4} = \frac{11}{36}$.

Conclusion

With this strategy, the optimal payoff of player 1 has improved to $11/36 = 0.305556$.

4.3 Beyond what we show

We will try to extend this format to where both players can raise back and forth indefinitely. This will completely solve this version of Two-player Indian Poker.

References

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[2] D. J. Newman. *A model for "real" poker*, Operations Research, 7(5):557-560, 1959.

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5 The John von Neumann's Model

The von Neumann and Newman poker models, [1, 3], are simplified two-person poker models in which hands are modeled by real values drawn uniformly at random from the unit interval.

He considered the following $(0, 1)$ two-player game: There is an initial pot of size P (consisting of player's antes). Each player is dealt a 'hand', i.e., a real number x, respectively y, drawn independently and uniformly at random from the unit interval. After X has inspected x, he can make a bet of a chips or check. If he bets, Y can either call (i.e., match X's bet of a chips) or fold (i.e., concede the pot to X), where of course he can base this decision on his own hand y. If X checks or if his bet is called by Y, a showdown occurs, i.e., both hands are revealed and the player with the higher hand value wins the pot and all bets made. Note that only X is allowed to make a bet. Thus he has an advantage over Y, who can only react.

In the above model, one can either prescribe the amount a that X bets if he opts to bet, or let X choose the size of his bet freely as a function of his hand x. The former case is the game originally studied by von Neumann. Here we assume $a = P$ (pot size bet) for simplicity. Von Neumann showed that in optimal play X has an expected payoff of $\frac{5}{9}P$. A key insight of his solution is that bluffing is a game-theoretic necessity: X will not only bet with his good hands, but also with very low-ranked hands. These bluff-bets aim at inducing his opponent to make more calls, even with only mediocre hands, which in turn enables X to extract more value from Y with his good hands. The case of variable bet size $a(x)$ was studied by Newman, [2], in the 1950s. Certainly, in this setup player X has an even greater advantage - Newman proved an expected payoff of $\frac{4}{7}P$ in optimal play.

5.1 Solution to von Neumann's model

Assume each player put 1 into the pot.

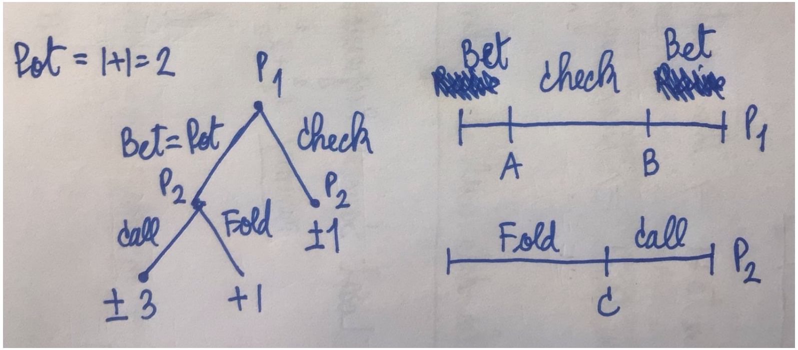

Figure 7: The setup for this problem

Assume $A < C < B$. The payoff is

$$P = \int_0^A C - 3(1-C)\,dx + \int_A^B x - (1-x)\,dx + \int_B^1 C + 3(x-C) - 3(1-x)\,dx.$$

Use calculus to solve for the optimal solution.

$$\begin{aligned} 4C - 3 &= 2A - 1 \\ 2B - 1 &= C + 3(B-C) - 3(1-B) \\ 4A &= 2(1-B) \end{aligned}$$

We got $A = 1/9, B = 7/9$ and $C = 5/9$. The payoff is $1/9$.

5.2 Solution to Newman's model

This time the bet $R \in (0, \infty)$.

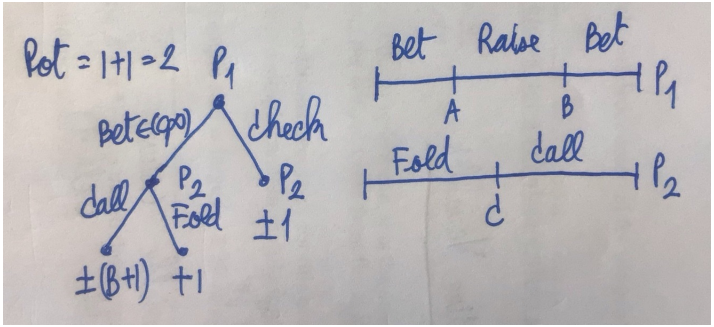

Figure 8: The setup for this problem

For $1 \leq i \leq k$, let $R_i > 0$ be the raise on (A_{i-1}, A_i) and on (B_{i-1}, B_i) for player 1. The correspond strategy to raise R_i is C_i (as in the picture) for player 2. We also let $A_0 = 0, B_0 = 1, A_k = A, B_k = B$ and $C_k = C$.

Assume $A_i < C_i < B_i$ for each i. The payoff is

$$P = \sum_{i=1}^{k} \int_{A_{i-1}}^{A_i} C_i - (R_i + 1)(1 - C_i)\, dx + \int_{A_k}^{B_k} x - (1 - x)\, dx$$
$$+ \sum_{i=1}^{k} \int_{B_i}^{B_{i-1}} C_i + (R_i + 1)(x - C_i) - (R_i + 1)(1 - x)\, dx.$$

We use calculus to find the relation of A_i, B_i, C_i. Some of the values and relations can be found right away.

Note $R_k = 0$.

$$\frac{\partial P}{\partial A_k} = 0 \implies C_k - (R_k + 1)(1 - C_k) = 2A_k - 1 \implies A_k = C_k.$$

$$\frac{\partial P}{\partial B_k} = 0 \implies 2B_k - 1 = C_k + B_k - C_k - 1 + B_k \implies 0 = 0.$$

For each i, $1 \leq i \leq k-1$,

$$\frac{\partial P}{\partial A_i} = 0 \implies C_i - (R_i + 1)(1 - C_i) = C_{i+1} - (R_{i+1} + 1)(1 - C_{i+1}).$$

Solving this successively to get

$$C_i - (R_i + 1)(1 - C_i) = C_k - (R_k + 1)(1 - C_k) \implies C_i = \frac{R_i + 2C_k}{R_i + 2}.$$

Payoff

For the **first part** of $x \in (0, A_k)$, the payoff is

$$P_1 = \int_0^{A_k} C - (R+1)(1-C)\, dx \int_0^{A_k} 2C_k - 1\, dx = A_k(2C_k - 1).$$

For the **second part** of $x \in (A_k, B_k)$, the payoff is

$$P_2 = \int_{A_k}^{B_k} 2x - 1\, dx = B_k^2 - B_k - A_k^2 + A_k.$$

With this relation of C_i, the payoff for the **last part** at each $x \in (B_k, 1)$ becomes

$$\frac{R + 2C_k}{R+2} + (R+1)\left(x - \frac{R+2C_k}{R+2}\right) - (R+1)(1-x).$$

Hence, for a given x, the optimal value of raise satisfies the relation:

$$x = \frac{R^2 + 4R + 2C_k + 2}{(R+2)^2}.$$

Then the whole payoff for the interval $x \in (B_k, 1)$ becomes

$$\begin{aligned} P_3 &= \int_{B_k}^{1} \frac{R + 2C_k}{R+2} + (R+1)\left(x - \frac{R+2C_k}{R+2}\right) - (R+1)(1-x)\,dx \\ &= \int_0^{\infty} \frac{(3-2C_k)R^2 + 4R + 4C_k}{(R+2)^2} \cdot \frac{4(1-C_k)}{(R+2)^3}\,dR \\ &= \frac{(5+C_k)(1-C_k)}{12}. \end{aligned}$$

We stress again that $A_k = C_k$ and $B_k = x_{\{R=0\}} = \dfrac{C_k + 1}{2}$.

Hence the total payoff is

$$\begin{aligned} P &= P_1 + P_2 + P_3 \\ &= A_k(2C_k - 1) + B_k^2 - B_k - A_k^2 + A_k + \frac{(5 + C_k)(1 - C_k)}{12} \\ &= \frac{7}{6}C^2 - \frac{C}{3} + \frac{1}{6}. \end{aligned}$$

P has an optimal value at $C = \dfrac{1}{7}$ with value $\dfrac{1}{7}$.

Note: $A_k = C_k = 1/7$ and $B_k = 4/7$.

Remark: I hope all this calculations are rigorous.